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Existence and Uniqueness of the solutions to Stochastic Neutral Functional Differential Equations Using Fractional Brownian motion with Non-Lipschitz Coefficients

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ABSTRACT: In this paper we investigate the existence and uniqueness of mild solutions to neutral stochastic functional differential equations driven by a Brownian motion in a Hilbert space with non-Lipschitzian coefficients. The results are obtained by using the method of successive approximation and generalize the results that were reported by Bao and Hou[1].

Keywords: Stochastic neutral functional differential equation, mild solution, Fractional Brownian motion, Non-Lipschitz coefficients.
AMS Subject Classification: 60H05, 60H07.

I. INTRODUCTION

The study of existence, uniqueness and stability of mild solutions of stochastic neutral functional differential equations (SNFDEs) due to their range of applications in various sciences such as physics, mechanical engineering, control theory and economics where in, quite often the future state of such systems depends not only on the present state but also on its past history leading to SNFDEs rather than SDEs. Mao [11] discussed this kind SNFDEs is the following neutral SFDEs with finite delay which could be used in chemical engineering and aero elasticity introduce in kolmanovskil and myshkis [9]. Under the global Lipschitz and linear growth condition Taniguchi [21] and Luo [10] considered the existence and Uniqueness of mild solutions to SPFDEs. The theory of stochastic differential equations driven by a fractional Brownian motion (FBM) has been studied intensively in recent years [3, 5, 8, 14, 15] . Senguttuvan et al., studied the existence of stochastic Partial differential equations with neutral and delay conditions [17, 18, 19, 20]. In [7] Farrante and Rovira studied the Stochastic delay differential equations driven by fractional Brownian motion with Hurst parameter H $>1/2$.

Consider the stochastic Partial differential equations (SPDEs) driven by a fractional Brownian motion (FBM). SPDEs arise in many areas of applied mathematics. For this reason, the study of this type of equations has been receiving increased attention in the last few years. An existence and uniqueness result of mild solutions for a class of neutral stochastic differential equation with finite delay, driven by an FBM in a Hilbert space has been investigated [2] in Boufoussi and Hajji. The asymptotic behaviour of solutions for stochastic differential equations with FBM has been investigated by authors [2, 4, 6]. More over Nguyen [12] has studied the asymptotic behaviours of mild solutions to neutral stochastic differential equations driven by an FBM.

Motivated by the above papers, in this work we aim to extend the existence and Uniqueness of mild solution to cover a class of more general SNFDs described in the form

$D\big[[X(t)] + G(t,\, X_t)\big] = \,[AX(t) +$
Ft,Xtdt+σtdBHt, $0 \le t \le T,$

with the initial condition $X(t) = \xi(t),\ t \in [-\tau. 0]$ and $X_t = \,[X(t+\,\theta) : -\tau \le \theta \le 0\,]$ where A is the infinitesimal generator of an analytic semi group of bounded linear operators, $\big(S(t)\big)_{t\ge0,}$ in a Hilbert space X, $B^H\ is\ a\ Q -$ fractional Brownian motion real and separable Hilbert space Y, r, ρ :$[0, T]\ \to [0,\tau](\tau\ > 0)$ are continuous, F,G: $[0, T\,] \times X\ \to$ X, σ: $[0,\ T]\ \to L_2^0(Y,\ X)$are appropriate function and $\xi \in C\left([-\tau, 0];\ L^2(\Omega, \mathrm{X})\right)$. Here $L_2^0(Y, X)$ denotes the space of all Q- Hilbert- Schmidt operators from Y into X .

Unfortunately, for many practical situations, the nonlinear terms do not obey the global Lipschitz and linear growth condition, even the local Lipschitz condition. Motivated by the above papers, in this paper, we aim to extend the existence and uniqueness of mild solutions to cover a class of more general neutral stochastic functional differential equations driven by fractional Brownian motion with Hurst parameter½< H < 1under a non –Lipschitz condition with the Lipschitz condition being regarded as a special case and a weakened linear growth condition.

The rest of this paper is organised as follows. In section 2 we introduce some notations, concepts and basic results about fractional Brownian motion, Wiener integral over Hilbert spaces and we recall some preliminary results about analytic semi groups and fractional power associated to its generator. In section 3, the existence and uniqueness of mild solutions are proved.

II. PRELIMINARY RESULTS

In this section, we collect some notions, conceptions and lemmas on Wiener integrals with respect to an infinite dimensional fractional Brownian motion. In addition, we also recall some basic results about analytical semi-groups and fractional powers of their infinitesimal generators which will be used throughout this paper.

Let $(\Omega, \mathcal{F}, \mathbb{P})$be a complete probability space. Consider a time interval $[0,T]$ with arbitrary fixed horizon T and let $\{\beta^H(t),\ t \in [0,T]\}$ be the one-dimensional fractional Brownian motion with Hurst parameter H $\in \left(\frac{1}{2},1\right)$.This means by definition that β^H is a centered Gaussian process with covariance function:

$$R_H(t,s) = \mathbb{E}(\beta_t^H \beta_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

More over β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t \mathrm{K}_H(t,s)d\beta(s)$$

where$\beta = \{\beta^H(t), t \in [0,T]\}$ is a Wiener process and $\mathrm{K}_H(t,s)$ is the kernel given by

$$\mathrm{K}_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{1}{2}}\, du$$

for t $> s$, where $c_H = \sqrt{H(2H-1)/\beta\left(2-2H, H-\frac{1}{2}\right)}$ and $\beta(.,.)$ denotes the Beta function.We put $\mathrm{K} =_H (t,s) = 0$ if t $\leq s$.

We will denote by $\mathcal{H}$ the reproducing kernel Hilbert space of the FBM. In fact $\mathcal{H}$ isthe closure of set of indicator functions $\{\mathrm{I}_{[0,t]},\ t \in [0,T]\}$with respect to the scalar product

$$\langle \mathrm{I}_{[0,t]}, \mathrm{I}_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s).$$

The mapping$\mathrm{I}_{[0,t]} \to \beta^H(t)$ can be extended to an isometry between$\mathcal{H}$ andthe first Wiener chaos and we will denote by $\beta^H(\varphi)$the image of φ by the previous isometry.

We recall that for $\psi, \Phi \in \mathcal{H}$ their scalar product in $\mathcal{H}$ is given by

$$\langle \psi, \varphi \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T \psi(s)\, \varphi(t) |t-s|^{2H-2}\, ds\, dt.$$

Let us consider the operator $K_H^*$from $\mathcal{H}$ to $L^2([0,T])$ defined by

$$(K_H^*)(s) = \int_s^T \varphi(r) \frac{\partial \mathrm{K}}{\partial r}(r,s) dr$$

We refer [13] for the proof of the fact that K_H^* is an isometry between $\mathcal{H}$and $L^2([0,T])$.

Moreover for any$\in \mathcal{H}$, we have

$$\beta^H(\varphi) = \int_0^t (K_H^*)(t)\, d\beta(t).$$

It follows from[13] that the elements of $\mathcal{H}$ may be not functions but distributions of negative order. In order to obtain a space of functions contained in $\mathcal{H}$,we consider the linear space $|\mathcal{H}|$ generated b the measurable functions ψ such that

$$\|\psi\|_{|\mathcal{H}|}^2 = \alpha_H \int_0^T \int_0^T |\psi(s)|\, |\psi(t)|\, |t-s|^{2H-2}\, ds\, dt < \infty$$

where $\alpha_{\mathcal{H}} = H(2H-1)$. The space $|\mathcal{H}|$is a Banach space with the norm $\|\psi\|_{|\mathcal{H}|}$ and we have the following conclusions[13].

Lemma 2.1

Let $L^2([0,T]) \subseteq L^{1/H}([0,T]) \subseteq |\mathcal{H}| \subseteq \mathcal{H}$, and for any $\psi \in L^2([0,T])$ we have

$\|\psi\|_{|\mathcal{H}|}^2 \leq 2H\, T^{2H-1} \int_0^T |\psi(s)|^2 ds.$

Let X and Y be two real, separable Hilbert spaces and let$\mathcal{L}(Y,X)$be the space of bounded linear operator from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in Y, X and $\mathcal{L}(Y,X)$. Let Q $\in \mathcal{L}(Y,X)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace tr Q = $\sum_{n-1}^{\infty} \lambda_n < \infty$ where $\lambda_n \geq 0$ (n = 1,2,….) are non-negative real numbers and { e_n }(n = 1,2,….) is a complete orthonormal basis in Y. We define the infinite dimensional FBM on Y with covariance Q as

$$B^H(t) = B_Q^H(t) = \sum_{n-1}^{\infty} \sqrt{\lambda_n}\, e_{n\, B_Q^H},$$

WhereB_Q^H are real, independent FBM.This process is a Y- valued Gaussian, it starts from 0, has zero mean and covariance:

$$\mathrm{E}\langle B^H(t),\ x\rangle \langle B^H(s),\ y\rangle = R(s,t)\langle Q(x),\ y\rangle, \text{ for all } x,y \in Y \text{ and } t,s \in [0,T].$$

In order to define Wiener integrals with respect to the Q-FBM, we introduce the space $L_2^0 = L_2^0(Y,\ X)$ of all Q-Hilbert-Schmidt operators ψ: Y → X. We recall that $\psi \in \mathcal{L}(\mathrm{Y},\mathrm{X})$ is called a Q-Hilbert-Schmidt operator if

$$\|\psi\|_{L_2^0}^2 = \sum_{n-1}^{\infty} \left\|\sqrt{\lambda_n}\, \psi e_n\right\|^2 < \infty,$$

and that the space L_2^0 equipped with the inner product $\langle\varphi,\psi\rangle_{L_2^0} = \sum_{n-1}^{\infty}\langle\varphi e_n, \psi e_n\rangle$ is a separable Hilbert space.

Now, let $\varphi(s)$, $s \in [0,T]$ be a function with values in $L_2^0(Y,X)$. The Wiener integral of φ with respect to B^H is defined by

$$\int_0^t \varphi(s)\, dB^H(s) = \sum_{n-1}^{\infty}\sqrt{\lambda_n}\, \varphi(s) dB_n^H(s) =$$
n−1∞0tλnKH∗φen sBnHs (2.1)

whereβ_n is the standard Brownian motion used to present B_n^H.

Now we end this subsection by stating the following result in [17].

Lemma 2.2 If $\psi: [0,T] \to L_2^0(Y,X)$satisfies $\int_0^T \|\psi(s)\|^2_{L_2^0} < \infty$, then the above sum in (2.1) is well defined as a X-valued random variable and we have

$$\mathrm{E}\left\|\int_0^t \varphi(s)\, dB^H(s)\right\|^2 \leq 2Ht^{2H-1}\int_0^t \|\psi(s)\|^2_{L_2^0}\, ds\,.$$

Now we turn to state notations and basic facts about the theory of semi-groups and fractional power operators. Let A : D(A) → X be the infinitesimal generator of an analytic semi-group, $(S(t))_{t\geq o,}$of bounded linear operators on X. For the theory of strongly continuous semigroup, we refer to Pazy [16]. We will point out here some notations and properties that will be used in this work. It is well known that there exist M ≥ 1 and $\lambda \epsilon \mathbb{R}$ such that $\|S(t)\| \leq M\, e^{\lambda t}$ for every t ≥ 0.

If $(S(t))_{t\geq o}$is a uniformly bounded and analytic semigroup such that $0 \in \rho(A)$,where $\rho(A)$ is the resolvent set of A, then it is possible to define the fractional power $(-A)^\alpha$ for $0 \leq \alpha \leq 1$, as a closed linear operator on its domain D $(-A)^\alpha$. Furthermore, the subspace $(-A)^\alpha$ is dense in X, and the expression $\|h\|_\alpha = \|(-A)^\alpha h\|$ defines a norm in D $(-A)^\alpha$. If X_α represents the space D $(-A)^\alpha$ endowed with the norm $\|\cdot\|_\alpha$,then the following properties are well known [10,Theorem 6.13].

Lemma 2.3 Suppose that the preceding conditions are satisfied.

(1) Let 0< α ≤1. Then X_α is a Banach space.

(2) If $0 < \beta \leq \alpha$ then the injection $X_\alpha \to X_\beta$ is continuous.

(3) For every $0 < \beta \leq 1$ there exists $M_\beta > 0$ such that

$$\|(-A)^\beta\, S(t)\| \leq M_\beta t^\beta e^{-\lambda t} \quad ,t > 0,\ \lambda > 0.$$

Lemma 2.4([3]: Lemma 1) For u, v ∈ X, and 0 < c <1,

$$\|u\| \leq \frac{1}{1-c}\|u-v\|^2 + \frac{1}{c}\|v\|^2.$$

III. EXISTENCE AND UNIQUENESS

In this section we study the existence and uniqueness of mild solution for Eq.(1.1).For this equation we assume that the following conditions hold.

(H1) A is the infinitesimal generator of an analytic semi group, $(S(t))_{t\geq o,}$ of bounded linear operators on X. Further, to avoid unnecessary notations, we,suppose that

$0 \in \rho(A)$, and that, (see Lemma 2.3,)

$$\|S(t)\| \leq \mathrm{M} \text{and} \left\|(-A)^\beta S(t)\right\| \leq \frac{\mathrm{M}1-\beta}{t^{1-\beta}},$$

for some constants $\mathrm{M}, \mathrm{M}_\beta$and every t $\in [0,T]$.

(H2) The function f satisfies the following non-Lipschitz conditionfor any

x,y $\in X$and t≥ 0,

$$\|f(t,x) - f(t,y)\|^2 \leq k\,(\|x-y\|^2),$$

where k is a concave nondecreasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$ such that $\kappa(0) = 0$,

$\kappa(u) > 0$ and$\int \int_{0^+} du/\kappa(u) = \infty$ e.g k $\sim u^\alpha$, ½ <$\alpha < 1$. We further assume that

there is an $\mathrm{M}' > 0$ such that

$\sup_0 \leq t \leq T\|f(t,0)\| \leq \mathrm{M}'$.

(H3) There exist constants $1/2 < \alpha \leq 1$, $\mathrm{K}_1 \geq 0$such that the function g is X_α-valued

and satisfies for any x, y∈ X and $t \geq 0$,

$$\|(-A)^\alpha G(t,x) - (-A)^\alpha G(t,y)\| \leq \mathrm{K}_1\|x-y\|,$$
$$\|(-A)^\alpha\|\mathrm{K}_1 < 1$$

We further assume that G(t,0) ≡ 0 for t ≥ 0 and the function $(-A)^\alpha$ is continuous in the quadratic mean sense:

$$\lim \mathbb{E}_{t\to s}\left\|(-A)^\alpha g(t,x(t)) -\right.$$
−Aαgs,xs2 =0.

(H4) The function $\sigma: [0,+\infty) \to L_2^0(Y,X)$ satisfies

$$\int_0^T \|\sigma(s)\|^2_{L_2^0}\, ds < \infty, \qquad \forall\, \mathrm{T} > 0$$

Definition 3.1: A X-valued process $x(t)$ is called a mild solution of (1.1) if

$x \in ([-\tau,\mathrm{T}], \mathbb{L}^2(\Omega,\mathrm{X}))$for $t \in [-\tau, 0]$, $x(t) = \varphi(t)$, and for $t \in [0,\mathrm{T}]$

Satisfies

$$\mathrm{X(t)} = \mathrm{T(t)}[\xi(0) + G(0,\xi)] - G(t,X_t) - \int_0^t AT(t-s)G(s,X_s ds) + \int_0^t T(t-s)F(s,Xs)ds + \int_0^t T(t-s)\sigma(s)d\,B^H(s)$$

Lemma 3.1[10]: Let T > 0 and c> 0. Let k: $\mathbb{R}^+$ to $\mathbb{R}^+$ be a continuous nondecreasing function such that $\kappa(t) > 0$ for all t >0. Let u(.) be a Borel measurable bounded nonnegative function n $[0,\mathrm{T}]$. If

$u(t) \le c + \int_0^t v(s)k(u(s))ds$ for all $0 \le t \le T$.

$$\mathfrak{u}(t) \le J^{-1}\left(J(c) + \int_0^t \mathfrak{v}(s)ds\right),$$

holds for all such $t \in [0,T]$ that

$$J(c) + \int_0^t \mathfrak{v}(s)ds \in \text{Dom}\,(J^{-1}),$$

where $J(r) = \int_0^r ds/k(s)$, on $r > 0$, and J^{-1} is the inverse function of J. In Particular, if, c = 0 and $\int_{0^+}^r ds/\kappa(s) = \infty$, then $\mathfrak{u}(t) = 0$ for all $t \in [0,T]$.

To complete our main results, we need to prepare several lemmas which will be utilize in the sequel. Note that $G(t, 0) \equiv 0$ and

$$\|(-A)^\alpha G(t,x) - (-A)^\alpha G(t,y)\| \le K_1 \|x - y\|.$$

Then we easily get that $\left\|(-A)^\alpha G(t,x)\right\|^2 \le K_1^2\|x\|^2$. Thus by [2], we can introduce the following successive approximating procedure: for each integer n = 1, 2, 3, ……,

$x^n(t) = T(t)[\xi(0) + G(0,\xi)] - G(t,X_t^n) - \int_0^t AT(t-s)G(s,X_s^n ds) +$ 0tTt−sFs,Xsn−1ds $+ \int_0^t T(t-s)\sigma(s)d\,B^H(s)$ (3.1)

and for n = 0,

$$x^0(t) = S(t)\xi(0), t \in [0, T].$$

While for n = 1,2,……..

$$x^n(t) = \xi(t), t \in [-\tau, 0].$$

Lemma 3.2: Let the hypothesis (H1)- (H4) hold. Then there is a positive constant C_1, which is independent of $n \ge 1$, such that for any $t \in [0,T]$,

$\mathbb{E} \sup_{0\le t\le T}\|x^n(t)\|^2 \le C_1$. (3.2)

Proof:For $0 \le t \le T$, it follows easily from (3.1) that

$$\mathbb{E} \sup_{0 \le t \le T}\|x^n(t) + G(t,X_t^n)\|^2$$
$$\le 4\mathbb{E} \sup_{0 \le t \le T}\|T(t)(\xi(0) +$$ G(0,ξ2
$$+ \quad 4\mathbb{E} \sup_{0 \le t \le T} \left\|\int_0^t AS(t-\right.$$ sG(s,Xsn)ds2
$$+4\mathbb{E} \sup_{0 \le t \le T} \left\|\int_0^t T(t-\right.$$ sFs, xn−1 ds2
$$+4\mathbb{E} \sup_{0 \le t \le T} \left\|\int_0^t T(t-\right.$$ sσsdBHs2
$$= 4(I_1 + I_2 + I_3 + I_4).$$

(3.3)

Note from [16] that $(-A)^{-\alpha}$ $for\ 0 < \alpha \le 1$ is a bounded operator. Employing the assumption (H3), it follows that

$$I_1 \le 2\left[\mathbb{E} \sup_{0 \le t \le T}\|S(t)\,\xi(t)\|^2 + \mathbb{E} \sup_{0 \le t \le T}\|T(t)(-A)^{-\alpha}(-A)^\alpha G(0,\xi)\|^2\right]$$
$$\le 2\left(1 + K_1^2\left\|(-A)^{-\alpha}\right\|^2\right).\|\xi\|_C^2 \quad (3.4)$$

Applying the Holder's inequality and taking into account Lemma 2.3 as well as (H3), and the fact that $1/2 < \beta \le 1$, we obtain

$$I_2 = \mathbb{E} \sup_{0 \le t \le T}\left\|\int_0^t (-A)^{1-\alpha}T(t-s)(-A)^\alpha G(s,X_s^n)ds\right\|^2$$
$$\le TK_1^2\left\|(-A)^{-\alpha}\right\|^2 \mathbb{E} \int_0^t \mathbb{E}\|x^n\|^2 ds.$$

(3.5)

On the other hand, in view of (H2), we obtain that

$$I_3 \le T\mathbb{E}_{0 \le t \le T}^{\sup}\int_0^t \|T(t-s)F(s,X_s^{n-1}) - F(s,0) + F(s,0)\|^2 ds$$
$$\le 2\,T\left[\mathcal{M}T + \int_0^t \mathbb{E}\kappa(\|X_s^{n-1}\|_C^2)\,ds\right] \quad (3.6)$$

Next by Lemma 2.2, we have

$$I_4 \le 2\mathcal{M}^2 H\,T^{2H-1}\int_0^T \|\sigma(s)\|_{L_2^0}^2\,ds < \infty \quad (3.7)$$

Since $\kappa(u)$ is concave on $u \ge 0$, there is a pair of positive constants a,b such that

$$\kappa(u) \le a + bu.$$

Putting (3.4) to (3.8) into (3.3) yields that, for some positive constants C_2 and C_3,

$$\mathbb{E}_{0 \le t \le T}^{\sup}\|x^n(t) + G(t,X_t^n)\|^2 \le C_2 + C_3\left[\int_0^t \mathbb{E}\|X_s^{n-1}\|^2 ds + \int_0^t \mathbb{E}\|X_s^n\|^2 ds\right. \quad (3.9)$$

While for $\left\|(-A)^{-\alpha}\right\|^2 < K_1$ By Lemma 2.4,

$$\mathbb{E} \sup_{0 \le t \le T}\|x^n(t)\|^2 \le \frac{1}{1 - K_1\|(-A)^{-\alpha}\|}\,\mathbb{E} \sup_{0 \le t \le T}\|x^n(t) + G(t,X_t^n)\|^2 + \frac{1}{K_1\left\|(-A)^{-\alpha}\right\|}\,\mathbb{E} \sup_{0 \le t \le T}\|G(t,X_t^n)\|^2$$
$$\le \frac{1}{1 - K_1\|(-A)^{-\alpha}\|}\,\mathbb{E} \sup_{0 \le t \le T}\|x^n(t) + G(t,X_t^n)\|^2 + K_1\left\|(-A)^{-\beta}\right\|\mathbb{E}\,\|\xi\|_C^2 + K_1\left\|(-A)^{-\alpha}\right\|\,\mathbb{E} \sup_{0 \le t \le T}\|x^n(t)\|^2$$

which further implies that

$$\mathbb{E}\sup_{0\le t\le T}\|x^n(t)\|^2 \le \frac{1}{(1-K_1\|(-A)^{-\alpha}\|)^2}\mathbb{E}\sup_{0\le t\le T}\|x^n(t)+G(t,X_t^n)\|^2$$

$$+\frac{K_1\|(-A)^{-\alpha}\|}{1-K_1\|(-A)^{-\alpha}\|}\mathbb{E}\|\xi\|_C^2.$$

Thus, by (3.9) we have

$$\mathbb{E}\sup_{0\le t\le T}\|x^n(t)\|^2 \le \left[\frac{K_1\|(-A)^{-\alpha}\|}{1-\|(-A)^{-\alpha}\|}+\frac{2C_{3r}}{(1-K_1\|(-A)^{-\alpha}\|)^2}\right]\mathbb{E}\|\xi\|_C^2$$

$$+\frac{C_3}{(1-K_1\|(-A)^{-\alpha}\|)^2}\left[\,2\int_0^T\mathbb{E}\sup_{0\le r\le s}\|x^{n-1}(r)\|^2ds+\right.$$

+0TEsup 0≤r≤sxnr2ds+C21−K1−A−α2

Observing that

$$\max_{1\le n\le \kappa}\mathbb{E}\sup_{0\le t\le T}\|x^{n-1}(t)\|^2 \le \mathbb{E}\|\xi\|_C^2+\max_{1\le n\le\kappa}\mathbb{E}\sup_{0\le t\le T}\|x^n(t)\|^2$$

We then derive that, for some positive constants C_4 and C_5

$$\max_{1\le n\le\kappa}\mathbb{E}\sup_{0\le t\le T}\|x^n(t)\|^2 \le C_4+C_5\mathbb{E}\int_0^T\max_{1\le n\le\kappa}\mathbb{E}\sup_{0\le r\le s}\|x^n(s)\|^2ds$$

Now, the application of the well-known Gronwall's inequality yields that

$$\max_{1\le n\le\kappa}\mathbb{E}\sup_{0\le t\le T}\|x^n(t)\|^2 \le C4+e^{C5T}.$$

The required assertion (3.2) is obtained since k is arbitrary.

Lemma 3.3: Let the condition (H1)-(H4) be satisfied. For $\alpha\in\left(\frac{1}{2},1\right]$, further assume that

$$\frac{2K_1^2\mathcal{M}_{1-\alpha}^2\gamma^{-2\alpha}\Gamma^{2\alpha-1}}{1-K_1\|(-A)^{-\alpha}\|}+K_1\left\|(-A)^{-\alpha}\right\|<1, \quad (3.10)$$

where $\Gamma(\cdot)$ is the Gamma function and $M_{1-\alpha}$ is a constant in Lemma 2.3. Then there exists a positive constant $\bar{C}$ such that, for all $0\le t\le T$ and $n,m\ge 0$

$$\mathbb{E}\sup_{0\le t\le T}\|x^{n+m}(s)-x^n(s)\|^2 \le \bar{C}\int_0^t\kappa\left(\mathbb{E}\sup_{0\le u\le s}\|x^{n+m-1}(u)-x^{n-1}(u)\|^2\right)ds.$$

(3.11)

Proof: From (3.1), It is easy to see that for any $0\le t\le T$

$$\mathbb{E}\sup_{0\le t\le T}\|x^{n+m}(s)-x^n(s)+G(s,x^{n+m}(s))-G(s,x^n(s))\|^2$$

$$\le 2\,\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t AT(s-r)[G(r,X_r^{n+m})-G(r,X_r^{n})]dr\right\|^2+$$

$$+2\,\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t T(s-r)\left[F\left(r,X_r^{n+m-1}\right)-\right.\right.$$

Fr,Xrn−1 dr2

Following from the proof of Lemma 3.2, there exists a positive C_6 satisfying

$$2\,\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t T(s-r)\left[F\left(r,X_r^{n+m-1}\right)-F\left(r,X_r^{n-1}\right)\right]ds\right\|^2 \le C_6\int_0^t\kappa\left(\mathbb{E}\sup_{0\le u\le s}\|x^{n+m-1}(r)-x^{n-1}(r)\|^2\right)ds.$$

The last inequality holds from the Jensen's inequality. Now by the condition (H3), Lemma 2.3 and Holder's inequality,

$$\mathbb{E}\sup_{0\le t\le T}\left\|\int_0^t AT(s-r)[G(r,X_r^{n+m})-G(r,X_r^{n})]dr\right\|^2$$

$$\le \mathbb{E}\sup_{0\le t\le T}\left(\int_0^s\|(-A)^{1-\alpha}T(s-r)[(-A)^{\alpha}G(r,X_r^{n+m})-(-A)^{\alpha}G(r,X_r^{n})]\,dr\|^2\right)$$

$$\le \mathbb{E}\sup_{0\le t\le T}\left(\int_0^s K_1\frac{M_{1-\alpha}^2e^{-\gamma(s-u)}}{(s-u)^{1-\alpha}}\|X_r^{n+m}-X_r^{n}\|du\right)^2$$

$$\le \mathbb{E}\sup_{0\le t\le T}\int_0^s K_1^2\frac{M_{1-\alpha}^2e^{-\gamma(s-u)}}{(s-u)^{1-\alpha}}du\int_0^s e^{-\gamma(s-u)}\|X_r^{n+m}-X_r^{n}\|^2du$$

$$\le K_1^2M_{1-\alpha}^2\gamma^{-2\alpha}\Gamma^{2\alpha-1}\,\mathbb{E}\sup_{0\le t\le T}\int_0^s e^{-\gamma(s-u)}\|X_r^{n+m}-X_r^{n}\|^2du$$

$$\le K_1^2M_{1-\alpha}^2\gamma^{-2\alpha}\Gamma^{2\alpha-1}\,\mathbb{E}\sup_{0\le t\le T}\|x^{n+m}(s)-x^n(s)\|^2.$$

On the other hand, Lemma 2.4 and (H3) give that

$$\mathbb{E}\sup_{0\le t\le T}\|x^{n+m}(s)-x^n(s)\|^2 \le \frac{1}{1-K_1\|(-A)^{-\alpha}\|}\mathbb{E}\sup_{0\le t\le T}\left\|x^{n+m}(s)-x^n(s)+G\left(s,x^{n+m}(s)-G(s,x^n(s))\right)\right\|^2$$

$+K_1 \left\| (-A)^{-\alpha} \right\| \mathbb{E} \sup_{0 \le t \le T} \|x^{n+m}(s) -$
xn s2. (3.12)

So the desired assertion (3.11) follows from (3.12) We can now state the main result of this paper.

Theorem3.1. Under the conditions of Lemma 3.3, then Eq. (1.1) admits a unique mild solution.

Proof:

Uniqueness: Let x and y be two mild solutions to equation (1.1).In the same way as Lemma (3.3) was done, we can show that for some $\overline{K} > 0$

$$\mathbb{E} \sup_{0 \le t \le T} \|x(s) - y(s)\|^2 \le \overline{K} \int_0^t \kappa\left(\mathbb{E} \sup_{0 \le t \le T} \|x(r) - y(r)\|\right) ds$$

This together with Lemma (3.1) leads to

$$\mathbb{E} \sup_{0 \le t \le T} \|x(s) - y(s)\|^2 = 0.$$

Consequently x=y which implies the uniqueness. The proof of theorem is complete.

Existence: By Lemma (3.3) there exists a positive $\overline{C}$ such that for $t \in T$ and $n, m \ge 1$,

$$\mathbb{E} \sup_{0 \le s \le t} \|x^{n+1}(s) - x^{m+1}(s)\|^2 \le \overline{C} \int_0^t \kappa\left(\mathbb{E} \sup_{0 \le r \le s} \|x^n(s) - x^m(s)\|^2\right) ds.$$

Integrating both sides and applying Jensen's inequality gives that

$$\int_0^t \mathbb{E} \sup_{0 \le l \le s} \|x^{n+1}(l) - x^{m+1}(l)\|^2 ds \le \overline{C} \int_0^t \int_0^s \kappa\left(\mathbb{E} \sup_{0 \le u \le s} \|x^n(r) - x^m(r)\|^2\right) dl ds.$$

$= \overline{C} \int_0^t s \int_0^s \kappa\left(\mathbb{E} \sup_{0 \le u \le s} \|x^n(r) -\right.$
xmr21s dlds.

$$\le \overline{C}\, t \int_0^t \kappa\left(\int_0^s \mathbb{E} \sup_{0 \le u \le s} \|x^n(r) - x^m(r)\|^2 \frac{1}{s}\, d\,l\right) ds.$$

Then

$$h_{n+1,m+1}(t) \le \overline{C} \int_0^t \kappa\left(h_{n,m}(s)\right) ds,$$

where

$$h_{n,m}(t) = \frac{\int_0^t \mathbb{E} \sup_{0 \le l \le s} \|x^{n+1}(l) - x^{m+1}(l)\|^2 ds}{t}$$

While by Lemma 3.2, it is easy to see that $\sup_{n,m} h_{n,m}(t) < \infty$, so letting h(t): $= \lim sup_{n,m\to\infty} h_{n,m}(t)$ and taking into account the Fatou's lemma,we yield that

$$h(t) \le \overline{C} \int_0^t \kappa\big(h(s)\big).$$

Now, applying the Lemma 3.1 immediately reveals h(t) = 0 for any $t \in [0,T]$.This further means $\{x^n(t), n \in \mathbb{N}\}$ is a Cauchy sequence in L^2.So there is a $x \in L^2$ such that

$$\lim_{n\to\infty} \int_0^t \mathbb{E} \sup_{0 \le l \le s} \|x^n(s) - x(s)\|^2 ds = 0.$$

In addition, By Lemma 3.2, it is easy to follow that $\mathbb{E}\|x(t)\|^2 \le C_1$. In what follows, we claim that x (t) is a mild solution to (1.1). On one hand, by (H3),

$$\mathbb{E}\|G(t, X_t^n) - G(t, X_t)\|^2 = \mathbb{E}\left\|(-A)^{-\alpha}\left[(-A)^{\alpha} G(t, X_t^n) - (-A)^{\alpha} G(t, X_t)\right]\right\|^2$$

$$\le \|(-A)^{-\alpha}\|^2 K_1^2\, \mathbb{E} \sup_{0 \le l \le s} \|x^n(s) - x(s)\|^2 \to 0,$$

whenever $n \to \infty$. On the other hand, by (H3) and Lemma 2.3, compute for $t \in [0,T]$

$\mathbb{E}\left\|\int_0^t AT(t-s)G(s, X_s^n) - G(s, X_s)\, ds\right\|^2$
$= T\mathbb{E} \int_0^t \left\|(-A)^{1-\alpha} T(t -\right.$
s-A−α Gs,Xsn−-A−α Gs,Xs2ds
$\le TK_1^2 \int_0^T \mathbb{E} \sup_{0 \le u \le s} \|x^n(r) - x(r)\|^2 ds$

$\to 0$, as $\to \infty$.

While, applying (H2) the Holder's inequality and $[9, \mathit{Theorem}\ 1.2.6]$ and letting $n \to \infty$, we can also claim that for $t \in [0,T]$

$$\mathbb{E}\left\|\int_0^t T(t-s)[F(s, X_s^{n-1}) - F(s, X(s))]ds\right\|^2 \to 0$$

Hence, taking limits on both sides of (3.1),

$$X(t) = T(t)[\xi(0) + G(0,\xi)] - G(t, X_t \int_0^t AT(t-s)G(s, X_s ds + \int_0^t T(t-s)F(s, X_s)\, ds + \int_0^t T(t-s)\sigma(s)\, ds +$$

Remark 3.1: If G = 0, that is $K_1 = 0$, then, obviously, the condition (3.11) must be satisfied. Consequently, our results can be reduced to some results in [1]. In other words, in this special case ,we generalize [1]

Remark3.2: In this work, we consider the existence and uniqueness of mild solutions to SNFDEs driven by a fractional Brownian motion under a non-Lipschitz condition with the Lipschitz condition being regarded as special case and a weakened linear growth assumption. Therefore, some of the results in [2] are improved to cover a class of more general SNFDEs driven by a fractional Brownian motion.

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