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Zagreb Indices for the Hanging Cycle Graph $H_{m,n}$

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ABSTRACT: Zagreb indices are the parameters defined using sum and product of degrees of vertices, joining an edge, in a graph the roots of this concept come from Chemical graph theory, in recent years lot of work is published on Zagreb indices of standard graphs and graph operations. In this paper we establish the relation for Zagreb indices of Hanging Cycle graph $H_{m,n}$ with its component graphs and also find their corresponding polynomials.

Keywords: Hyper Zagreb Indices, Hanging Cycle Graph $H_{m,n}$, Zagreb Indices,
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I. INTRODUCTION

The concepts of connectivity in Chemical Graph Theory, which define relationships between the structure of a molecule and its properties. One important parameter, topological index, which characterizes molecular graph and remain invariant under graph automorphism are called Zagreb Indices. These parameters introduced by Gutman I [1] are defined by using sum and product of degrees of vertices joining an edge.

Consider a subset of E(G) denoted as $E_{a,b}$={(u,v) ϵ E(G) / d(u) = a and d(v) = b }. Partitioning the edge set E(G) into disjoint sets $E_{a,b}$ with all possible choices of pairs a,b we can determine the Zagreb and hyper Zagreb indices and their corresponding polynomials for first and second kind. The Zagreb indices**,** hyper Zagreb indices and their corresponding polynomials we use definitions given by Gutman [1] stated as follows**:**

$$M_1(G) = \sum_{E_{uv \in E(G)}} [d_G(u) + d_G(v)]$$

$$M_2(G) = \sum_{uv \in V(G)} [d_G(u) d_G(v)]$$

$$HM_1(G) = \sum_{uv \in E} [d_G(u) + d_G(v]^2$$

$$HM_2(G) = \sum_{uv \in E} [d_G(u) d_G(v)]^2$$

$$M_1(G, x) = \sum_{uv \in E(G)} x^{[d_G(u)+d_G(v)]}$$

$$M_2(G, x) = \sum_{uv \in E(G)} x^{[d_G(u)d_G(v)]}$$

$$HM_1(G, x) = \sum_{uv \in E} x^{[d_G(u)+d_G(v)]^2}$$

$$HM_2(G, x) = \sum_{uv \in E} x^{[d_G(u)d_G(v)]^2}$$

In this paper we present results for Zagreb indices, hyper Zagreb indices and their polynomials for product graphs, $H_{m,n}$.For simplicity of notation we write $d_G(u) = a$ and $d_G(v) = b$, in all further proofs. we refer [4], [5],[6],[10],and [9].
The definition of Cross product of we refer to Kavitha B N & Indrani Pramod Kelkar research papers [8] and [7].

II. Zagreb indices for $H_{m,n}$

The hanging cycle graph $H_{m,n}$ is the graph obtained by cross product of star S_{m+1} and cycle C_n, where $m \geq 3$ and $n \geq 3$ with number of vertices and edges

$\left|V(H_{m,n})\right| = n(m+1)$ and
$\left|E(H_{m,n})\right| = n(2m+1)$.

We can partition $V(H_{m,n})$ in to two disjoint subsets V_3 and V_{m+2} as

$V_3 = \{v \in V(H_{m,n});\ d(v) = 3\}; |V_3| = mn$
$V_{m+2} = \{v \in V(H_{m,n}); d(v) = m+2\};$
$|V_{m+2}| = n$

Next, the edge set $E(B_{m,n})$ can be partitioned into three disjoint subsets based on the degree of incident vertices as follows:

Theorem 2.1. The first Zagreb indices and their polynomial of $\mathbf{H_{m,n}}$ are

$M_1(H_{m,n}) = n[m(m+13)+4]$ if $m \geq 3$ & $n \geq 3$
$M_1(H_{m,n}, x) = mx^{2(m+2)} + mnx^{m+5} + mnx^6$
if $m \geq 3$ & $n \geq 3$

Proof. The first Zagreb indices of $H_{m,n}$ is

$$M_1(H_{m,n}) = \sum_{uv\in V(G)} [a+b]$$
$$= \sum_{E_{3,3}} [a+b] + \sum_{E_{3,m+2}} [a+b] + \sum_{E_{m+2,m+2}} [a+b]$$
$$= mn[3+3] + mn[3+m+2] + n[m+2+m+2]$$
$$= 13mn + m^2n + 4n$$
$$= mn(m+13) + 4n$$
$$= n[m(m+13)+4] \text{ ------------- (i)}$$

Next the first Zagreb polynomial of $H_{m,n}$ is

$$M_1(H_{m,n}, x) = \sum_{v_1v_2\in E(G)} x^{[a+b]}$$
$$= \sum_{E_{3,3}} x^{[a+b]} + \sum_{E_{3,m+2}} x^{[a+b]} + \sum_{E_{m+2,m+2}} x^{[a+b]}$$
$$= mnx^{[3+3]} + mnx^{[3+m+2]} + nx^{[m+2+m+2]}$$
$$= mnx^6 + mnx^{m+5} + nx^{2m+4}$$
$$= nx^{2(m+2)} + mnx^{m+5} + mnx^6 \text{ -------(ii)}$$

Hence equation (i) and (ii)

$M_1(H_{m,n}) = n[m(n+13)+4]$ if $m \geq 3$ & $n \geq 3$
$M_1(H_{m,n}, x) = mx^{2(m+2)} + mnx^{m+5} + mnx^6$
if $m \geq 3$ & $n \geq 3$ ■

Theorem 2.2. The second Zagreb indices and their polynomial of $H_{m,n}$ are

$M_2(H_{m,n}) = mn(4m+19) + 4n$
if $m \geq 3$ & $n \geq 3$
$M_2(H_{m,,n}, x) = nx^{(m+2)^2} + mn\left[x^{3(m+2)} + x^9\right]$
if $m \geq 3$ & $n \geq 3$

Proof. The second Zagreb indices of $H_{m,n}$ is

$$M_2(H_{m,n}) = \sum_{uv\in V(G)} [a.b]$$
$$. = \sum_{E_{3,3}} [a.b] + \sum_{E_{3,m+2}} [a.b] + \sum_{E_{m+2,m+2}} [a.b]$$
$$= mn[3\times 3] + mn[3\times(m+2)] + n[(m+2)\times(m+2)]$$
$$= 4m^2n + 19mn + 4n$$
$$= mn(4m+19) + 4n \text{ --------- (iii)}$$

Next the second Zagreb polynomial of $H_{m,n}$ is

$$M_2(H_{m,n}, x) = \sum_{uv\in E(G)} x^{[a.b]}$$
$$= \sum_{E_{3,3}} x^{[a.b]} + \sum_{E_{3,m+2}} x^{[a.b]} + \sum_{E_{m+2,m+2}} x^{[a.b]}$$
$$= mnx^{[3\times 3]} + mnx^{[3\times(m+2)]} + nx^{[(m+2)\times(m+2)]}$$
$$= mnx^9 + mnx^{3(m+2)} + nx^{(m+2)^2}$$
$$= nx^{(m+2)^2} + mn[x^{3(m+2)} + x^9] \text{---(iv)}$$

Hence equation (iii) & (iv) show that

$M_2(H_{m,n}) = mn(4m+19) + 4n$ i fm ≥ 3 & $n \geq 3$
$M_2(H_{m,n}, x) = nx^{(m+2)^2} + mn\left[x^{3(m+2)} +\right.$ **x9 if m≥3 & *n≥3*** ■

Theorem 2.3. The first hyper-Zagreb indices and their polynomial of $H_{m,n}$ are

$HM_1(H_{m,n}) = mn[m^2 + 14m + 77] + 16n$
$m \geq 3$ *and* $n \geq 3$
$HM_1(H_{m,n}, x) = mn\left[x^{[m+5]^2} + x^{36}\right] + nx^{[2m+4]^2}$

Proof: The first hyper-Zagreb indices $H_{m,n}$ is

$$HM_1(H_{m,n}) = \sum_{uv\in E}[a+b]^2$$

$$= \sum_{E_{3,3}}[a+b]^2 + \sum_{E_{3,m+2}}[a+b]^2 + \sum_{E_{m+2,m+2}}[a+b]^2$$

$$= mn[3+3]^2 + mn[3+m+2]^2 + n[m+2+m+2]^2$$

$$= m^3n + 14m^2n + 77mn + 16n$$

$$HM_1(H_{m,n}) = mn[m^2 + 14m + 77] + 16n \text{ --(v)}$$

Next for the first hyper-Zagreb-polynomial $H_{m,n}$ is

$$HM_1(H_{m,n}, x) = \sum_{uv\in E} x^{[a+b]^2}$$

$$= \sum_{E_{3,3}} x^{[a+b]^2} + \sum_{E_{3,m+2}} x^{[a+b]^2} + \sum_{E_{m+2,m+2}} x^{[a+b]^2}$$

$$= mnx^{[3+3]^2} + mnx^{[3+m+2]^2} + nx^{[m+2+m+2]^2}$$

$$HM_1(H_{m,n}, x) = mn\left[x^{[m+5]^2} + x^{36}\right] + nx^{[2m+4]^2} \text{-------(vi)}$$

Hence equation (v) & (vi) show that

$$HM_1(H_{m,n}) = mn[m^2 + 14m + 77] + 16n$$

if $m \geq 3$ and $n \geq 3$

$$HM_1(H_{m,n}, x) = mn\left[x^{[m+5]^2} + x^{36}\right] + nx^{[m+4]^2}$$

if $m \geq 3$ and $n \geq 3$ ■

Theorem 2.4: The second hyper-Zagreb indices and their polynomial of $H_{m,n}$

$$HM_2(H_{m,n}) = m^2n(m^2 + 17m + 60) + n(mn + 16)$$

$$HM_2(H_{m,n}, x) = nx^{(m+2)^4} + mn[x^{(3(m+2))^2} + x^{81}]$$

Proof: The second hyper-Zagreb indices of $H_{m,n}$ is

$$HM_2(H_{m,n}) = \sum_{uv\in E}[a.b]^2$$

$$= \sum_{E_{3,3}}[a.b]^2 + \sum_{E_{3,m+2}}[a.b]^2 + \sum_{E_{m+2,m+2}}[a.b]^2$$

$$= mn[3\times 3]^2 + mn[3\times m+2]^2 + n[(m+2)\times (m+2)]^2$$

$$= m^4n + 17m^3n + 60m^2n + 149mn + 16n$$

$$= m^2n(m^2 + 17m + 60) + n(149m + 16) \text{------- (vii)}$$

Next, for second hyper-Zagreb polynomial of we have

$$HM_2(H_{m,n}, x) = \sum_{uv\in E} x^{[a.b]^2}$$

$$= \sum_{E_{3,3}} x^{[a.b]^2} + \sum_{E_{3,m+2}} x^{[a.b]^2} + \sum_{E_{m+2,m+2}} x^{[a.b]^2}$$

$$= mnx^{[3\times 3]^2} + mnx^{[3\times(m+2)]^2} + nx^{[(m+2)\times(m+2)]^2}$$

$$= nx^{(m+2)^4} + mn[x^{(3(m+2))^2} + x^{81}] \text{- (viii)}$$

Hence equation (vii) & (viii) show that

$$HM_2(H_{m,n}) = m^2n(m^2 + 17m + 60) + n(mn + 16)$$

$$HM_2(H_{m,n}, x) = nx^{(m+2)^4} + mn[x^{(3(m+2))^2} + x^{81}]$$ ■

III. Results

In the paper of Cangul et al [23] we have results for M_1, $M_2$and we find the results for hyper Zagreb indices for star, path and cycle graph as given below:

	S_{m+1}	C_n
M_1	m^2+m	4n
M_2	m^2	4n
HM_1	$m(m+1)^2$	16n
HM_2	m^3	16n

Expressing the results for product graphs in terms of their component graph parameters we observe that

(i) if $m \geq 3$ and $n \geq 3$

$$M_1(H_{m,n}) = n[m(m+13)+4]$$
$$= n(m^2+m)+(3m+1)(4n)$$
$$= n\,M_1(S_{m+1})+(3m+1)M_1(P_n)$$
$$M_2(H_{m,n}) = mn(4m+19)+4n$$
$$= 4n(m^2+m)+\left(\frac{15}{4}m+1\right)(4n)$$
$$= 4nM_2(S_{m+1})+\left(\frac{15}{4}m+1\right)M_2(P_n)$$
$$HM_1(H_{m,n}) = mn[m^2+14m+77]+16n$$
$$= nm(m+1)^2+mn(12m+$$
76+16n
$$= n\,HM_1(S_{m+1})+\left(\frac{19}{4}m+1\right)$$

$$HM_1(C_n)+12m^2n$$

$$HM_2(H_{m,n}) = m^2n(m^2+17m+60)$$
$$+n(mn+16)$$
$$= m^3(m+n+16n)+mn(60m+$$
n+16n
$$= HM_2(S_{m+1})n(m+1)+(m^3$$
$$+1)HM_2(C_n))+mn(60m+n)$$

IV CONCLUSION

$$HM_1(H_{m,n}) \geq |V(P_n)|\,HM_1(S_{m+1})$$
$$+|V(S_{m+1})|\,HM_1(C_n)$$

$$HM_2(H_{m,n}) \geq |V(P_n)|\,HM_2(S_{m+1})$$
$$+|V(S_{m+1})|\,HM_2(C_n)$$

Thus, from above results we can conclude the Zagreb indices of product graphs $H_{m,n}$ with respective Zagreb indices of its factor graphs satisfy Vizing conjecture like results, where $\boldsymbol{\alpha}$ represents various Zagreb indices.

$$\alpha(H_{m,n}) = \alpha(S_{m+1} \times C_n) \geq |V(P_n)|\,\alpha(S_{m+1}) + |V(S_{m+1})|\,\alpha(C_n).$$

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