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Study of Solitary Waves in a Silicon Waveguide with Two Photon Absorption effect using Hirota's bilinear method

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ABSTRACT

In a Silicon waveguide, we have studied solitary wave solutions with the pulse propagation equation that governs the dynamics of solitons by employing the powerful analytical Hirota's bilinear method. The bright and dark soliton solutions are obtained in such a medium under some restricted conditions. We have noticed that Two Photon absorption effect, one of the important features in Silicon waveguide, influences much higher on dark soliton propagation than that on bright solitons. Soliton interaction is an extremely important issue for various practical applications in optical communication systems. By using Hirota's bilinear method we obtain the bright two and three soliton solutions and dark two soliton solutions to study interaction among them. Our results show that by carefully selecting soliton and medium parameters, soliton-soliton interaction can be controlled which is crucial for various applications in nonlinear systems.

Keywords: Optical solitons, Nonlinear Schrodinger Equation, Two Photon Absorption, Hirota method.

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I. Introduction

Silicon photonics has gained significant attention due to their potential application in optical communication for particularly producing optoelectronic devices ([1], [2]). Si is optimum medium for IC integrated devices due to its most interesting linear and nonlinear optical properties in spectral area covering to mid-infrared from near regions ([3]-[5]). Even, the tight strong mode confinement property results in Si waveguide for reducing cross-section to ultra small size ($< 0.1\mu m^2$). Si exhibits a large third order nonlinear susceptibility compared to air/silica and This characteristic enables the formation of solitons in a very small waveguide dimension([6]-[9]). Because of this large Kerr nonlinear parameter in addition with the strong tight mode confinement leads to

the further enhancement of nonlinear properties at moderately low optical power of few mW and this makes more fascinating area of research for assessing nonlinear phenomenon like XPM(crossphase modulation), SPM(self-phase modulation), SRC(stimulated Raman scattering) along with FWM(four wave mixing) ([10]-[15]).One key feature of Si is dispersion tailoring where particularly group velocity dispersion (GVD) may be altered by waveguide's geometry because of their ultra-short dimensions ([16]). This allows for soliton formation though Si has normal GVD at interest wavelength and its value is three times larger than the conventional optical fiber.

The experimental observation of optical soliton inside small SOI(Silicon-on-Insulator) waveguide (only 5mm long) was first reported in ([9]) that could result in novel SOI waveguide application associated with optical switching ([17]) along with optical interconnects ([18]). Optical soliton in Si waveguide is a delicate balance between dispersion and nonlinear effects ([19]). However, the unique feature of Si such as FCA(free carrier absorption), TPA(two-photon absorption) impact on pulse shapes ([16]). Further it is noticed that TPA restricts SPM's extent by using nonlinear absorption in SOI waveguide ([8]).

TPA's impact over soliton propagation in Si waveguide is studied numerically in our previous work ([20]). Dark as well as bright soliton are having unique characteristics during propogation in optical communication systems. Bright soliton means pulse with 0 intensity background where soliton is a localized peak in intensity, while dark soliton appears as a drop in intensity within an infinite CW background where the soliton is a localized decrease in intensity. For this paper, we documented exact solitary wave solutions for bright as well as dark solitons with TPA effect in Si waveguide by utilizing powerful analytical Hirota bilinear approach ([21], [22]) in anomalous dispersion regime. In the optical soliton communication system, soliton interaction is an extremely important issue that needs to be considered for the transmission of information purpose. The interaction of solitons has been studied widely in silica based optical fiber ([23], [24]). However, we have noticed that there has not been much discussion on soliton soliton interaction in Si waveguide. Very few reports have been published on bright and dark solitons interactions in SOI optical waveguide in recent years ([25]-[27]). We utilized powerful Hirota' bilinear technique to study soliton-soliton interaction which is widely used in nonlinear systems ([28]-[33]). For the first time of our knowledge, we have studied soliton-soliton interaction analytically in Si waveguide for both bright and dark solitons. We have noticed that soliton soliton interactions may be controlled or manipulated through a judicious choice of soliton and medium parameters.

II. Model

NLSE(Nonlinear Schrödinger Equation) provides formula to govern propagation of pulse in silicon waveguide, that describes pulse envelope evolution in time and space. The NLSE is given by

$$\frac{\partial A}{\partial z}+\frac{1}{2}\rho A+i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2}=i\gamma|A|^2A-\frac{c\beta_T}{2n_2\omega_0}\gamma|A|^2A \tag{1}$$

where A,β_2,ρ as well as γ denotes slowly varying field amplitude, second order dispersion coefficient, linear loss parameter, TPA coefficient and nonlinear Kerr coefficient correspondingly. We use dimensionless variables as $U=\frac{A}{\sqrt{P_0}},\xi=\frac{z}{L_D},\tau=\frac{t}{T_0},N^2=\frac{\gamma P_0T_0^2}{|\beta_2|},u=NU$ here $T_0,P_0\&N$ are pulse width, initial pulse power and order of the soliton respectively. Hence, the normalized form of Eq. (1) is

$$\frac{\partial u}{\partial \xi}=\frac{i}{2}\frac{\partial^2 u}{\partial \tau^2}+i(1+ir)|u|^2u-\rho_0u \tag{2}$$

Here $r=\frac{c\beta_T}{2n_2\omega_0}=$ dimensionless TPA parameter, $\rho_0=\frac{\rho T_0^2}{2|\beta_2|}$ denotes dimensionless loss parameter that comprises free-carrier absorption along with linear scattering loss. We are interested to study the exact 1 & 2 solitary wave solutions using Hirota's bilinear method.

III. Bright One-Soliton Solution

We take $\alpha=1+ir$ is dimensionless nonlinear parameter that comprises both Kerr nonlinearity and TPA effect. We neglect the loss parameter as we know it only decreases the amplitude of the pulse envelope during propagation. Then Eq. (2) can be put in below given form as

$$i\frac{\partial u}{\partial \xi}+\frac{1}{2}\frac{\partial^2 u}{\partial \tau^2}+\alpha|u|^2u=0 \tag{3}$$

Now, we apply the following transformation $u=\frac{g(\xi,\tau)}{f(\xi,\tau)}$, here $g(\xi,\tau)$ denotes a complex differentiable function & $f(\xi,\tau)$ is a real one. Because of such transformation Eq. (3) turns into

$$f\left[iD_\xi+\frac{1}{2}D_\tau^2\right]g.f-2g\left[\frac{1}{2}D_\tau^2f.f-\alpha g^*g\right]=0 \tag{4}$$

where $D_\xi\&D_\tau$ are known as Hirota bilinear operators determined in [21-22] as

$$D_\xi^mD_\tau^n g(\xi,\tau)\cdot f(\xi,\tau)=\left(\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\xi'}\right)^m\left(\frac{\partial}{\partial\tau}-\frac{\partial}{\partial\tau'}\right)^n g(\xi,\tau)f(\xi',\tau')\bigg|_{\xi=\xi',\tau=\tau'} \tag{5}$$

Eq. (4) can be decoupled as

$$\left[iD_\xi + \frac{1}{2}D_\tau^2\right]g.f = 0 \text{ and } \left[\frac{1}{2}D_\tau^2 f.f - \alpha g^* g\right] = 0 \tag{6}$$

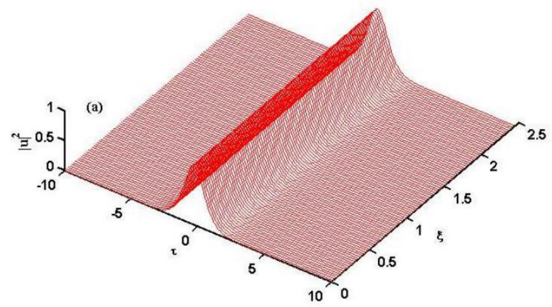
For obtaining 1 soliton solution, it is assumed $g = \chi g_1, f = 1 + \chi^2 f_1$, here χ = arbitrary parameter. Substituting the in Eq.(6) hence then gathering terms using identical powers in χ, we get

$$\left[iD_\xi + \frac{1}{2}D_\tau^2\right]g_1 \cdot 1 = 0, D_\tau^2 f_1 \cdot 1 = \alpha g_1 g_1^*, \left[iD_\xi + \frac{1}{2}D_\tau^2\right]g_1 \cdot f_1 = 0, D_\tau^2 f_1 \cdot f_1 = 0 \tag{7}$$

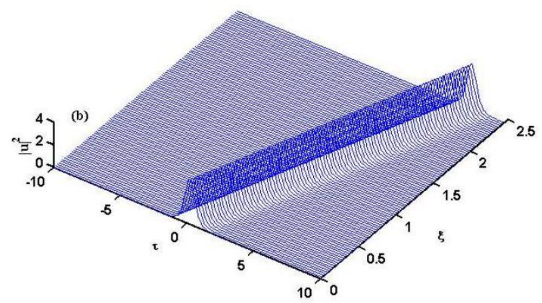
It is simple to verify that the solution consistent to system of Eq. (7) $g_1 = e^{\eta_1}, f_1 = A_1 e^{\eta_1 + \eta_1^*}$ where $\eta_1 = P_1\tau + \Omega_1(\xi) + \phi_0$, in which P_1 is an arbitrary complex parameter, ϕ_0 is a real constant and $\Omega_1(\xi)$ is a differentiable function given by $\Omega_1(\xi) = \frac{iP_1^2\xi}{2}$. Here, A_1 is a complex parameter which can be evaluated from (7) as $A_1 = \frac{\alpha}{(P_1+P_1^*)^2}$. Without loss of generality, setting $\chi = 1$, we obtain the following explicit form of the bright one soliton solution of Eq. (2) as

$$u = \frac{g_1}{1+f_1} = \frac{e^{\eta_1}}{1 + \frac{\alpha}{(P_1 + P_1^*)^2}e^{\eta_1+\eta_1^*}} \tag{8}$$

It is clear that the fundamental bright soliton solution is depend on the parameters P_1 and α. In Fig. 1(a) and Fig. 1(b) we plot the spatio-temporal evolution of the fundamental soliton for $P_1 = 1$ and $P_1 = 2 + 3i$ for $\alpha = 1$ with $r = 0$ and $\phi_0 = 1$. It is observed that the phase of soliton gets changed due to different values of P_1, however without affecting the stable and robust soliton propagation.



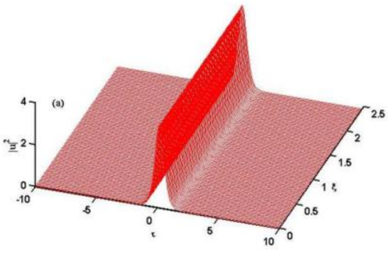
(a)



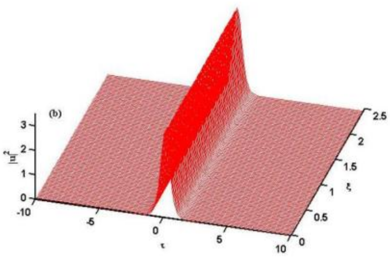
(b)

Figure 1: Spatio-temporal evolution of bright one soliton for $\alpha = \mathbf{1}, \phi_0 = 1$ with (a) $P_1 = 1$, (b) $P_1 = 2 + 3i$.

Moreover, we find that α which signifies the effect of TPA parameter just controls the intensity of the soliton pulse. In Fig. 2(a) and Fig. 2(b) we plot the Spatio temporal evolution of fundamental bright soliton for $r = 0.2$ and $r = 0.8$ with $P_1 = 2, \phi_0 = 1$ respectively. The soliton amplitude or intensity slightly decreases with the increase in the value of the TPA parameter. It is clearly visible in the two-dimensional intensity profile in Fig. 2(c) for different values of $r = 0,0.4,0.8$. Thus, the TPA effect only decreases the intensity profile of soliton propagation in time domain.



(a)



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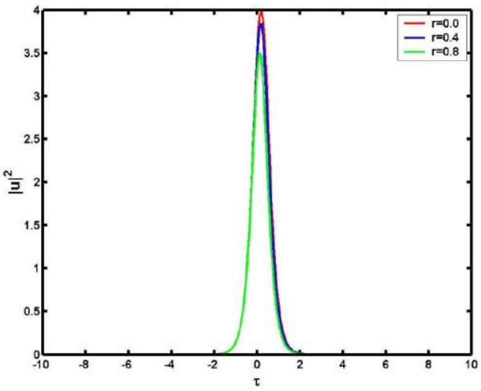
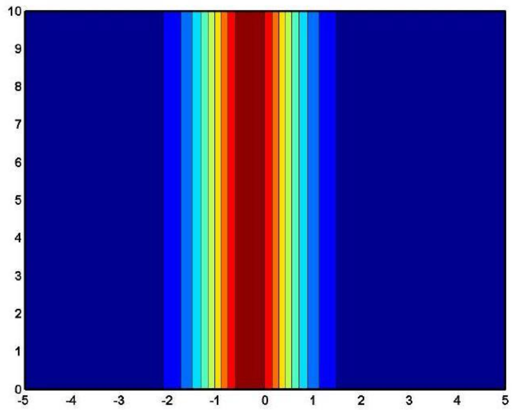


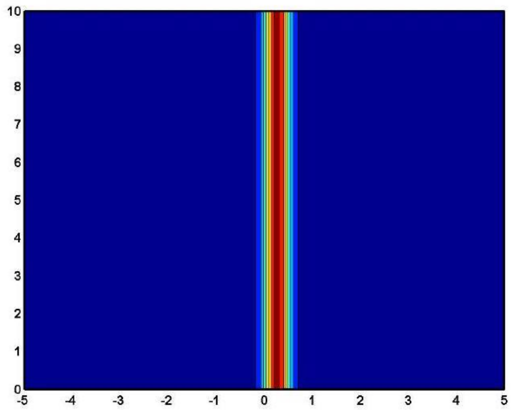
(c)

Figure 2: Spatio-temporal evolution of bright one soliton for $\phi_0 = 1$ and $P_1 = 2$ with (a) $\alpha = 1 + 0.2i$, (b) $\alpha = 1 + 0.8i$ (c) two-dimensional intensity profile for different normalized TPA parameters $r = 0{,}0.4{,}0.8$.

Contour plot provides valuable insights into the behavior of soliton dynamics under different conditions. In our work , it is observed that when the contour lines are set apart as shown in Fig.



(a)



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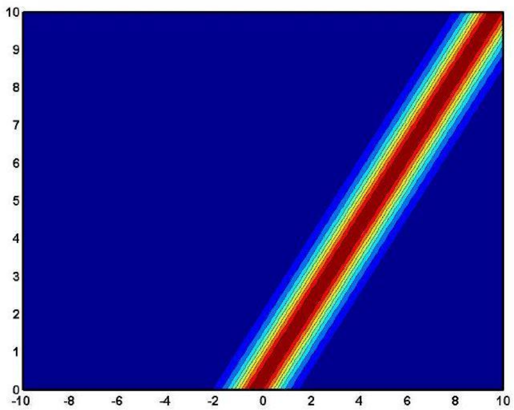


(c)

Figure 3: Contour plot of bright one soliton solution with $\phi_0 = 1$ and $\alpha = 1 + 0.2i$ for (a) $P_1 = 1$ (b) $P_1 = 4$ (c) $P_1 = 1 + i$.

3(a) with $P_1 = 1$ for one bright soliton, it indicated that the intensity of bright soliton changes rapidly. In Fig. 3(b) with $P_1 = 4$, where the contour lines are close together, it suggests that the intensity of soliton profile changes slowly. It is further noticed that in Fig. 3(c) for $P_1 = 1 + i$ in the contour plot the phase of soliton gets changed maintaining its stable and robust propagation.

IV. Bright Two-Solition Solution

Soliton interaction is an extremely crucial and relevant issue for various practical applications in optical communication systems. In the high bit rate and long-distance communication systems, multiple signals propagate simultaneously which makes soliton interactions essential to consider in transmitting the information through the systems. To study the soliton interactions in Si waveguide, we find two soliton solutions using Hirota method and for that we assume

$$g = \chi g_1 + \chi^3 g_2, f = 1 + \chi^2 f_1 + \chi^4 f_2 \quad (9)$$

where $g_1 = e^{\eta_1} + e^{\eta_2}, \eta_j = P_j\tau + iP_j^2\xi/2 + \phi_{0j}, j = 1,2$. Here P_j are arbitrary complex parameters and ϕ_{0j} are real constants. Substituting (9) in (6) and then solving the set of partial differential equations rigorously, we can obtain the exact form of the bright two soliton solution as

$$u = \frac{g_1 + g_2}{1 + f_1 + f_2}$$

$$= \frac{e^{\eta_1} + e^{\eta_2} + B_{12}e^{\eta_1+\eta_2+\eta_2^*} + B_{21}e^{\eta_1+\eta_2+\eta_1^*}}{1 + A_{11}e^{\eta_1+\eta_1^*} + A_{12}e^{\eta_1+\eta_2^*} + A_{21}e^{\eta_2+\eta_1^*} + A_{22}e^{\eta_2+\eta_2^*} + C_2e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*}} \qquad (10)$$

Where

$$f_1 = A_{11}e^{\eta_1+\eta_1^*} + A_{12}e^{\eta_1+\eta_2^*} + A_{21}e^{\eta_2+\eta_1^*} + A_{22}e^{\eta_2+\eta_2^*}$$

$$g_2 = B_{12}e^{\eta_1+\eta_2+\eta_2^*} + B_{21}e^{\eta_1+\eta_2+\eta_1^*} \qquad (11)$$

$$f_2 = C_2e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*}$$

With

$$A_{11} = \frac{\alpha}{(P_1+P_1^*)^2}, A_{21} = \frac{\alpha}{(P_2+P_1^*)^2}, A_{12} = \frac{\alpha}{(P_1+P_2^*)^2}, A_{22} = \frac{\alpha}{(P_2+P_2^*)^2}$$

$$B_{12} = \frac{\alpha(P_1-P_2)^2}{(P_1+P_2^*)(P_2+P_2^*)}, B_{21} = \frac{\alpha(P_1-P_2)^2}{(P_2+P_1^*)(P_1+P_1^*)} \qquad (12)$$

$$C_2 = \frac{\alpha^2(P_2-P_1)^2(P_1^*-P_2^*)^2}{(P_1+P_1^*)^2(P_2+P_1^*)^2(P_1+P_2^*)^2(P_2+P_2^*)^2}$$

By judiciously choosing soliton parameters in Eq. (10), we can study the various aspects of two-soliton interaction.

From Fig. 4(a) and Fig. 4(b) it is clear that α just control the amplitude or intensity of the two-soliton interaction just like the fundamental soliton propagation in Si waveguide. In both cases we observe two sets of parallel solitons propagate stably and they do not interact with each other. From the two-dimensional intensity profile of bright soliton propagation in Fig. 4(c), it shows that TPA parameter only decreases the intensity maintaining its stable nature. Fig. 5(a) depicts that the soliton interaction is dependent on the phase parameter P_j for a given separation of the solitons, decided by the difference of ϕ_{0j}. Our study further shows that the soliton-soliton interaction could be enhanced by reducing the separation of two solitons by ϕ_{01} and ϕ_{02} as shown in Fig. 5(b).

For bright two soliton solutions, we have studied contour plots which reveal interesting insights like one soliton solution. In Fig. 6(a), the separated contour lines indicate rapid changes in the

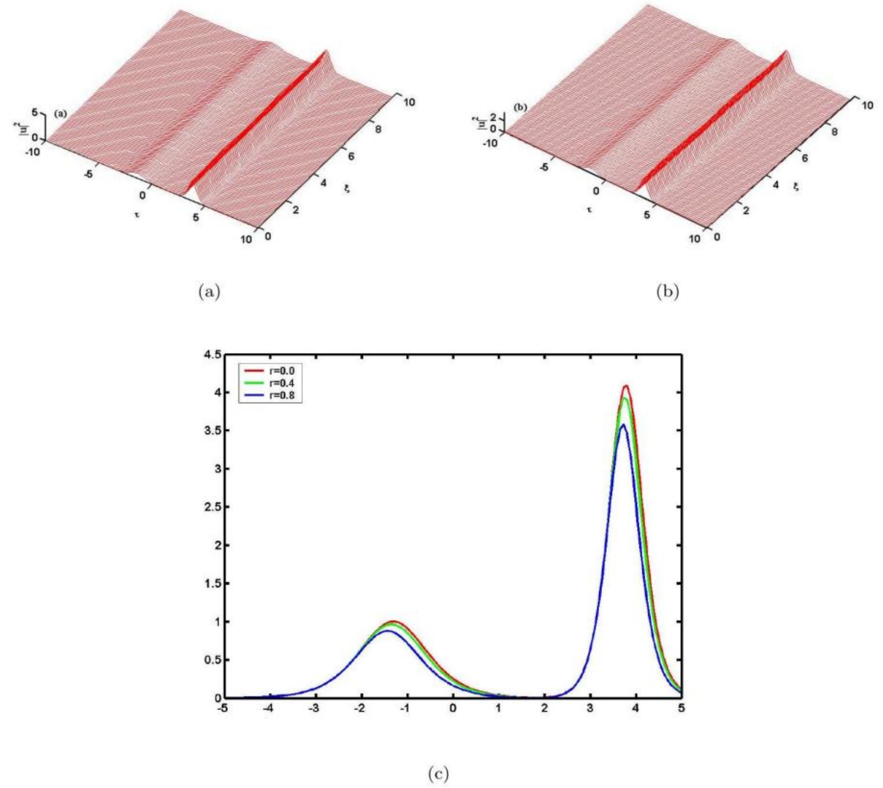
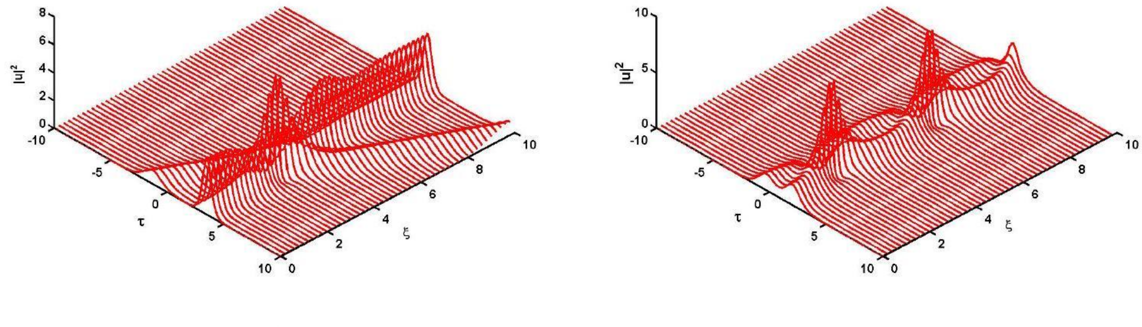


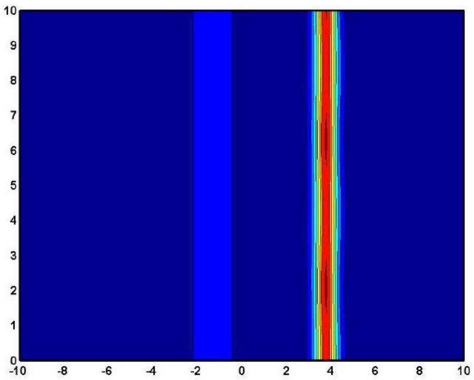
Figure 4: Spatio-temporal evolution of bright two soliton interaction (a) $\alpha = 1$, (b) $\alpha = 1 + 0.8i$ (c) two-dimensional intensity profile of bright two soliton for $r = 0.0, r = 0.4, r = 0.8$ with the parameters $\phi_{01} = 2, \phi_{02} = -4, P_1 = 1, P_2 = 2$.

intensity of the soliton profiles for $P_1 = 1$ and $P_2 = 2$ and $\phi_{01} = 2, \phi_{02} = -4$. In contrast, in Fig. 6(b) keeping same values of P_1 and P_2 values and reducing the separation of two solitons by $\phi_{01} = 2$ and $\phi_{02} = 3$, we noticed that contours are more close to each other implying intensity changes steadily and soliton-soliton interaction is also enhanced. This highlights the importance of soliton parameters and their separation in controlling soliton dynamics and interactions.

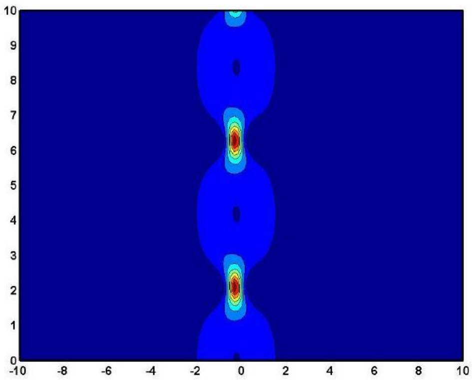


(a) (b)

Figure 5: Spatio-temporal evolution of bright two soliton interaction with $\alpha = 1 + 0.2i$ (a) $P_1 = 1 + i, P_2 = 2$ and $\phi_{01} = 2, \phi_{02} = -4$ (b) $P_1 = 1, P_2 = 2$ and $\phi_{01} = 2, \phi_{02} = 3$.



(a)



(b)

Figure 6: Contour plot of bright two soliton solution with $\alpha = 1 + 0.2i$ and $P_1 = 1, P_2 = 2$ for (a) $\phi_{01} = 2, \phi_{02} = -4$, (b) $\phi_{01} = 2, \phi_{02} = 3$

V. Bright Three-Solition Solution

Applying the same mathematical framework as we have used to obtain one and two soliton solutions, you can extend the approach to construct a three-soliton solution for equation (3).

With suitable choice of the soliton parameters and using equation (15) in equation (14), we can study the various aspects of three soliton-soliton interaction. We assume that

$$g = \chi g_1 + \chi^3 g_2 + \chi^5 g_3, f = 1 + \chi^2 f_1 + \chi^4 f_2 + \chi^6 f_3 \tag{13}$$

where $g_1 = e^{\eta_1} + e^{\eta_2} + e^{\eta_3} = \sum_{j=1}^{3} e^{\eta_j}, \eta_j = P_j\tau + iP_j^2\xi/2 + \phi_{0j}, j = 1,2,3$. Here P_j are arbitrary complex parameters and ϕ_{0j} are real constants. Substituting (13) in (6) and then solving the resulting partial differential equations recursively, we can derive the explicit form of the bright three soliton solution as

$$u = \frac{g}{f} = \frac{g_1 + g_2 + g_3}{1 + f_1 + f_2 + f_3} \tag{14}$$

where

$$f_1 = A_{11}e^{\eta_1+\eta_1^*} + A_{12}e^{\eta_1+\eta_2^*} + A_{13}e^{\eta_1+\eta_3^*} + A_{21}e^{\eta_2+\eta_1^*} + A_{22}e^{\eta_2+\eta_2^*} + A_{23}e^{\eta_2+\eta_3^*} + A_{31}e^{\eta_3+\eta_1^*} + A_{32}e^{\eta_3+\eta_2^*} + A_{33}e^{\eta_3+\eta_3^*}$$

$$g_2 = B_{121}e^{\eta_1+\eta_2+\eta_1^*} + B_{122}e^{\eta_1+\eta_2+\eta_2^*} + B_{123}e^{\eta_1+\eta_2+\eta_3^*} + B_{231}e^{\eta_2+\eta_3+\eta_1^*} + B_{232}e^{\eta_2+\eta_3+\eta_2^*} + B_{233}e^{\eta_2+\eta_3+\eta_3^*} + B_{311}e^{\eta_3+\eta_1+\eta_1^*} + B_{322}e^{\eta_3+\eta_2+\eta_2^*} + B_{323}e^{\eta_3+\eta_2+\eta_3^*} \tag{15}$$

$$f_2 = C_{1212}e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*} + C_{1223}e^{\eta_1+\eta_2+\eta_2^*+\eta_3^*} + C_{1231}e^{\eta_1+\eta_2+\eta_3^*+\eta_1^*} + + C_{2323}e^{\eta_2+\eta_3+\eta_2^*+\eta_3^*} + C_{2331}e^{\eta_2+\eta_3+\eta_3^*+\eta_1^*} + C_{3112}e^{\eta_3+\eta_1+\eta_3^*+\eta_2^*+\eta_2^*} + C_{3123}e^{\eta_3+\eta_1+\eta_2^*+\eta_3^*} + C_{3131}e^{\eta_3+\eta_1+\eta_3^*+\eta_1^*}$$

$$g_3 = D_{31}e^{\eta_1+\eta_2+\eta_3+\eta_1^*+\eta_2^*} + D_{32}e^{\eta_1+\eta_2+\eta_3+\eta_2^*+\eta_3^*} + D_{33}e^{\eta_1+\eta_2+\eta_3+\eta_3^*+\eta_1^*}$$

$$f_3 = E_3e^{\eta_1+\eta_2+\eta_3+\eta_1^*+\eta_2^*+\eta_3^*}$$

With

$$A_{11} = \frac{\alpha}{(P_1 + P_1^*)^2}, A_{12} = \frac{\alpha}{(P_1 + P_2^*)^2}, A_{13} = \frac{\alpha}{(P_1 + P_3^*)^2},$$
$$A_{21} = \frac{\alpha}{(P_2 + P_1^*)^2}, A_{22} = \frac{\alpha}{(P_2 + P_2^*)^2}, A_{23} = \frac{\alpha}{(P_2 + P_3^*)^2}, \tag{16}$$
$$A_{31} = \frac{\alpha}{(P_3 + P_1^*)^2}, A_{32} = \frac{\alpha}{(P_3 + P_2^*)^2}, A_{33} = \frac{\alpha}{(P_3 + P_3^*)^2}$$

$$\mathrm{B}_{121} = \frac{\alpha(P_2 - P_1)^2}{(P_1 + P_1^*)^2(P_2 + P_1^*)^2},\ B_{122} = \frac{\alpha(P_2 - P_1)^2}{(P_1 + P_2^*)^2(P_2 + P_2^*)^2},$$
$$B_{123} = \frac{\alpha(P_2 - P_1)^2}{(P_1 + P_3^*)^2(P_2 + P_3^*)^2},\ B_{231} = \frac{\alpha(P_3 - P_2)^2}{(P_2 + P_1^*)^2(P_3 + P_1^*)^2}, \tag{17}$$
$$\mathrm{B}_{232} = \frac{\alpha(P_3 - P_2)^2}{(P_2 + P_2^*)^2(P_3 + P_2^*)^2}, B_{233} = \frac{\alpha(P_3 - P_2)^2}{(P_2 + P_3^*)^2(P_3 + P_3^*)^2},$$
$$\mathrm{B}_{311} = \frac{\alpha(P_1 - P_3)^2}{(P_3 + P_1^*)^2(P_1 + P_1^*)^2}, B_{312} = \frac{\alpha(P_1 - P_3)^2}{(P_3 + P_2^*)^2(P_1 + P_2^*)^2}$$
$$\mathrm{B}_{313} = \frac{\alpha(P_1 - P_3)^2}{(P_3 + P_3^*)^2(P_1 + P_3^*)^2},$$

$$C_{1212} = \frac{\alpha^2(P_1-P_2)^2(P_1^*-P_2^*)^2}{(P_1+P_1^*)^2(P_1+P_2^*)^2(P_2+P_1^*)^2(P_2+P_2^*)^2}, C_{1223} = \frac{\alpha^2(P_1-P_2)^2(P_2^*-P_3^*)^2}{(P_1+P_2^*)^2(P_1+P_3^*)^2(P_2+P_2^*)^2(P_2+P_3^*)^2},$$
$$C_{1231} = \frac{\alpha^2(P_1-P_2)^2(P_3^*-P_1^*)^2}{(P_1+P_3^*)^2(P_1+P_1^*)^2(P_2+P_3^*)^2(P_2+P_1^*)^2},\ C_{2312} = \frac{\alpha^2(P_2-P_3)^2(P_1^*-P_2^*)^2}{(P_2+P_1^*)^2(P_2+P_2^*)^2(P_3+P_1^*)^2(P_3+P_2^*)^2},$$
$$C_{2323} = \frac{\alpha^2(P_2-P_3)^2(P_2^*-P_3^*)^2}{(P_2+P_2^*)^2(P_2+P_3^*)^2(P_3+P_2^*)^2(P_3+P_3^*)^2}, C_{2331} = \frac{\alpha^2(P_2-P_3)^2(P_3^*-P_1^*)^2}{(P_2+P_3^*)^2(P_2+P_1^*)^2(P_3+P_3^*)^2(P_3+P_1^*)^2} \tag{18}$$
$$C_{3112} = \frac{\alpha^2(P_3-P_1)^2(P_1^*-P_2^*)^2}{(P_3+P_1^*)^2(P_3+P_2^*)^2(P_1+P_1^*)^2(P_1+P_2^*)^2}, C_{3123} = \frac{\alpha^2(P_3-P_1)^2(P_2^*-P_3^*)^2}{(P_3+P_2^*)^2(P_3+P_3^*)^2(P_1+P_2^*)^2(P_1+P_3^*)^2},$$
$$C_{3131} = \frac{\alpha^2(P_3 - P_1)^2(P_3^* - P_1^*)^2}{(P_3 + P_3^*)^2(P_3 + P_1^*)^2(P_1 + P_3^*)^2(P_1 + P_1^*)^2},$$

$$D_{31} = \frac{\alpha^2(P_1 - P_2)^2(P_2^* - P_3^*)^2(P_3 - P_1)^2(P_1^* - P_2^*)^2}{(P_1 + P_1^*)^2(P_1 + P_2^*)^2(P_2 + P_1^*)^2(P_2 + P_2^*)^2(P_3 + P_1^*)^2(P_3 + P_2^*)^2},$$
$$D_{32} = \frac{\alpha^2(P_1 - P_2)^2(P_2^* - P_3^*)^2(P_3 - P_1)^2(P_2^* - P_3^*)^2}{(P_1 + P_2^*)^2(P_1 + P_3^*)^2(P_2 + P_2^*)^2(P_2 + P_3^*)^2(P_3 + P_2^*)^2(P_3 + P_3^*)^2}, \tag{19}$$
$$D_{31} = \frac{\alpha^2(P_1 - P_2)^2(P_2^* - P_3^*)^2(P_3 - P_1)^2(P_3^* - P_1^*)^2}{(P_1 + P_3^*)^2(P_1 + P_1^*)^2(P_2 + P_3^*)^2(P_2 + P_1^*)^2(P_3 + P_3^*)^2(P_3 + P_1^*)^2}$$

and

$$E_3 = \frac{\alpha^3 (P_1 - P_2)^2 (P_2 - P_3)^2 (P_3 - P_1)^2}{(P_1 + P_1^*)^2 (P_1 + P_2^*)^2 (P_1 + P_3^*)^2 (P_2 + P_1^*)^2 (P_2 + P_2^*)^2} \times \frac{(P_1^* - P_2^*)^2 (P_2^* - P_3^*)^2 (P_3^* - P_1^*)^2}{(P_2 + P_3^*)^2 (P_3 + P_1^*)^2 (P_3 + P_2^*)^2 (P_3 + P_3^*)^2} \tag{20}$$

We have studied the propagation of parallel three solitons through the medium using equation (14) without any interaction. From Fig. 7(a) and Fig. 7(b) it is clear that α plays the same role as the amplitude or intensity profile of the three-soliton interaction decreases just like the fundamental and two soliton-soliton propagation in Si waveguide. In both cases we observe three sets of parallel bright solitons propagate stably and they do not interact with each other. From the two-dimensional intensity profile of bright three soliton propagation in Fig. 7(c), it shows that TPA parameter only decreases the intensity maintaining its stable nature.

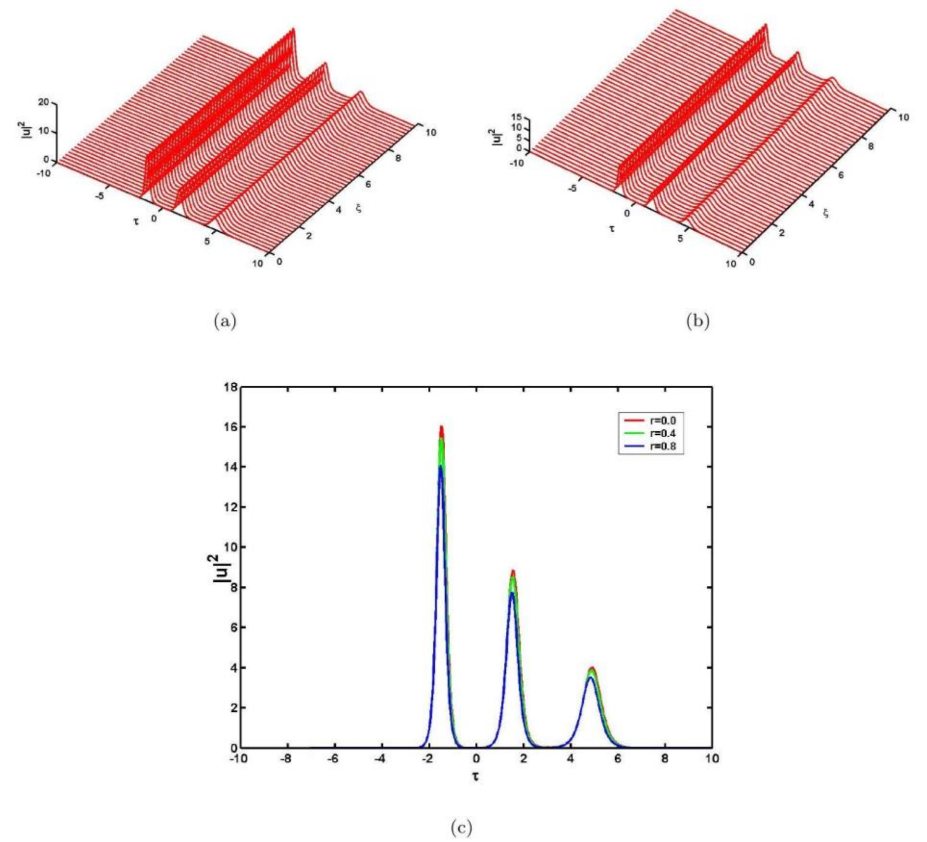
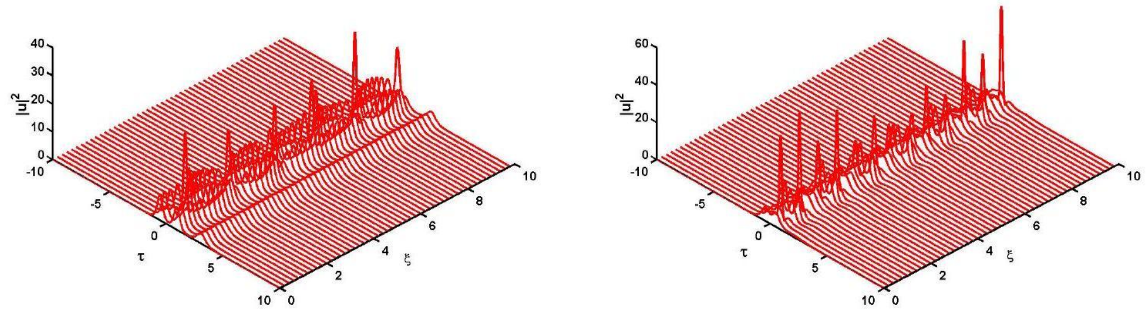


Figure 7: Spatio-temporal evolution of bright three soliton interaction (a) $\alpha = 1$, (b) $\alpha = 1 + 0.8i$ (c) two-dimensional intensity profile of bright three soliton for $r = 0.0, r = 0.4, r = 0.8$ with the parameters $\phi_{01} = -3, \phi_{02} = 1, \phi_{03} = 8$ and $P_1 = 2, P_2 = 3, P_3 = 4$.

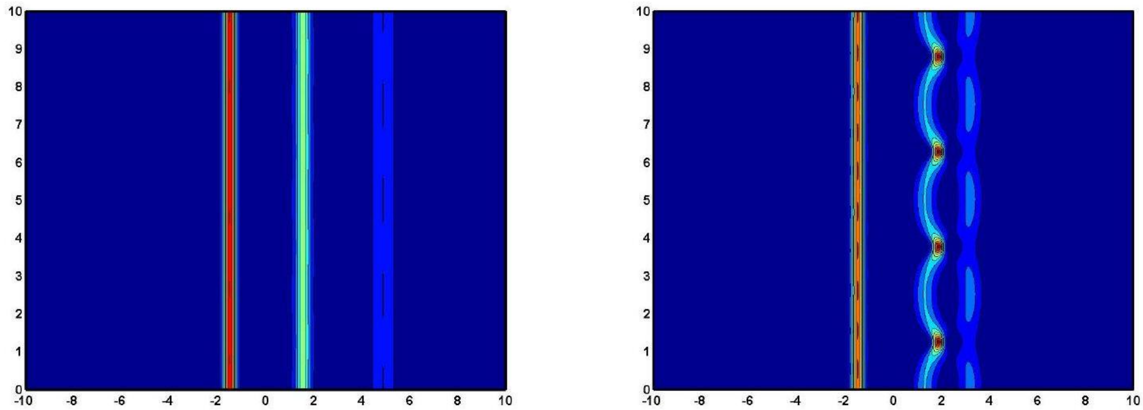
Next, we have illustrated the interactions among the three solitons by their relative distances in Fig. 8(a). It depicts that the soliton interaction is dependent on the phase parameter P_j for a given separation of the solitons, decided by the difference of ϕ_{0j}. The left sided soliton and the middle one interact more strongly due to their closer proximity whereas the right sided soliton and the middle one interact weakly due to their larger separation. Our study further shows that, the soliton-soliton interaction could be enhanced by reducing the separation of two solitons by ϕ_{01} and ϕ_{02} as shown in Fig. 8(b). The contour plot for three parallel soliton and



(a) (b)

Figure 8: Spatio-temporal evolution of bright three soliton interaction with $\alpha = 1 + 0.8i$ and $\phi_{01} = 0.6, \phi_{02} = 2.8, \phi_{03} = 3.2$ for the parameters (a) $P_1 = 2, P_2 = 3, P_3 = 4$ (b) $P_1 = 1, P_2 = 3, P_3 = 5$.

three interacted solitons are also shown by choosing some suitable parameters shown in the Fig. 9(a) and Fig. 9(b).



(a) (b)

Figure 9: Contour plot of bright three soliton interaction with $\alpha = 1 + 0.8i$ and $P_1 = 2, P_2 = 3, P_3 = 4$ for the parameters (a) $\phi_{01} = -3, \phi_{02} = 1, \phi_{03} = 8$ and (b) $\phi_{01} = 0.5, \phi_{02} = 1, \phi_{03} = 8$.

VI. Dark One-Solition Solution

To find exact dark one soliton solution of Eq. (1), we rewrite Eq. (2) in the following form:

$$i\frac{\partial u}{\partial \xi} + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} - \beta |u|^2 u = 0 \tag{21}$$

where $\beta = 1$ - ir with $r < 1$. Physically, it implies that in the self-defocusing medium with anomalous dispersion regimes dark solitons may exist. Applying the transformation as earlier, we can write Eq. (21) in the following bilinear form:

$$f\left[iD_\xi + \frac{1}{2}D_\tau^2\right]g.f - g\left[\frac{1}{2}D_\tau^2 f.f - 2\beta g^* g\right] = 0 \tag{22}$$

To construct the exact dark soliton solution of Equation.(21), we decouple Eq.(22) as

$$\begin{aligned} S_1 g.f &= 0 \\ S_2 f.f &= -2\beta g^* g \end{aligned} \tag{23}$$

where the Hirota bilinear operators S_1 and S_2 are given as $S_1 = iD_\xi + \frac{1}{2}D_\tau^2 - \lambda$ and $S_2 = D_\tau^2 - \lambda$, with λ is a constant to be determined. To get the exact dark one-soliton solution we assume

$$g = g_0(1 + \chi g_1), f = 1 + \chi f_1 \tag{24}$$

Using the definition (5) we can easily find the solution as $g_0 = \rho e^{i\phi}$ where $\phi = -[(2\gamma^2 + \lambda)\xi + 2\gamma\tau] - \varphi$ in which γ and φ are real constants and ρ is a complex parameter with the restrictions $|\rho|^2 = \lambda/2\beta$. Then collecting the coefficients of χ and χ^2 we obtain

$$\begin{gathered} \Re(g_1 \cdot 1 + 1 \cdot f_1) = 0 \\ S_2(1 \cdot f_1 + f_1 \cdot 1) = -2\beta|\rho|^2(g_1 + \overline{g_1}) \end{gathered} \tag{25}$$

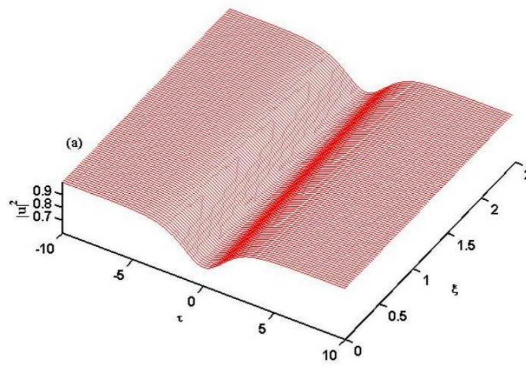
$$\begin{gathered} \Re(g_1 \cdot f_1) = 0 \\ S_2(f_1 \cdot f_1) = -2\beta|\rho|^2(\overline{g_1} g_1) \end{gathered} \tag{26}$$

where $R = iD_\xi + \frac{1}{2}D_\tau^2 - i\gamma D_\tau$. It is easy to verify that the system of Eqs. (25) and (26) admit the following solutions

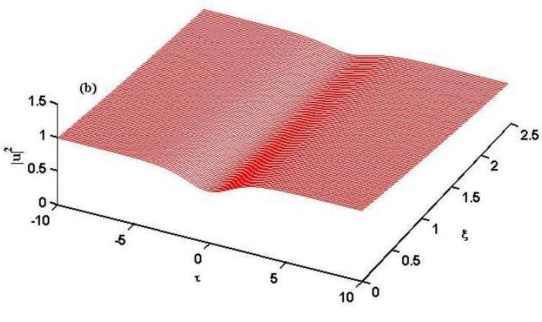
$$g_1 = A_1 e^{\eta_1} \text{ and } f_1 = e^{\eta_1} \tag{27}$$

with $\eta_1 = P_1\tau - \Omega_1\xi + \phi_0$ in which P_1, Ω_1 and ϕ_0 are real constants and A_1 is complex parameter which can be evaluated from (25) as $A_1 = -\frac{[P_1^2/2 + i(\Omega_1 + \gamma P_1)]^2}{[P_1^4/4 + (\Omega_1 + \gamma P_1)^2)]}$ From Eq. (25) we get some additional constraints as $\Omega_1 + \gamma P_1 = P_1(4\beta|\rho|^2 - P_1^2)^{\frac{1}{2}}/2$ with $4\beta|\rho|^2 - P_1^2 > 0$.
Setting $\chi = 1$, we obtain the following explicit form of the dark one-soliton solution of Eq. (22) as

$$u = \frac{g}{f} = \rho\left(\frac{1 + A_1 e^{\xi_1}}{1 + e^{\xi_1}}\right)e^{i\phi} \tag{28}$$



(a)



(b)

Figure 10: Intensity profile of the dark one-soliton solution with the parameters $\varphi = 1, \gamma = 1, \rho = 1 + 2i, P_1 = 1$ and $\phi_0 = 1.5$ (a) $\beta = 1$ (b) $\beta = 1 - 0.04i$

The dark one soliton propagation is stable and undistorted inside the medium without TPA parameter as shown in Fig. 10(a). But in Fig. 10(b) we have observed that TPA parameter does not effect on the stable propagation of soliton but its amplitude decreases and width of the soliton increases. With a slight increase of the TPA parameter, dark soliton propagation is more effected in intensity profile compare with bright soliton. We also observed that further increase of the TPA parameter by 0.1 we do not get the stable soliton propagation and soliton can be shifted along the temporal axis without affecting its stable propagation by changing ϕ_0.

VII. Dark Two-Soliton Solution

For constructing the dark two soliton solution of Eq. (21) we assume that

$$g = g_0(1 + \chi g_1 + \chi^2 g_2), f = 1 + \chi f_1 + \chi^2 f_2 \tag{29}$$

Where g_0 is same as in the case of dark one-soliton solution. Applying the same procedure as in the earlier case we find the set equations for different orders of χ and then applying a straightforward but tedious algebraic calculation we get the following set of solutions as

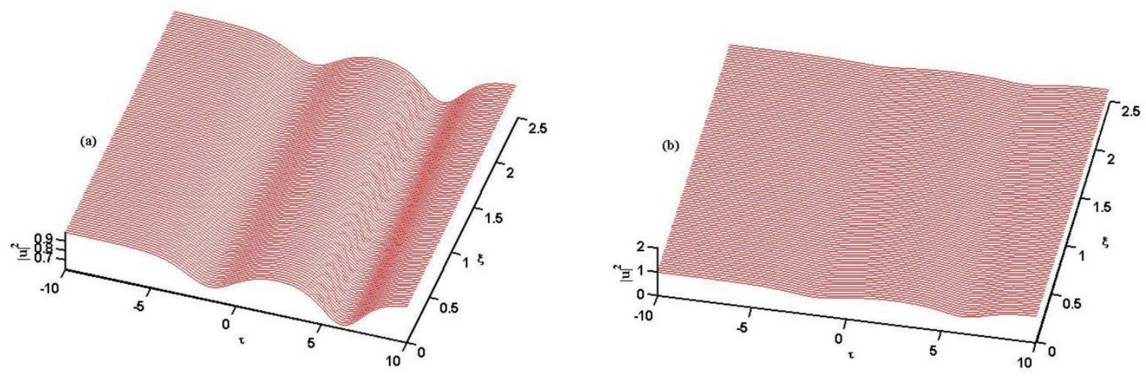
$$\begin{gathered} g_1 = A_1 e^{\eta_1} + A_2 e^{\eta_2}, f_1 = e^{\eta_1} + e^{\eta_2} \\ g_2 = B_{12} A_1 A_2 e^{\eta_1+\eta_2}, f_2 = B_{12} e^{\eta_1+\eta_2} \end{gathered} \tag{30}$$

where $\eta_j = P_j\tau - \Omega_j\xi + \phi_{0j}$ and $A_j = -\frac{\left[P_j + i\left(4\beta|\rho|^2 - P_j^2\right)^{\frac{1}{2}}\right]}{\left[P_j - i\left(4\beta|\rho|^2 - P_j^2\right)^{\frac{1}{2}}\right]}$ with $j = 1,2$ where we have used the constraint $\Omega_j + \gamma P_j = P_j\left(4\beta|\rho|^2 - P_j^2\right)^{\frac{1}{2}}/2$. Taking $k_j = \left(4\beta|\rho|^2 - P_j^2\right)^{\frac{1}{2}}$, we can rewrite $A_j = -\left[\left(P_j + ik_j\right)/\left(P_j - ik_j\right)\right]$ and $\Omega_j = \left(k_j/2 - \gamma\right)P_j$. The expression for B_{12} is

$$B_{12} = \frac{i(A_1 - A_2)(P_1k_1 - P_2k_2) - (A_1 + A_2)(P_1 - P_2)^2}{i(1 - A_1A_2)(P_1k_1 + P_2k_2) + (1 + A_1A_2)(P_1 + P_2)^2} \tag{31}$$

Finally, using the above Eqs., we can write the exact dark two-soliton solution as

$$u = \frac{g}{f} = \rho\frac{1 + A_1e^{\eta_1} + A_2e^{\eta_2} + B_{12}A_1A_2e^{\eta_1+\eta_2}}{1 + e^{\eta_1} + e^{\eta_2} + B_{12}e^{\eta_1+\eta_2}}e^{i\phi} \tag{32}$$



(a) (b)

Figure 11: Intensity profiles of the dark two soliton solutions with $\rho = 1 + 5i, \gamma = 1.5, P_1 = 1, P_2 = 1.2, \varphi = 2, \phi_{01} = 1.5$ and $\phi_{02} = -3$ (a) $\beta = 1$, (b) $\beta = 1 - 0.04i$

We note that in order to have dark two soliton solution we should always have $P_1 \neq P_2$. In Fig. 11(a) and Fig. 11(b) we depict the stable dark two soliton propagation with and without TPA effect. It is observed that the two dark solitons do not interact with each other with changing of TPA parameter likely in case of bright soliton interaction.

VIII. Conclusions

In this paper, we have obtained bright and dark solitary wave solutions in Si waveguide with TPA effect by using Hirota's bilinear method. We have found the necessary conditions under which bright or dark solitons may exist in such a medium. It is observed that TPA parameter does not effect on the stable soliton propagation and only controls the amplitude or intensity profile of soliton pulse. We have also noticed that the TPA parameter influences much higher on dark soliton propagation than that on bright solitons. Though we know that dark solitons are more stable in presence of loss and spreads more slowly against loss as compared with bright solitons but in Si waveguide dark solitons will be considered as less attractive for TPA effect in Si for high-speed optical communication system. Our work also demonstrates the control of soliton-soliton interactions in Si waveguide using Hirota's bilinear method. We also presented contour plots for one, two and three bright solitons solutions which provide valuable insights into the intensity profile and soliton-soliton interactions. By carefully selecting soliton and medium parameters, one can manipulate the interactions between solitons, which is crucial for various applications in nonlinear systems.

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