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# Short Note on Continuous Fourier series

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**ABSTRACT:**
This paper studies two data analytic methods: Fourier transforms and wavelets. Fourier transforms approximate a function by decomposing it into sums of sinusoidal functions, while wavelet analysis makes use of mother wavelets. Both methods are capable of detecting dominant frequencies in the signals

## I. INTRODUCTION

For a function with period T, a continuous Fourier series can be expressed as [1.1-1.5]

$$f(t)=a_0+\sum_{k=1}^{\infty} a_k \cos(kw_0t)+b_k \sin(kw_0t)$$

(1.1)

The unknown fourier coefficients $a_0$, $a_k$ and $b_k$ can be computed as

$$a_0=\frac{1}{T}\int_0^T f(t)dt \qquad (1.2)$$

Thus, $a_0$ can be interpreted as the "average" function value between the period interval [0,T].

$$a_k=\left(\frac{2}{T}\right)\int_0^T f(t)\cos(kw_0t)dt$$

(1.3)

$=a_{-k}$ (hence $a_k$ is an "even" function)

$$b_k\left(\frac{2}{T}\right)\int_0^T f(t)\sin(kw_0t)dt$$

(1.4)

$=-b_{-k}$ ( hence $b_k$ is an "odd" function)

**Derivation of formulas for $a_0$, $a_k$ and $b_k$:**

Integrating both sides of equation 1 with respect to time, one gets

$$\int_0^T f(t)dt=\int_0^T a_0dt+\int_0^T\sum_{k=1}^{\infty} a_k\cos(kw_0t)dt+\int_0^T\sum_{k=1}^{\infty} a_k\sin(kw_0t)dt$$

(1.5)

The second and third terms on the right hand side of the above equations are both zeros, due to the result stated in equation (1.1)

Thus,

$$\int_0^T f(t)dt=$$

$$\int_0^T f(t)dt=\int\left[a_0t\right]_0^T \qquad (1.6)$$

$=a_0T$

Hence,

$$a_0=\frac{1}{T}\int_0^T f(t)dt \qquad (1.7)$$

Now, if both sides of equation (1.1) are multiplied by $\sin(mw_0t)$ and then integrated with respect to time, one obtains

$$\int_0^T f(t)\times\sin(mw_0t)dt=\int_0^T a_0\sin(mw_0t)dt+\int_0^T\sum_{k=1}^{\infty} a_k\cos(kw_0t)dt\sin(mw_0t)dt$$
$$+\int_0^T\sum_{k=1}^{\infty} b_k\sin(kw_0t)dt\sin(mw_0t)dt$$

(1.8)

Due to equation (1.1) and (1.3) , the first and second terms on the right hand side (RHS) of equation (1.8) are zero

Due to equation (1.4) , the third RHS term of equation (1.8) is also zero, with the exception when k=m, which will become (by referring to equation (1.2))

$$\int_0^T f(t)\sin(kw_0t)dt=0+0+\int_0^T b_k\sin^2(kw_0t)dt$$

$$=b_k \text{ x } \frac{T}{2}$$

Thus,

$$b_k=\left(\frac{2}{T}\right)\int_0^T f(t)\sin(kw_0t)dt$$

Similar derivation can be used to obtain $a_k$, as shown in Equation (1.3)
for finding Fourier coefficients $a_0$, $a_k$, and $b_k$

Based upon the derived formulas for $a_0$, $a_k$ and $b_k$ (shown in equations 1.2-1.4)
Example 1
Using the continuous Fourier series to approximate the following periodic function (T=2π seconds) shown in Figure 1.

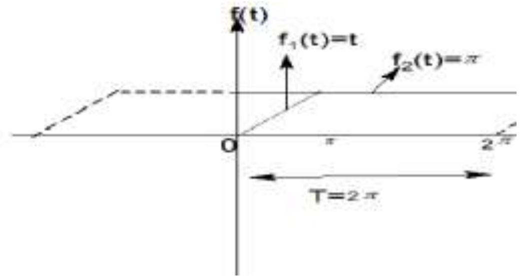



Figure 1 A Periodic Function (Between 0 and 2π)

$$f(t) = \begin{cases} t \; for \; 0 < t \le \pi \\ \pi \; for \; \pi \le t < 2\pi \end{cases}$$

specifically, find the Fourier coefficients $a_0$, $a_1$,……………..,$a_8$ and $b_1$,………..,$b_8$
Solution:
The unknown Fourier coefficients $a_0$, $a_k$ and $b_k$ can be computed based on equations (1.2-1.4) as following:

$$a_0 = \left(\frac{1}{T}\right)\int_0^{2\pi} f(t)dt$$

$$a_0 = \frac{1}{(2\pi)} \times \left\{ \int_0^{\pi} t dt + \int_{\pi}^{2\pi} \pi dt \right\}$$

$$a_0 = 2.35619$$

$$a_k = \left(\frac{2}{T}\right)\int_0^{T=2\pi} f(t)\cos(kw_0t)dt$$

$$a_k = \left(\frac{2}{2\pi}\right) \times \left\{ \int_0^{\pi} t\cos\left(k \times \frac{2\pi}{T} \times t\right)dt + \int_0^{2\pi} \pi \times \cos\left(k \times \frac{2\pi}{T} \times t\right)dt \right\}$$

$$a_k = \left(\frac{1}{\pi}\right) \times \left\{ \int_0^{\pi} t\cos(kt)dt + \int_0^{2\pi} \pi \times \cos(kt)dt \right\}$$

The "integration by part" formula can be utilized to compute the first integral on the right=hand side of the above equation.

For k = 1,2,……….8 the Fourier coefficients $a_k$ can be computed as

$a_1$ = -0.6366257003116296
$a_2$= $-5.070352857678721 \times 10^{-6} \approx 0$
$a_3$= -0.07074100153210318
$a_4$= $-5.07320092569666 \times 10^{-6} \approx 0$
$a_5$= -0.025470225589332522
$a_6$= $-5.070265333302604 \times 10^{-6} \approx 0$
$a_7$= -0.0012997664818977102
$a_8$= $-5.070188612604695 \times 10^{-6} \approx 0$

Similarly,

$$b_k = \left(\frac{2}{T}\right)\int_0^{T=2\pi} f(t)\sin(kw_0t)dt$$

$$b_k = \left(\frac{1}{\pi}\right) \times \left\{ \int_0^{\pi} t\sin(kt)dt + \int_0^{2\pi} \pi \times \sin(kt)dt \right\}$$

$b_1$ = -0.9999986528958207
$b_2$= -.04999993232285269
$b_3$= -0.3333314439509194
$b_4$= -0.24999804122384547
$b_5$= -0.19999713794872364
$b_6$= -0.1666635603759553
$b_7$= -0.14285324664625462
$b_8$= -0.12499577981019251

Any periodic function f(t) , such as the one shown in Figure 1 can be represented by the Fourier series as

$$f(t) = a_0 = \sum_{k=1}^{\infty}\{a_k\cos(kw_0t) + b_k\sin(kw_0t)\}$$

where $a_0$, $a_k$ and $b_k$ have already been computed( for k=1,2,…….., 8)
and $w_0$= 2πf

$$= \frac{2\pi}{T}$$

$$= \frac{2\pi}{2\pi}$$

$$= 1$$

Thus, for k=1, one obtains

$$f_1(t) = a_0 + a_1\cos(t) + b_1\sin(t)$$

For k=1→2 one obtains

$$f_2(t) = a_0 + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t)$$

For k=1→4, one obtains

$f_4(t)=a_0+a_1\cos(t)+b_1\sin(t)+a_2\cos(2t)+b_2\sin(2t)+a_3\cos(3t)+b_3\sin(3t)+a_4\cos(4t)+b_4\sin(4t)$

Plots for $f_1(t)$, $f_2(t)$ and $f_4(t)$ as shown in figure 2.

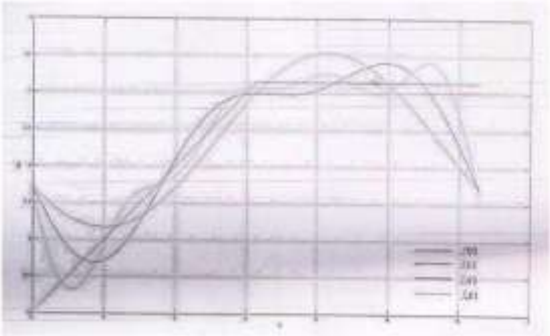


**Figure 2** Fourier Approximated Functions (

For example 1)

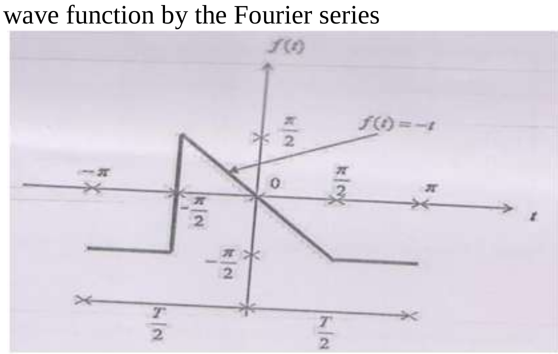
It can be observed from Figure 2 that as more terms are included in the Fourier series, the approximated Fourier functions are more closely resemble the original periodic function as shown in Figure 1.

Example 2

The periodic triangular wave function f(t) is defined as

$$f(t)=\begin{cases} \frac{-\pi}{2} & \text{for } -\pi<t<\frac{-\pi}{2} \\ -t & \text{for } \frac{-\pi}{2}<t<\frac{\pi}{2} \\ \frac{-\pi}{2} & for\ \frac{\pi}{2}<t<\pi \end{cases}$$

Find the Fourier coefficients $a_0$, $a_1$ …..$a_8$ and $b_1$……$b_8$ and approximate the periodic triangular wave function by the Fourier series




Solution

The unknown Fourier Coefficients $a_0$, $a_k$ and $b_k$ can be computed based on equations (1.2-1.4) as follows

$$a_0=\left(\frac{1}{T}\right)\int_{-\pi}^{\pi} f(t)dt$$

$$a_0=\left(\frac{1}{2\pi}\right)\times\left\{\int_{-\pi}^{\frac{-\pi}{2}}\left(\frac{-\pi}{2}\right)dt+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(-t)dt+\int_{\frac{\pi}{2}}^{\pi}\left(\frac{-\pi}{2}\right)dt\right\}$$

$a_0= -0.78539753$

$$a_k=\left(\frac{2}{T}\right)\int_{-\pi}^{\pi} f(t)\cos(kw_0t)dt$$

where

$$w_0=\frac{2\pi}{T}=\frac{2\pi}{2\pi}=1$$

Hence,

$$a_k=\left(\frac{2}{T}\right)\int_{-\pi}^{\pi} f(t)\cos(kt)dt$$

or

$$a_k=\left(\frac{2}{2\pi}\right)\left\{\int_{-\pi}^{\frac{-\pi}{2}}\left(-\frac{\pi}{2}\right)\cos(kt)dt+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(-t)\cos(kt)dt+\int_{\frac{\pi}{2}}^{\pi}\left(-\frac{\pi}{2}\right)\cos(kt)dt\right\}$$

Similarly,

$$b_k=\left(\frac{2}{T}\right)\int_{-\pi}^{\pi} f(t)\sin(kw_0t)dt=\left(\frac{2}{T}\right)\int_{-\pi}^{\pi} f(t)\sin(k_0t)dt$$

or,

$$b_k=\left(\frac{2}{2\pi}\right)\left\{\int_{-\pi}^{\frac{-\pi}{2}}\left(-\frac{\pi}{2}\right)\sin(kt)dt+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(-t)\sin(kt)dt+\int_{\frac{\pi}{2}}^{\pi}\left(-\frac{\pi}{2}\right)\sin(kt)dt\right\}$$

The "integration by part" formula can be utilized to compute the second integral on the right –hand side of the above equation s for $a_k$ and $b_k$.

For k=1,2………8 the Fourier coefficients $a_k$ and $b_k$ can be computed and summarized as following in Table 1

Table 1: Fourier coefficients $a_k$ and $b_k$ for various k values

| **K** | **$a_k$** | **$b_k$** |
|---|---|---|
| 1 | 0.999997 | -0.63661936 |
| 2 | 0.00 | -0.49999932 |
| 3 | -0.3333355 | 0.7073466 |
| 4 | 0.00 | 0.2499980 |
| 5 | 0.1999968 | -0.02546389 |
| 6 | 0.00 | -0.16666356 |

| | | |
|---|---|---|
| 7 | -0.14285873 | 0.0126991327 |
| 8 | 0.00 | 0.12499578 |

The periodic function (shown in Example 1) can be approximated by Fourier series as

$$f(t) = a_0 + \sum_{k=1}^{\infty}\{a_k\cos(kt)+b_k\sin(kt)\}$$

Thus, for k=1, one obtains:

$$f_1(t) = a_0 + a_1\cos(t) + b_1\sin(t)$$

For k=1→2, one obtains:

$$f_2(t) = a_0+a_1\cos(t)+b_1\sin(t)+a_2\cos(2t)+b_2\sin(2t)$$

Similarly, for k=1→4 one has:

$$f_4(t)=a_0+a_1\cos(t)+b_1\sin(t)+a_2\cos(2t)+b_2\sin(2t)+a_3\cos(3t)+b_3\sin(3t)+a_4\cos(4t)+b_4\sin(4t)$$

Plots for functions, $f_1(t)$, $f_2(t)$ and $f_4(t)$ are shown in Figure 4.

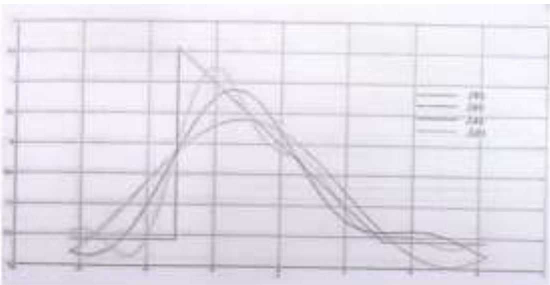


**Figure 4** Fourier approximated functions for Example 2.

It can be observed from Figure 4 that as more terms are included in the Fourier series, the approximated Fourier functions closely resemble the original periodic function.

**1.2 Complex form of the Fourier Series:**

Using Euler's identity, $e^{ix} = \cos(x)+i\sin(x)$, and $e^{-ix} = \cos(x)-i\sin(x)$, the sine and cosine can be expressed in the exponential form a s

$$\sin x=\frac{e^{ix}-e^{-ix}}{2i}$$ ="odd" function, since sin(x)=-sin(-x) (1.10)

$$\cos x=\frac{e^{ix}+e^{-ix}}{2i}$$ ="even" function, since cos(x) = cos(-x) (1.11)

Thus, the Fourier series(expressed in equation 1.1) can be converted into the following form

$$f(t)=a_0+\sum_{k=1}^{\infty}a_k\left(\frac{e^{ikw_0t}+e^{-ikw_0t}}{2}\right)+b_k\left(\frac{e^{ikw_0t}-e^{-ikw_0t}}{2i}\right)$$

or

$$f(t)=a_0+\sum_{k=1}^{\infty}e^{ikw_0t}\left(\frac{a_k}{2}+\frac{b_k}{2i}*\frac{i}{t}\right)+e^{-ikw_0t}\left(\frac{a_k}{2}-\frac{b_k}{2i}*\frac{i}{t}\right)$$

or, since $i^2 = -1$, one obtains

$$f(t)=a_9+\sum_{k=1}^{\infty}e^{ikw_0t}\left(\frac{a_k-ib_k}{2}\right)+e^{-ikw_0t}\left(\frac{a_k+ib_k}{2}\right)$$

Define the following constants

$$C_0= a_0 \quad (1.14)$$

$$C_1=\frac{a_k-ib_k}{2} \quad (1.15)$$

Hence:

$$C_{-k}=\frac{a_{-k}-ib_{-k}}{2} \quad (1.16)$$

Using the even and odd properties shown in Equations (1.3) and (1.4) respectively, equation (1.16) becomes

$$C_{-k}=\frac{a_k+ib_k}{2} \quad (1.17)$$

$$f(t) = C_0+\sum_{k=1}^{\infty}C_ke^{ikw_0t}+\sum_{k=1}^{\infty}C_{-k}e^{-ikw_0t}$$

$$=\sum_{k=0}^{\infty}C_ke^{ikw_0t}+\sum_{k=-1}^{-\infty}C_{-k}e^{ikw_0t}$$

$$=\sum_{k=0}^{\infty}C_ke^{ikw_0t}+\sum_{k=-\infty}^{-1}C_ke^{ikw_0t}$$

$$=\sum_{k=-\infty}^{\infty}C_ke^{ikw_0t}$$

The coefficient $c_k$ can be computed, by substituting equations (1.3) and (1.4) into equations (1.15) to obtain

$$C_k =\left(\frac{1}{2}\right)\left(\frac{2}{T}\right)\left\{\int_0^T f(t)\cos(kw_0t)dt-i\int_0^T f(t)\sin(kw_0t)dt\right\}$$
$$=\left(\frac{1}{T}\right)\left\{\int_0^T f(t)\times[\cos(kw_0t)dt-i\sin(kw_0t)]dt\right\}$$

Substituting Equations (1.10, 1.11) into the above equation, one gets

$$C_k = \left(\frac{1}{T}\right) \left\{ \int_0^T f(t) \times e^{-ikw_0 t} dt \right\}$$

(1.20)

Thus, equations (1.18) and (1.20) are the equivalent complex version of equations (1.1)-(1.4)

## REFERENCES

[1] E. Oran Bri gham. The Fast Fourier Transform. Prentice-Hall, Inc, 1974.
[2] J. Stoer and R. Bulirsch. Introduction to Numerical Analysis. Springer-Verlag New York, Inc. 1993.
[3] WikiPedia, Fourier Series. http://en.wikipedia.org/wiki/Fourier_series
[4] WikiPedia, Fourier Series. http://en.wikipedia.org/wiki/Wavelets
[5] Stewart, James. Calculus Early Transcendentals, 4th Edition.
[6] Fourier Analysis, Course Notes, Modeling II.3] Anders Vretbald. Fourier Analysis and its Applications. Springer-Verlag New York, Inc. 2003.

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