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Cartesian hyperbole

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ABSTRACT
We rework Descartes' solution to Pappus' problem about products of distances to concurrent lines. Also, we imagine these lines as those that delimit a Mesoamerican pyramid and are projected onto a vertical plane. We seek a geometric interpretation for Cartesian parameters and discriminant. We consider an abundance concept versus scarcity. We recall the concept of conservation to find the shape of the potential energy of a moving object on a conic in polar coordinates. We compare Cartesian parameters with Newtonian ones. We define relative Cartesian potential energy ($U_{rel,D} = -\sqrt{\delta}\phi_2 \cdot \frac{1}{r}$). We imagine annual evolution curves under the concept of contrasting abundance versus scarcity. We take the case of Xochicalco and its pyramid, *The Feathered Serpent*, to interpret the evolution of the predominant behavior of its successive agricultural cycles as an inverse problem in time and imagine the nonexistent annual records as present in the various construction stages of the pyramids.
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I. INTRODUCTION

We speak of “hyperbole”, alluding to its dual meaning: on the one hand, to indicate abundance or excess, as opposed to elliptical, which refers to scarcity or lack. And it’s Cartesian because Descartes had already proposed a solution by introducing, starting from a point, three additional auxiliary lines, with a relationship of proportionality between similar triangles. Thus, we have an abundance of lines, seven, and triangles; however, although we observe an absence of curvature, its composition gives rise to the curvature of a conic, which contains the point of origin of the three auxiliary lines.

Our method is based on the Principle of Duality, which is extracted and highlighted as a cornerstone of the culture and mythology of the Mesoamerican Indigenous Peoples and its hypothetical relationship with the *“Arhuaca culture”* (that lives in the “Sierra Nevada of Santa Marta”, the highest coastal mountain in the world, located in northern Colombia). As can be seen, for example, in the case of the *“God Quetzalcoatl”* (central figure or deity in Mesoamerican mythology) and his representation as a feathered serpent, “who knows the hidden secrets of the subsoil and the air”. Or in “*Mictlán*, where death and life are combined”; (*Mictlán*, final destiny of souls in the Mexican mythology). This principle later reaches us under the signature of the philosopher Hegel, as "Being and Nothingness"; and in Lefebvre H. " Unity Law of the Contradictories" ([1]).

We start from the figure of a Mesoamerican pyramid. We project it onto a vertical plane, highlighting the three concurrent lines that delimit it and are cut by a fourth line, corresponding to the roof of the pyramid, which we orient from the north - blue - *Tlaloc* to the south - red - *Huitzilopochtli* (*Tlaloc* and *Huitzilopochtli* as Mexican deities). Then, the three auxiliary lines mentioned above are constructed, emerging from a point and mediated by the proportion between similar triangles, where said point will be immersed in a conic curve. The base document can be found in the section or chapter entitled: René Descartes (1596-1650), of reference ([2]).

This geometric and philosophical approach allows us to establish a bridge between the worldview of the Mesoamerican Indigenous Peoples and the foundations of modern rational thought. The pyramid, as an architectural and conceptual symbol, not only represents a physical structure, but also a metaphor for the duality and interconnection between the earthly and the divine. By projecting the pyramid onto a vertical plane, the concurrent lines and the fourth line representing the roof symbolize the intersection between the material and spiritual worlds, a concept deeply rooted in Mesoamerican cosmogony ([3]).

The construction of the three auxiliary lines, emerging from a common point and mediated by the proportion of similar triangles, reflects the harmony and balance present in nature and in dualistic thought. This point of convergence, immersed in a conic curve, suggests a connection with the mathematical principles René Descartes developed in his analytic geometry, where geometric figures and algebraic equations intertwine to accurately describe the universe.

The reference to Descartes in the background paper ([2]) is not accidental. Descartes, considered the father of modern philosophy, sought to unify knowledge through reason and mathematics, something that resonates with the quest of Indigenous Peoples to understand the universe through duality and proportion. Thus, the Duality Principle is not only manifested in Mesoamerican culture, but also finds echoes in Western thought, from Hegel to Lefebvre, who explored dialectics and the unity of opposites.

In this sense, our method not only seeks to rescue and reinterpret ancestral knowledge, but also to integrate it with the conceptual tools of modernity. In doing so, we propose a holistic vision that reconciles the wisdom of Indigenous Peoples with contemporary philosophical and scientific currents, opening new perspectives for understanding the complexity of the world and our relationship with it.

1.1 Quadratic form

From qualitative relationships we will move on to quantitative ones. Pappus's problem could be stated as follows: Given several fixed straight lines, we want to find the geometric locus of the points whose product of the distances from some of the lines results in the product of the distances to the remaining straight lines, ([4]). In the case of four concurrent lines, we call this statement "the Cartesian hypothesis".

With the projection of the edges of a pyramid, three lines that are cut by a fourth are produced, horizontal secant corresponding to the roof of the pyramid. The projection of the fourth edge overlaps that of the intermediate edge, or is hidden by the body of the pyramid. This gives rise to three points on the roof line, which we mark as $\{E, A, G\}$ and also two longitudinal parameters: $k = d(A, E)$, and $l = d(A, G)$, are originated (Figure 1, *Chichén Itzá* Projected Pyramid; 'legendary Mayan city in Mexico'), ([2], [4]). At the base of the projected edges, the three points $\{F, D, H\}$ are indicated. And from a hypothetical point $\{C\}$, three more lines are drawn: $\{C, F\}, \{C, D\}, \{C, H\}$, to those three points at the base mentioned above, and a 4th $\{C, B\}$, with $\{B\}$ now located on the straight line of the roof. Three more higher points are obtained $\{T, R, S\}$, with T in $\{G, H\} \cap \{C, B\}$, R in $\{D, A\} \cap \{C, B\}$, and S in $\{F, E\} \cap \{C, B\}$, which could be reduced to a single point, however this hypothesis is not necessary. In summary, seven additional points are obtained: $\{F, D, H, B, T, R, S\}$ and two longitudinal variables: $x = d(B, A)$ and $y = d(C, B)$.

Next, we consider the triangles: $\{C, D, F\}$ with inscribed subtriangle $\{C, B'(B), H'(H)\}$, where CB' is a rotation of CB and CH' another rotation of CH; the $\{C, T, H\}$ with subtriangle $\{B, T, G\}$, angles $\hat{2}$ in T, $(\hat{2}1)$ in G, $\hat{2}2$ in H; the $\{C, D, R\}$ with sub $\{B, A, R\}$, angles $\hat{3}$ in R, $(\hat{3}1)$ in A, $(\hat{3}2)$ in D and the $\{C, F, S\}$, with subtriangle $\{B, E, S\}$, angles $\hat{4}$ in S, $(\hat{4}1)$ in E, and $(\hat{4}2)$ in F. We start then with eight triangles, in groups of two, the upper angles are denoted by $i \in \{2,3,4\}$, the lateral angles by $j \in \{1,2\}$, the first one on the right, the last two on the left. In particular, we highlight the angle β of inclination between CB and the horizontal roof BA.

Denoting $D_i = \{B, H, D, F\}$ by applying the law of sines, the four segments from C are expressed as linear combinations of the parameters and linear variables: $CD_i = \frac{\sin i}{\sin i2} \cdot y + \frac{\sin i1}{\sin i2} \cdot l_i, l_i = \{l - x, k - x, x\}$, under the Cartesian hypothesis,

$$\frac{CD_1}{CD_2} = \frac{CD_3}{CD_4} \tag{1}$$

what unfolds in:

$$\frac{y}{\frac{\sin \hat{2}}{\sin \hat{2}2}y + \frac{\sin \hat{2}1}{\sin \hat{2}2}(l-x)} = \frac{\frac{\sin \hat{3}}{\sin \hat{3}2}y + \frac{\sin \hat{3}1}{\sin \hat{3}2}x}{\frac{\sin \hat{4}}{\sin \hat{4}2}y + \frac{\sin \hat{4}1}{\sin \hat{4}2}(k-x)} \tag{2}$$

and translates into the quadratic equation:

$$\left(\frac{\sin \hat{4}}{\sin \hat{4}2} - \frac{\sin \hat{2}}{\sin \hat{2}2} \cdot \frac{\sin \hat{3}}{\sin \hat{3}2}\right) y^2 = \left(\frac{\sin \hat{2}}{\sin \hat{2}2} \cdot \frac{\sin \hat{3}1}{\sin \hat{3}2} - \frac{\sin \hat{2}1}{\sin \hat{2}2} \cdot \frac{\sin \hat{3}}{\sin \hat{3}2} + \frac{\sin \hat{4}1}{\sin \hat{4}2}\right) xy + \left(-\frac{\sin \hat{2}1}{\sin \hat{2}2} \cdot l \cdot \frac{\sin \hat{3}}{\sin \hat{3}2} - \frac{\sin \hat{4}1}{\sin \hat{4}2} \cdot k\right) y + \left(\frac{\sin \hat{2}1}{\sin \hat{2}2} \cdot l \cdot \frac{\sin \hat{3}1}{\sin \hat{3}2}\right) x + \left(-\frac{\sin \hat{2}1}{\sin \hat{2}2} \cdot \frac{\sin \hat{3}1}{\sin \hat{3}2}\right) x^2 \tag{3}$$

or, in the usual notation, $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$, being:

$$A = \frac{\sin\hat{2}1}{\sin\hat{2}2}\cdot\frac{\sin\hat{3}1}{\sin\hat{3}2} \quad (4)$$

$$B = -\left(\frac{\sin\hat{2}}{\sin\hat{2}2}\cdot\frac{\sin\hat{3}1}{\sin\hat{3}2} - \frac{\sin\hat{2}1}{\sin\hat{2}2}\cdot\frac{\sin\hat{3}}{\sin\hat{3}2} + \frac{\sin\hat{4}1}{\sin\hat{4}2}\right)$$

$$C = \left(\frac{\sin\hat{4}}{\sin\hat{4}2} - \frac{\sin\hat{2}}{\sin\hat{2}2}\cdot\frac{\sin\hat{3}}{\sin\hat{3}2}\right)$$

$$D = -\left(\frac{\sin\hat{2}1}{\sin\hat{2}2}\cdot l\cdot\frac{\sin\hat{3}1}{\sin\hat{3}2}\right)$$

$$E = -\left(-\frac{\sin\hat{2}1}{\sin\hat{2}2}\cdot l\cdot\frac{\sin\hat{3}}{\sin\hat{3}2} - \frac{\sin\hat{4}1}{\sin\hat{4}2}\cdot k\right)$$

Conversely, from the general quadratic equation, we can reconstruct the segments CD_i and reproduce the proportionality between the four original segments of the hypothesis. In conclusion, the quadratic equation and the Cartesian hypothesis correspond to each other.

If we move point C to another location, changes must be made to the positions of other points such as D_i, $\{B, D, F, H\}$, thereby altering the lengths CB, but maintaining the product or proportionality of the Cartesian hypothesis. If we observe the successive movements of point C, counterclockwise, we can see the successive positions of point B, first to the right until it connects with vertex V' (right), then point B returns to the left, passes through its original position, until it connects with the other vertex V (left). Therefore, the major axis is observed anti-projected on the fixed horizontal line that contains B, although we can also anti-project it from the minor axis. Similarly, point D moves until it connects with vertex O, (lower), then returns until it connects with vertex O' (upper), so we describe it as the anti-projection of the minor axis and also, we could relate it to the major axis. Then we observe H and the original line is described as anti-projection of the minor axis, or with the major. Finally, we observe F, we see its line as an antiprojection of the minor axis and alternately, with the major. In conclusion, for each of the four given fixed lines, these can be decomposed into two classes, with one version that links them to the major axis and the other version in the class related to the minor axis.

1.2 Cartesian parameters

The general form of the quadratic contains five coefficients, $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$, but to place ourselves in the context of Descartes, $C = 1$, we take them in reference to the third of them (C), so the result are four coefficients, $y^2 + (Bx + E)y + Ax^2 + Dx = 0$. The discriminant is introduced: $|\delta| = B^2 - 4AC$, or $A = \frac{1}{4}B^2 - \frac{1}{4}|\delta|$, ([5], [6]); so, the characteristic of abundance will be incorporated into the coefficient A. By the formula of the roots, the relation is represented by: $y = -\frac{1}{2}(Bx + E) \pm \left(\left(\frac{1}{2}(Bx + E)\right)^2 - Ax^2 - Dx\right)^{1/2}$, then as squares, for $\delta \neq 0$, the relation becomes: $\left(y + \frac{1}{2}(Bx + E)\right)^2 - \frac{1}{4}|\delta|\left(x + \frac{2}{|\delta|}\left(\frac{1}{2}BE - D\right)\right)^2 = \phi_1^2$, denoting $\phi_1^2 = \frac{1}{4}E^2 - \frac{1}{|\delta|}\left(\frac{1}{2}BE - D\right)^2$, or in standard form, with $\phi_1 \neq 0$,

$$\frac{\left(y+\frac{1}{2}(Bx+E)\right)^2}{\phi_1^2} - \frac{\left(x+\frac{2}{|\delta|}\left(\frac{1}{2}BE-D\right)\right)^2}{\frac{4}{|\delta|}\phi_1^2} = 1 \quad (5)$$

This covers at least two classes: with $|\delta| = +\delta > 0$, hyperbolas are obtained; and with $|\delta| = -\delta, \delta < 0$, ellipses result, and we denote $\phi_i^2 = \frac{1}{4}E^2 + \frac{1}{|\delta|}\left(\frac{1}{2}BE - D\right)^2$.

Now we incorporate the aspect of orthogonality. We had already highlighted the angle β of inclination between the line CB and the horizontal roof BA, which corresponds to the angle between the axis containing the variable x and that of y, which are not orthogonal. In the notation chosen by Descartes, the coefficients of the quadratic are expressed as: $\frac{B}{C} = -2\frac{n}{z}$, $\frac{E}{C} = -2m$, $-\frac{D}{C} = +2m\frac{n}{z} + O$, $\left(\frac{n}{z}\right)^2 - \frac{A}{C} = \frac{p}{m}$, being $\cos\beta = \frac{\frac{n}{z}x}{x} = \frac{n}{z}$. So, the quadratic equation takes the form: $\left(\left(\frac{n}{z}\right)^2 - \frac{p}{m}\right)x^2 - 2\frac{n}{z}xy + y^2 + \left(-2m\frac{n}{z} - O\right)x - 2my = 0$, or in terms of the angle β:

$$\left(\cos^2\beta - \frac{p}{m}\right)x^2 - 2\cos\beta xy + y^2 + (-2m\cos\beta - O)x - 2my = 0 \quad (6)$$

The discriminant is: $\frac{1}{4}|\delta| = \frac{1}{4}\left((-2\cos\beta)^2 - 4\left(\cos^2\beta - \frac{p}{m}\right)1\right) = \frac{p}{m}$. The sign of the discriminant is specified by the sign in front of $\frac{p}{m}$, so we can incorporate both signs using the absolute value: $|\delta|$ Now the sign of $\frac{1}{4}(B^2 - 4AC)$ is that of: $\frac{p}{m} > 0$ and corresponds to hyperbolas, or $-\frac{p}{m} < 0$, and we would have ellipses or antibolas. We will call the relation $\frac{1}{4}|\delta| = \frac{p}{m}$ the Cartesian discriminant.

For the variables x, y to be in the usual Cartesian form, the angle must be $\beta = \frac{\pi}{2}$, the orthogonality, then $B = 0$, and the standard form remains:

$$\frac{\left(y+\frac{1}{2}E\right)^2}{\phi_1^2} - \frac{\left(x-\frac{2}{|\delta|}D\right)^2}{\frac{4}{|\delta|}\phi_1^2} = 1 \qquad (7)$$

We now seek a geometric interpretation for the Cartesian parameters. The first coordinate of the center, point M, is $x_M = -\frac{2}{\delta}D$, but $-\frac{D}{C} = +2m\frac{n}{z} + O = 2m\cos\beta + O$, which is reduced to $D = -O$, and with $\frac{\delta}{4} = \frac{p}{m} > 0$, the positive Cartesian discriminant, results $x_M = \frac{1}{2}\frac{m}{p}O$. The second coordinate of the center $y_M = \frac{1}{2}E$, but $E = -2m, y_M = -m$, so the coordinates of the center, point M, are: $(x_M, y_M) = \left(\frac{1}{2}\frac{m}{p}O, -m\right)$. We note that the semi-axis a must be in accordance with the $+$ sign, in this case with the y axis. The semi-axis (a) associated with y is $\phi_1^2 = \frac{1}{4}E^2 - \frac{1}{|\delta|}\left(\frac{1}{2}BE - D\right)^2 = m^2 - O^2\frac{m}{4p}$, which happens for $\delta > 0$. For eccentricity: $\varepsilon = \frac{c}{a} = \left(\frac{b^2}{a^2} + 1\right)^{\frac{1}{2}} = \left(1 + \frac{m}{p}\right)^{\frac{1}{2}}$, or $\varepsilon^2 - 1 = \frac{m}{p}$, so the term $\frac{m}{p}$ is the supplement, with respect to 1, of the square of the eccentricity. Finally, in standard form it becomes, with

$\delta_c = \frac{p}{m}$, and $\varepsilon = (1 + \delta_c^{-1})^{1/2} > 1$:

$$\frac{(y+m)^2}{a^2} - \frac{\left(x-\frac{1}{2}\frac{m}{p}O\right)^2}{b^2} = 1 \ , \ a = \phi_1 \ , \quad b = \sqrt{\frac{m}{p}}\phi_1 \ , \ \varepsilon = \left(1 + \frac{m}{p}\right)^{1/2} \qquad (8)$$

The foci and vertices are located on an axis parallel to the y axis, the vertices $\frac{(y+m)^2}{a^2} = 1, y = -m \pm \phi_1$. The center at $(x_M, y_M) = \left(\frac{1}{2}\frac{m}{p}O, -m\right)$. The directrix at: $\bar{y} = \frac{a}{\varepsilon} = \frac{\phi_1}{\left(1+\frac{m}{p}\right)^{1/2}}$. For the straight side: $q_1 = \frac{b^2}{a} = \frac{\frac{m}{p}\phi_1^2}{\phi_1} = \frac{m}{p}\phi_1$. The eccentricity ε could also be understood as the proportional mean between 1 and $1 + \frac{m}{p}, \frac{1+\frac{m}{p}}{\varepsilon} = \frac{\varepsilon}{1}$. The eccentricity ε can be represented by the height of a right triangle inscribed in a circle with diameter $2 + \frac{m}{p}$ and equal to its hypotenuse.

In the other class, with $|\delta| = -\delta, \delta < 0$ ellipses or antibolas are obtained, then, from the expression $\frac{\left(y+\frac{1}{2}E\right)^2}{\phi_2^2} - \frac{\left(x-\frac{2}{|\delta|}D\right)^2}{\frac{4}{|\delta|}\phi_2^2} = 1$, it is obtained $\frac{\left(y+\frac{1}{2}E\right)^2}{\phi_2^2} - \frac{\left(x-\frac{2}{-\delta}D\right)^2}{\frac{4}{-\delta}\phi_2^2} = 1$,

$$\frac{\left(y+\frac{1}{2}E\right)^2}{\phi_2^2} + \frac{\left(x+\frac{2}{\delta}D\right)^2}{\frac{4}{\delta}\phi_2^2} = 1 \qquad (9)$$

with $\phi_2^2 = \frac{1}{4}E^2 + \frac{1}{\delta}\left(\frac{1}{2}BE - D\right)^2 = m^2 + \frac{m}{4p}O^2$, and remains: $\frac{\left(x+\frac{m}{2p}O\right)^2}{\frac{m}{p}\phi_2^2} + \frac{(y-m)^2}{\phi_2^2} = 1$. For eccentricity, $\varepsilon = \frac{c}{a} = \left(1 - \left(\frac{b}{a}\right)^2\right)^{1/2} = \left(1 - \frac{p}{m}\right)^{1/2}$, so $\frac{p}{m}$ is the complement, with respect to 1, of the eccentricity. Eccentricity could also be seen as a proportional average between 1 and $1 - \frac{p}{m}$, $\frac{1}{\varepsilon} = \frac{\varepsilon}{1-\frac{p}{m}}$, and $\varepsilon = (1 - \delta_c)^{1/2} < 1$,

$$\frac{\left(x + \frac{m}{2p}O\right)^2}{a^2} + \frac{(y-m)^2}{b^2} = 1 \ , \quad a = \sqrt{\frac{m}{p}}\phi_2 \ , \quad b = \phi_2 \ , \quad \varepsilon = \left(1 - \frac{p}{m}\right)^{1/2} \qquad (10)$$

Starting from the relationship: $|\delta| = B^2 - 4AC$, that is: $\frac{1}{4}|\delta| = \frac{p}{m}$, the Cartesian discriminant; we can interpret m as a reference value, so p represents $\frac{1}{4}|\delta|$. For the center of the hyperbole $\left(\frac{x_M}{m}, \frac{y_M}{m}\right) = \left(O\frac{1}{2p}, -1\right)$. The semi-axis a: $\left(\frac{\phi_2}{m}\right)^2 = 1 - \frac{1}{4}\frac{O^2}{pm}$. The eccentricity $\varepsilon = \left(1 + \frac{1}{p/m}\right)^{1/2} > 1$. The straight side: $q_2 = \frac{\phi_2^2}{\sqrt{\frac{m}{p}}\phi_2} = \left(\frac{p}{m}\right)^{1/2}\phi_2$.

It is interesting to consider the arithmetic combinations of the two types of eccentricities. On the one hand, if $\varepsilon_1 \cdot \varepsilon_2 = 1$ we have: $(1 + \delta_c^{-1})(1 - \delta_c) = \delta_c^{-1} - \delta_c = 1$, which leads to: $\delta_c^{-1} = \frac{1+\sqrt{5}}{2} = 1.61803 \ldots$, $\delta_c = \frac{-1+\sqrt{5}}{2} = 0.61803 \ldots$, which is known as the golden number. However, in the case of addition, it would be: $\varepsilon_1 + \varepsilon_2 = 1$; but this relationship is not possible, because $\varepsilon_1 > 1$ and $\varepsilon_2 > 0$.

1.2.1 Polar conic

The conic ellipse in the Cartesian variables (x, y), (orthogonal, $\beta = \frac{\pi}{2}$) is now needed in its representation in polar coordinates (orthogonal). The ellipse has its focus (left) at the origin L and the major axis of the ellipse on the horizontal axis of x. We move on to the polar expression of the horizontal projection: $r\cos\theta = x$; and of the vertical: $r\sin\theta = y$, ([7]). The horizontal center is: $c + r\cos\theta = x + c$ and the vertical is $y = 0$. The antibola is: $\frac{(r\cos\theta+c)^2}{a^2} + \frac{(r\sin\theta)^2}{b^2} = 1$, , with $\varepsilon = \frac{c}{a}$ and $q = \frac{b^2}{a}$, the straight side; we obtain $\left(\frac{q}{r} - \varepsilon\cos\theta\right)^2 = +1$ or:

$$\frac{q}{r} = +1 + \varepsilon\cos\theta \qquad (11)$$

It is verified that $r(\pi/2) = q = \frac{b^2}{a}$, is the straight side (on the line CB). The vertex V in: $r(0) = \frac{q}{1+\varepsilon}$ and the other, V' in: $r(\pi) = \frac{q}{1-\varepsilon}$. The major axis $2a = r(0) + r(\pi) = \frac{q}{1+\varepsilon} + \frac{q}{1-\varepsilon} = q\left(\frac{2}{1-\varepsilon^2}\right)$, (on the straight line $VLMV'$) or $a = \frac{q}{1-\varepsilon^2}$. he location of a focus is: $c = \varepsilon a = q\frac{\varepsilon}{1-\varepsilon^2}$. The minor semi-axis: $b = q\frac{1}{(1-\varepsilon^2)^{1/2}}$, (above OMO'). In the case of hyperbolas, however, $\frac{q}{r} = -1 + \varepsilon\cos\theta$ and both classes are included under the option: $\frac{q}{r} = \pm 1 + \varepsilon\cos\theta$.

1.3 Abundance

We consider a special case of the general quadratic and its possible Cartesian interpretation. The forms $y^2 = 2kx + |\gamma|x^2$ or $+|\gamma|x^2 - y^2 + 2kx = 0$ contain abundance or excess, or scarcity or defect, depending on the sign of γ. In the general quadratic $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$ and with the discriminant $\delta = B^2 - 4AC$, it is observed that the coefficients must be: $(A, B, C, D, E) = (|\gamma|, 0, -1, 2k, 0)$; then $\delta = 4|\gamma|$; therefore, the abundance parameter is: $|\gamma| = \frac{1}{4}\delta$, a quarter of the discriminant.

On the other hand, in the Cartesian form $\left(\cos^2\beta - \frac{p}{m}\right)x^2 - 2\cos\beta xy + y^2 + (-2m\cos\beta - O)x - 2my = 0$, with $\beta = \pi/2$ and with a change of signs so that $C = -1$, and $+\frac{|p|}{m}x^2 - Ox - y^2 - 2my = 0$, with the discriminant $\delta = B^2 - 4AC = 4\frac{|p|}{m}$, or $\frac{\delta}{4} = \frac{|p|}{m}$; thus, the Cartesian discriminant $\frac{\delta}{4} = \frac{p}{m}$ measures abundance and $+\frac{\delta}{4} = -\frac{p}{m}$ measures scarcity, or $\frac{\delta}{4} = 0$, its absence; or in a single expression: $|\gamma| = \frac{1}{4}\delta = \frac{|p|}{m}$.

In the case of absence of abundance: $\frac{|p|}{m}x^2 - Ox - y^2 - 2my = 0$, or $(y + m)^2 = +m^2 + Ox$, then $\frac{k}{2} = \frac{O}{4}$ is the parameter and the vertex: $(h_1, h_2) = \left(-\frac{m^2}{O}, -m\right)$. The directrix is in $\bar{x} = (x - h_1) = \left(x + \frac{m^2}{O}\right) = -\frac{k}{2} = -\frac{O}{4}$ or $x_d = -\frac{O}{4} - \frac{m^2}{O}$.

However, in standard form, $\frac{\left(y+\frac{1}{2}(Bx+E)\right)^2}{\phi^2} - \frac{\left(x+\frac{2}{|\delta|}\left(\frac{1}{2}BE-D\right)\right)^2}{\frac{4}{|\delta|}\phi^2} = 1$, with $\phi_i^2 = \frac{1}{4}E^2 + \frac{1}{|\delta|}\left(\frac{1}{2}BE - D\right)^2 = \frac{k^2}{|\gamma|}$; it is obtained $\frac{y^2}{\frac{k^2}{|\gamma|}} - \frac{\left(x-\frac{k}{|\gamma|}\right)^2}{\frac{k^2}{|\gamma|^2}} = 1$, or $a = \frac{k}{|\gamma|^{1/2}}$, $b = \frac{k}{|\gamma|}$, the straight side: $q = \frac{b^2}{a} = \frac{k}{|\gamma|^{3/2}}$ and $\varepsilon^2 = 1 + \frac{q}{a} = 1 + \frac{1}{|\gamma|}$. And we observe the two classes: if $|\gamma| = +\gamma = \frac{p}{m}$ they are hyperbolas $(\varepsilon > 1)$. If $|\gamma| = -\gamma = +\frac{p}{m}$, they are ellipses $(\varepsilon < 1)$.

For the economic interpretation, the category "Income" (or ground rent), could be represented by $p > 0$ as excess over average profit m, ([8]), ([9]). Thus, in the context of abundance, the quotient of the two Cartesian parameters of the Cartesian discriminant gives rise to two classes according to predominance: of abundance or of scarcity. This quotient can represent a surplus of income with respect to average profit, such that in the first case, the positive one, it leads to an open annual evolution, the hyperbolas; in contrast to the second, the negative one, with a closed annual evolution, the ellipses.

1.4 Conservation

We consider a linear and Euclidean space Y. We define the bilinear form through a scalar product: $a(u, v) = \langle u, v\rangle$ and the norm is given by: $\|u\|_a = \sqrt{a(u, u)}$. We think about a differentiable vector field $X^k\frac{\delta}{\delta x^k}$, in the space Y. The "Lie" derivative with respect to the vector field is represented by L_X. Based on the bilinear form a, we define the Lagrangian functional L,

$$L(u) = a\big((I - L_X)u, u\big) = a(u - L_X u, u) \qquad (12)$$

Then, $L(u) = a\left(u - X^k\frac{\partial}{\partial x^k}u, u\right) = \langle u, u\rangle_a - \langle X^k\frac{\partial}{\partial x^k}u, u\rangle_a$ ([10]).

The differential of the Lagrangian function, as the differential of a product, is $dL = \langle du, u\rangle_a + \langle u, du\rangle_a - \langle X^k \frac{\partial}{\partial x^k} du, u\rangle_a - \langle X^k \frac{\partial}{\partial x^k} u, du\rangle_a$, so $dL = -\frac{\partial}{\partial x^k}\langle X^k du, u\rangle_a + \langle du, u + X^k \frac{\partial}{\partial x^k} u\rangle_a + \langle u - X^k \frac{\partial}{\partial x^k} u, du\rangle_a$. The unconditional extreme condition produces: $\langle du, X^{k*} \frac{\partial}{\partial x^k} u - u\rangle_a + \langle X^k \frac{\partial}{\partial x^k} u + u, du\rangle_a = 0$, then $X^k \frac{\partial}{\partial x^k} u + u = 0$, and $X^{k*} = -X^k$. Therefore, $\frac{\partial}{\partial x^k}\langle X^k u, u\rangle_a = \langle X^k \frac{\partial}{\partial x^k} u, u\rangle_a + \langle X^k u, \frac{\partial}{\partial x^k} u\rangle_a = \langle X^k \frac{\partial}{\partial x^k} u + u, u - X^k \frac{\partial}{\partial x^k} u\rangle_a$; but since the first factor is zero, the divergence also turns out to be zero,

$$\frac{\partial}{\partial x^k}\langle X^k u, u\rangle_a = 0 \qquad (13)$$

The coordinates are listed from $k = 0$ to $k = n - 1$. We associate proportionally $x^k|_{k=0}$over time and with $J^0 = \langle X^0 u, u\rangle_a$, $J^k = \langle X^k u, u\rangle_a, k \geq 1$, we obtain the well-known continuity equation for the J field:

$$\text{div}J = -\frac{\partial}{\partial t} J^0 \qquad (14)$$

A convex functional has the characteristic that any of its chords (or subtense) is above the graph of the functional. Thus, height differences can be defined on its graphs, and for each chord, we can assign the maximum height difference (or supremum). The Lagrange functional (L) is convex with respect to the generalized velocity variable $(q^{\cdot k})$. Therefore, we consider the corresponding generalized moment (p_k) as the slope of a chord passing through the origin, which intersects the graph of the functional at two points, which determine an interval on the coordinate axis of the generalized velocities $[\dot{q}_1^k, \dot{q}_2^k]$. Thus, the supremum of the height differences between the chord and the functional is reached, making it possible to define the image of the new functional, now dependent on the slope of the chord, or the corresponding generalized moment. This is known as the Hamilton functional (H),

$$\hat{L} = H(q, p, t) = p^k \dot{q}^k - L \qquad (15)$$

This transformation of the functional L dependent on $(\dot{q}^k)$, into the functional H dependent on (p_k), is called the Legendre transform $(\hat{L} = H)$. The unconditional end condition yields: $\frac{\partial}{\partial \dot{q}^k}(p_k \dot{q}^k - L)$, or $p_k = \frac{\partial}{\partial \dot{q}^k} L$, which coincides with the definition of momentum in Lagrange's equations $\left(\frac{\partial}{\partial \dot{q}^k} L = +p_k\right)$. On the other hand, if there is $\frac{\partial}{\partial p_k}\dot{q}^k, \frac{\partial}{\partial p_k} H = \left(\frac{\partial}{\partial p_k} p_k\right)\dot{q}^k + p_k \frac{\partial}{\partial p_k}\dot{q}^k - \frac{\partial}{\partial p_k} L, q^{\cdot k} + \frac{\partial}{\partial \dot{q}^k} L \frac{\partial}{\partial p_k}\dot{q}^k - \frac{\partial}{\partial p_k} L, \dot{q}^k + \frac{\partial}{\partial p_k} L - \frac{\partial}{\partial p_k} L = \dot{q}^k = \frac{\partial}{\partial p_k} H$, and is known as a Hamilton equation: $\dot{q}^k = \frac{\partial}{\partial p_k} H$. Lagrange's equation (Law of Force) $\frac{\partial}{\partial q^k} L = \dot{p}_k, \frac{\partial}{\partial q^k}(p_k \dot{q}^k - H) = 0 - \frac{\partial}{\partial q^k} H, \dot{p}_k = -\frac{\partial}{\partial q^k} H$, Hamilton's second equation (or Law of Force). It also gives rise to two vector fields: $J_L = \left(\frac{\partial}{\partial q^k} L, \frac{\partial}{\partial \dot{q}^k} L\right)\begin{pmatrix}\frac{\partial}{\partial q^k} \\ \frac{\partial}{\partial \dot{q}^k}\end{pmatrix}$, and $J_H = \left(\frac{\partial}{\partial p^k} H, -\frac{\partial}{\partial q^k} H\right)\begin{pmatrix}\frac{\partial}{\partial q^k} \\ \frac{\partial}{\partial p_k}\end{pmatrix}$, ([11]). The continuity equation is div $J_L = -\frac{\partial}{\partial t} J_L^0$ and div $J_H = -\frac{\partial}{\partial t} J_H^0$.

However, with a change of sign in H, we have $J_{-H} \rightarrow \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial q^k} \\ \frac{\partial}{\partial p_k}\end{pmatrix} H = \left(-\frac{\partial}{\partial p_k}, +\frac{\partial}{\partial q^k}\right) H$, where slope switching is achieved by applying the matrix $J_2 \rightarrow \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Similarly, from the components of the differential of any differentiable function, one can define an open field of an appropriate manifold. If $df = \left(\frac{\partial}{\partial q^k} f, \frac{\partial}{\partial p_k} f\right)\begin{pmatrix} dq^k \\ dp_k \end{pmatrix}$, the field is:

$$Y_{df} = \left(-\frac{\partial}{\partial p_k} f, +\frac{\partial}{\partial q^k} f\right)\begin{pmatrix}\frac{\partial}{\partial q^k} \\ \frac{\partial}{\partial p_k}\end{pmatrix} \qquad (16)$$

Then we define the Poisson bracket of a pair of such functions, by:

$$\{f, g\} = Y_{df} g = \left(-\frac{\partial}{\partial p_k} f, +\frac{\partial}{\partial q^k} f\right)\begin{pmatrix}\frac{\partial}{\partial q^k} \\ \frac{\partial}{\partial p_k}\end{pmatrix} g \qquad (17)$$

and where the asymmetry of the bracket under the exchange of functions is observed: $Y_{df} g = -Y_{dg} f$. But also, in its evolution along an integral curve of the field X_{-dH}, we know that the rate of change of the function f is the Poisson bracket $\{f, H\}$ evaluated at the initial point of the curve. Therefore, if the bracket is zero, the function f remains constant on the curve, and vice versa. In particular, H also remains constant because it commutes with itself: $\{f, H\}|_{f=H} = 0$ ([12]).

More classically, if L does not explicitly depend on t, $\frac{d}{dt}L = \frac{\partial}{\partial q^k}L \cdot \dot{q}^k + \frac{\partial}{\partial \dot{q}^k}L \cdot \frac{d}{dt}\dot{q}^k$, the 2nd component is modified by the definition $\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}^k}L\right) = \frac{d}{dt}p_k$; while the first, by law forces $\frac{\partial}{\partial q^k}L = \dot{p}_k$, that is: $\frac{\partial}{\partial q^k}L = \dot{p}_k = \frac{d}{dt}p_k = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}^k}L\right)$, thus the temporal change of L is rewritten as: $\frac{d}{dt}L = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}^k}L\right) \cdot \dot{q}^k + \frac{\partial}{\partial \dot{q}^k}L \cdot \frac{d}{dt}\dot{q}^k = \frac{d}{dt}\left(\left(\frac{\partial}{\partial \dot{q}^k}L\right)\dot{q}^k\right)$; or $\frac{d}{dt}\left(\left(\frac{\partial}{\partial \dot{q}^k}L\right)\dot{q}^k - L\right) = 0$; and by definition $\frac{d}{dt}(p_k\dot{q}^k - L) = 0, \frac{d}{dt}(H) = 0$, then $\operatorname{div} J_H = 0$ and the J_H field is conservative.

1.5 Materiality

To add materiality to the geometry, we consider a moving object along an ellipse, in the spatial curvature or in the field created by another body located at a focus of the ellipse. We observe two physical variables, the first linked to angular momentum and the second to energy. The angular momentum $L = \vec{r} \times M\vec{v}$, in polar coordinates relative to a focus of the ellipse, $L = r\hat{r} \times M\left(v_r\hat{r} + v_\theta\hat{\theta}\right) = rMv_\theta\hat{r} \times \hat{\theta} = rMv_\theta\hat{z}$.

The ellipse in its polar form is $\frac{q}{r} = +1 + \varepsilon\cos\theta$, with the two parameters: q the straight side and ε the eccentricity. And, the two semiaxes are recovered by means of the two relations: $q = \frac{b^2}{a}$ and $\varepsilon = \left(1 - \left(\frac{b}{a}\right)^2\right)^{1/2}$. On the other hand, the velocity can be obtained by: $\frac{d}{d\theta}\left(\frac{q}{r}\right) = -\frac{q}{r^2}\frac{dr}{d\theta} = -\varepsilon\sin\theta$, $\frac{dr}{d\theta} = \frac{r^2\varepsilon}{q}\sin\theta$, $\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{r^2\varepsilon}{q}\sin\theta\frac{d\theta}{dt}$, then $\left(\frac{dr}{dt}\right)^2 = \left(\frac{r^2\varepsilon}{q}\sin\theta\right)^2\left(\frac{d\theta}{dt}\right)^2$. Thus, the square of the velocity is $\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \frac{r^4}{q^2}\left(\frac{q^2}{r^2} + \varepsilon^2\sin^2\theta\right)\left(\frac{d\theta}{dt}\right)^2$, ([7]).

According to the conservation of energy, if $H = H_c + U$, as the sum of the kinetic and potential energies, and with $H = constant$, then the kinetic energy is the complement of the potential energy: $H_c = H - U$. For the kinetic energy of the moving object: $\frac{H_c}{\frac{1}{2}M} = \left((q\varepsilon\sin\theta)^2\frac{r^4}{q^4} + r^2\right)\left(\frac{d\theta}{dt}\right)^2$. As we will see in the following paragraph, we have an expression for the angular velocity: $\frac{1}{2}r(\theta)^2\frac{d\theta}{dt} = \frac{A_e}{T}$, in terms of the area and period of the ellipse; thus, the square of the angular velocity is: $\left(\frac{d\theta}{dt}\right)^2 = \frac{1}{r^4}\left(\frac{2A_c}{T}\right)^2$. Therefore, $\frac{H_c}{\frac{1}{2}M\left(\frac{2A_c}{T}\right)^2} = \frac{1}{q^2}\left(\frac{q^2}{r^2} + \varepsilon^2\sin^2\theta\right)$. We seek to substitute this expression, which contains the angular position $\frac{1}{q^2}\left(\frac{q^2}{r^2} + \varepsilon^2\sin^2\theta\right) = \frac{1}{q^2}(\varepsilon^2 - 1) + \frac{2}{q}\frac{1}{r} = \frac{H}{\frac{1}{2}M\left(\frac{2A_e}{T}\right)^2} + \frac{2}{q}\frac{1}{r}$. Thus, the energy linked to positions, or potential energy, is: $U = -\frac{M}{q}\left(\frac{2A_c}{T}\right)^2 \cdot \frac{1}{r}$, so:

$$U(r) = -k_U \cdot \frac{1}{r} \quad , \quad k_U = \frac{M}{q}\left(\frac{2A_c}{T}\right)^2 > 0 \qquad (18)$$

On the other hand, since the potential energy depends only on the radial distance, it is a central field and therefore the angular momentum L is also constant, then $\frac{d}{dt}(rMv_\theta) = 0$. The area swept out by the radius vector is $\frac{dA}{dt} = \frac{1}{2}r \cdot r\frac{d\theta}{dt}$, as the area of an elementary triangle; then the conservation of angular momentum is reinterpreted in terms of the area swept out by the radius vector: $\frac{d}{dt}(rMv_\theta) = \frac{d}{dt}\left(M \cdot r \cdot r\frac{d\theta}{dt}\right) = 2M \cdot \frac{d}{dt}\left(\frac{dA}{dt}\right) = 0$, therefore, $\frac{dA}{dt} = constant = k_a$ and the conservation of areolar velocity results, known as Kepler's second law, under the statement: "*the vector radius sweeps out equal areas in equal times*".

Or, $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = k_A, A(t) = k_At$; but the area of the entire ellipse is: $A_c = \pi ab = k_AT$, where a and b are the semiaxes, while T is the period, then the areolar velocity constant is: $k_A = \frac{A_c}{T}$; and they are summarized as:

$$A(t) = A_c\frac{t}{T}, A_c = \pi ab \qquad (19)$$

Let's introduce a relative potential energy. From $L = rMv_\theta\hat{z}, \|L\| = \|rMv_\theta\hat{z}\|, \frac{L}{M} = rv_\theta = r \cdot r\frac{d\theta}{dt} = 2\left(\frac{1}{2}r^2\frac{d\theta}{dt}\right) = 2\frac{A_c}{T}$, then $A_c = \frac{1}{2}\frac{L}{M}T$, and $k_u = \frac{M}{q}\left(\frac{2A_c}{T}\right)^2 = \frac{1}{q}\frac{L^2}{M}$, or $q = \frac{1}{k_U}\frac{L^2}{M}$. We consider a reference kinetic energy: $K_r = \frac{1}{2}M(k_U/L)^2, K_rq = \frac{1}{2}k_U$ and define $U_{rel,N} = \frac{U(r)}{K_r}$.

$$U_{rel,N} = \frac{U(r)}{K_r} = -2q \cdot \frac{1}{r}, \quad k_U = 2K_r \cdot q > 0, \quad q = \frac{1}{k_U}\frac{L^2}{M} \qquad (20)$$

Now, we will make a comparison. For the antibola or ellipse we had, in the context of Descartes, that

$2a = \frac{\alpha}{z}\left(\sqrt{\frac{m}{p}}\right)^2 (4mp + O^2)^{1/2} = \sin\beta\left(\frac{m}{p}\right)(4mp + O^2)^{1/2}$ without orthogonality, but with it: $2a = \sin\beta\left(\frac{m}{p}\right)(4mp + O^2)^{1/2}\Big|_{\beta=\frac{\pi}{2}}, a = \frac{1}{2}\left(\frac{m}{p}\right)(4mp + O^2)^{1/2}$; and with $\phi_2 = \frac{1}{2}\left(\frac{m}{p}\right)^{1/2}(4pm + O^2)^{1/2}$, we have: $a_D = \sqrt{\frac{m}{p}}\phi_2$. Therefore, this Cartesian semi-axis corresponds to the Newtonian value: $q\frac{K_r}{|E|}$,

$$a_D = \sqrt{\frac{m}{p}}\phi_2 \to a_N = q\frac{K_r}{|E|} = \frac{k_U}{|E|} \qquad (21)$$

Likewise, $2b = csc\beta(4mp + O^2)^{1/2}$ in general, but in the orthogonal case: $2b = csc\beta(4mp + O^2)^{1/2}\Big|_{\beta=\frac{\pi}{2}} = (4mp + O^2)^{1/2} = 2\phi_2$; so there is correspondence with: $2q\left(\frac{K_r}{|E|}\right)^{1/2} = 2\left(\frac{L^2}{M}\cdot\frac{1}{2|E|}\right)^{1/2}, q = \frac{L^2}{M}\cdot\frac{1}{k_U}$, in the Newtonian context, then:

$$b_D = \phi_2 \to b_N = q\left(\frac{K_r}{|E|}\right)^{1/2} = \left(\frac{L^2}{M}\cdot\frac{1}{2|E|}\right)^{1/2} \qquad (22)$$

For eccentricity, $(1-\varepsilon^2)^{1/2} = \frac{csc\beta(4mp+O^2)^{1/2}}{\sin\beta\left(\frac{m}{p}\right)(4mp+O^2)^{1/2}} = \left(\frac{p}{m}\right)csc^2\beta$, in general; or in the orthogonal case: $\varepsilon = \left(1 - \left(csc^2\beta\frac{p}{m}\right)^2\right)^{1/2}\Bigg|_{\beta=\frac{\pi}{2}}$, then $\varepsilon = \left(1 - \left(\frac{p}{m}\right)^2\right)^{1/2}$. Thus, the relationship is established with $\varepsilon = \left(1 - \left(\frac{|E|}{K_r}\right)\right)^{1/2}$, or $\left(\frac{p}{m}\right)^2 \to \left(\frac{|E|}{K_r}\right)$, or $\frac{p}{m} \to \left(\frac{|E|}{K_r}\right)^{1/2}$. And as $\frac{b}{a} = (1-\varepsilon^2)^{1/2}$, we affirm that $(1-\varepsilon^2)^{1/2}$ it also measures abundance or excess, as a complement to eccentricity. Eccentricity could be seen as a proportional average between 1 and $1 - \frac{p}{m}, \frac{1}{\varepsilon} = \frac{\varepsilon}{1-\frac{p}{m}}$, and $\frac{1}{\varepsilon} = \frac{\varepsilon}{1-\left(\frac{b}{a}\right)^2}$.

In particular, for the square of the period we have the well-known Kepler 3rd:

$$T^2 = \pi^2\frac{2M}{\frac{1}{2}qK_r}\left(\frac{1}{2}q\frac{K_r}{|E|}\right)a^2 = 4\pi^2\frac{M}{qK_r}a^3 \qquad (23)$$

In summary, the parameters introduced by Descartes lead us to $\frac{p}{m}$, which can be understood as a plus value over a mean value and provides the discriminant $\left(\frac{\delta}{4} \neq 0\right)$, whose sign indicates two classes: $+$ for hyperbolas and $-$ for antibolas or ellipses; $\frac{p}{m}$ and its inverse $\left(\frac{p}{m}\right)^{-1}$, contribute respectively, with the supplement and the complement, with respect to 1 for eccentricities. The parameter O, together with $\left(\frac{p}{m}\right)^{-1}$, allow us to locate the coordinates of the center: $(x_M, y_M) = \left(\frac{1}{2}\frac{m}{p}O, -m\right)$. For the hyperbola, in the right triangle (MOL') between the origin, the vertex, and the point on the slope asymptote $\frac{b}{a} = \sqrt{\frac{m}{p}}$ and length $c = \left(1 + \frac{m}{p}\right)^{1/2}a$, with $\phi_1^2 = m^2 - \frac{m}{4p}O^2$, $a = \phi_1, b = \sqrt{\frac{m}{p}}\phi_1, q_1 = \frac{m}{p}\phi_1$. For the ellipse, $\phi_2^2 = m^2 + \frac{m}{4p}O^2$, with $a = \sqrt{\frac{m}{p}}\phi_2, b = \phi_2, \varepsilon = \frac{c}{a}\left(1 - \frac{p}{m}\right)^{1/2} < 1$, and $q_2 = \sqrt{\frac{p}{m}}\phi_2 = \frac{1}{2}\sqrt{\delta}\phi_2$. The Cartesian relative potential energy, $U_{rel,D} = -2q_{2D}\cdot\frac{1}{r} = -2\sqrt{\frac{p}{m}}\phi_2\cdot\frac{1}{r} = -\sqrt{\delta}\phi_2\cdot\frac{1}{r}$.

In the context of abundance, the quotient of the two Cartesian parameters that determine the discriminant gives rise to two classes according to predominance: abundance or scarcity. This quotient makes it possible to represent a surplus in the income relative to average profit, so that in the first case, the positive one, it leads to an open annual evolution; in contrast to the second, the negative one, with a closed annual evolution, the ellipse. And within the second, since we have established the period (annual), the semi-major axis is fixed; only the semi-minor axis can vary, giving rise to variations in the focal length. Thus, for lower values of the semi-minor axis, we have more elongated or flattened ellipses, symbolizing greater scarcity, while more rounded ellipses symbolize greater relative abundance. Therefore, a progressive evolution from flatter ellipses to more rounded ones is desirable, even reaching their breaking point and initiating hyperbolic-type stages; something that should happen early within the three decades of a working generation.

II. CONCLUSION

Descartes' parameters extend to the conservation of energy, which reflects the permanence of the laws of motion after time translations, as a very important type of symmetries of the equations that formulate them ([13]).

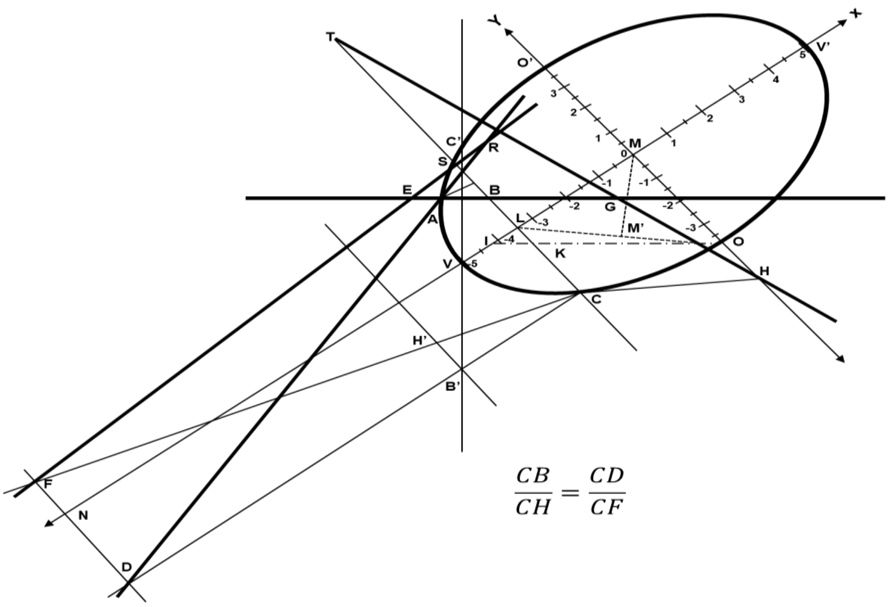
We can define a Cartesian, relative, potential energy, where the energy constant is determined by the Cartesian parameters. It also corresponds to a central field, which guarantees the conservation of angular momentum.

The quotient of the two Cartesian parameters $\left(\frac{p}{m}\right)$, which determine the Cartesian discriminant, can represent a surplus in income relative to average profit, such that in the first case, the positive one, it leads to an open annual evolution, the hyperbola; in contrast to the second, the negative one, with a closed annual evolution, the ellipse.

The principle of duality contained in mythology and engraved in its pyramids also transcends in the location of two foci, where the supplements or complements of the unit of its eccentricities are located, which arise from the quotient of the two Cartesian parameters mentioned, and corresponds to the root of total energy dimensionless by the relative kinetic energy, $\left(\frac{p}{m} \to \left(\frac{|E|}{K_r}\right)^{1/2}\right)$.

In particular, the Cartesian discriminant coincides with the "golden number" when the product of the eccentricities of the two classes is unity.

For each of the 4 given fixed lines, these can be broken down into two classes, with one version linked to the major axis and the other version, the one related to the minor axis.
Regarding the pyramids, agricultural cycles are necessarily annual; however, and of course, we do not have the annual records that would allow us to place successive cycles in one of the only two classes: elliptical or hyperbolic. But in general, due to problems of drought or other interethnic conflicts, we could imagine that they favor the presence of elliptical cycles, with their characteristics of scarcity. However, in the accumulated repetition, we can assume the predominance of one class over the other. If the predominance had been of the elliptical cycles, it would have been manifested through migration. On the other hand, if the predominance had fallen on the hyperbolic cycles, these would leave traces of abundance and permanence in the site. The pyramids are precisely the trace of permanence and the predominance of hyperbolic annual cycles. The construction periods of the pyramids were on the order of a few centuries, so we can imagine the nonexistent annual records as present in the various construction sections of the pyramids. For example, the pyramid of Xochicalco, dedicated to the "God Quetzalcoatl" ("the Feathered Serpent"), was built over approximately two centuries and has since been observed annually as what is now known as the spring equinox, marking the beginning of each annual agricultural cycle. The pyramid of Teopanzolco, built just over three centuries ago, saw its construction interrupted by the unexpected and abrupt arrival of the conquest.



$$\frac{CB}{CH} = \frac{CD}{CF}$$

Figure 1: Projected Pyramid

REFERENCES

[1] Lefebvre H., *Lógica Formal, Lógica Dialéctica*, Siglo XXI, México, (1970), pp. 346.
[2] Hawking S. W., *Dios Creó los Números*, Crítica, Barcelona, (2005), pp. 1030.
[3] Web site: www.deepseek-online.com
[4] Prieto R. S., *Historia de las Matemáticas*, IMC/Ediciones, Toluca, México, (1991), pp. 145.
[5] Thomas G. B., Finney R. L., *Cálculo con Geometría Analítica* (V1), Addison-Wesley-I, Argentina, (1987), pp. 1181.
[6] Voevodin V. V., *Álgebra Lineal*, Mir, Moscú, (1982), pp. 424.
[7] Landau L., Lifchitz E., *Mécanique*, Mir, Moscou, (1969), pp. 228.
[8] Pesenti A., *Lecciones de Economía Política*, Ediciones Quinto Sol, México, D.F. (1985), pp. 420.
[9] Dussel E., 16 *Tesis Economía Política*, Siglo XXI, (2018), pp. 423.
[10] Mercado J. R., *Problemas Inversos y el Principio de Causalidad Eficiente*, Tesis, UNAM, (1997), pp. 123.
[11] Abraham R., Marsden J. E., Ratiu T., Manifolds, *Tensor Analysis and Applications*, Springer-Verlag, N. York, (1988), pp. 654.
[12] Torres del Castillo, GF Torres. *Notas sobre variedades diferenciables*. Monografía del Centro de Investigación y de Estudios Avanzados del IPN, México (1981), pp. 159.
[13] Olver P. J., *Applications Lie* Groups to Differential Equations, Springer- Verlag, Berlin, (1993), pp. 513.

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