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# Camera Calibration: An Overview of Concept, Methods and Equations

Mohammad Hamed Saeifar[1*], Mousa Mohammad Nia[2]
*[1]- Remote Sensing Department, Afagh Research Institute, Tehran, I.R. Iran*
*[2]- Computer Department, Afagh Research Institute, Tehran, I.R. Iran*
*Corresponding Author: Mohammad HamedSaeifar*

**ABSTRACT**
Camera calibration is favored as an important issue in photogrammetry and computer vision literatures. The importance of this issue can be due to two reasons: firstly, every recently camera should be calibrated before being used to correct its lens distortion and interior orientation elements. In addition, it is a main preprocessing step at many vision applications. This paper aims to provide an overview of concept, objective, methods and mathematical equations for camera calibration.

***Keywords***:Camera Calibration, Photogrammetry, Computer Vision, Self–Calibration, Lens Distortion, Interior Orientation Elements, Exterior Orientation Elements.

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## I. INTRODUCTION

Many years, photogrammetry researchers and recently, computer vision researchers have taken camera calibration issue into consideration [1–17]. Every recently camera should be calibrated to correct its lens distortion and interior orientation elements. In addition, it is a main preprocessing step towards many important applications in photogrammetry and vision such as camera localization, 3D reconstruction, object localization, visual inspection and robot navigation [3, 5].The objective of the geometric calibration of a digital camera system is to reconstruct the precise geometry of the bundle of rays that entered the camera at the instant of exposure from the 2D measurement of points on the resulting imagery [18, 19]. We aim to provide an overview of concept, methods and equations for camera calibration.The reminder of the paper is organized as follows: In section 2, a concept of camera calibration is presented. Then, section 3 provides the objective of camera calibration. The camera calibration methods are introduced in section 4 in terms of photogrammetry and computer visioncategorizations. Finally, the mathematical equations for single-image calibration and multi-image self-calibrationare provided in section 5.

## II. CAMERA CALIBRATION CONCEPT

In photogrammetry literatures, the camera calibration can be defined as the process of determining geometric characteristics (called as "interior orientation elements") of a camera system [20]. Of course, Li (1999) provided a comprehensive definition for camera calibration: "it can be considered as a process of and compensating for systematic errors of the camera system and imaging process, through which the system's properties can be determined and the metric performance of the camera system be enhanced [19].Two important terms should be defined to better perceive of camera calibration concept: Interior orientation elements (IOE) and exterior orientation elements (EOE).IOE also called as intrinsic or internal parameters includes of principal point location at fidual marks coordinate system (xo, yo), principal distance (c) or focal length (f), skew factor (s) and aspect ratio ($\gamma$). In metric cameras, s=0 and $\gamma$=1 and there are only three unknown parameters i.e. (xo, yo, f) [21–23].Establishing relationship between image space (or image coordinate system) and object space (or object coordinate system) is called as "exterior orientation". It is accomplished by locating the camera center in the object coordinate system. For this purpose, three translation parameters (XO, YO, ZO) i.e. the position of camera center or image center at object coordinate system and three rotation parameters ($\omega$, $\varphi$, $\kappa$) i.e. the rotation of camera related to object coordinate system axes (X, Y, Z) are needed. These parameters are called as EOE [21].

## III. OBJECTIVE OF CAMERA CALIBRATION

As previously said, the objective of camera calibration is to compensate systematic errors affected on the geometric position of the image points [19]. Generally, this process is included of both elements related to lens system and elements related to camera focal plane which are as follows [19, 24]:

1–Principal point location (at fidual marks coordinate system);
2– Camera focal length/principal distance/constant (calibrated);
3– Lens distortion (both radial and decentering distortion);
4– Geometry of sensor plane (planarity, pixel size, orthogonality of axes);
5– Variation of the interior geometry of the camera system with a focusable lens;
6– Stability of the above elements.

The most of projects in the regards of camera geometric calibration have been performed considering chiefly the lens part of the camera and a few considered the sensor aspect [19].

## IV. CAMERA CALIBRATION METHODS

As said at Introduction, camera calibration has been favored by both of photogrammetry and computer vision researchers. In this section, we aim to provide the categorizations of camera calibration methods in two these fields.

### 4.1.Categorization in the Photogrammetry

A categorization of camera calibration methods has been provided by Faig (1998) in terms of photogrammetry as follows [19]:

#### 4.1.1. Pre/Post–Calibration

This group contains conventional laboratory, test-field and star calibration methods. For the calibration. For the calibrated system, calibration parameters remain constant or change according to a determined pattern during the subsequent evaluation. In this group, the calibration and the evaluation are processed separately [19].

#### 4.1.2. On–the–Job Calibration

In this group, the calibration and the evaluation are either combined into one process or carried out sequentially in which calibration parameters are treated as unknowns. In order to solve them, additional object-space control is needed [2, 25].

#### 4.1.3. Self–Calibration

This group may be sometimes confused with the group. However, it does not need additional object-space control to solve the calibration parameters [2]. This group makes use of the geometric strength of overlapping images to determine these parameters. Therefore, unlike the on-the-job method which can be implemented either in the case of single-image or of multi-image, self-calibration is only valid when there are overlapping images and the configuration of the image acquisition has a direct influence upon the final results [19].

Brown (1989) stated that there are several requirements to meet a successful self-calibration [2]:
1– Availability of at least three images of the object taken by a same camera;
2– Remaining stability of both the interior geometry of the camera and the point to be measured on the object during the measurement process;
3– Availability of the high strength of photogrammetric network and a high degree of convergence;
4– Availability of at least one image having a roll angle that is significantly different from the others;
5– Employing a relatively large number of well distributed points.

Having abovementioned cases, Brown claims “a satisfactory calibration of the camera can be performed as an integral part of the triangulation without any control points”. However, there is a problem with the aerial application of the self-calibrating bundle adjustment and it is to obtain images which have an adequate diversity of camera angles [2].

### 4.2. Categorization in the Computer Vision

In 3D computer vision, camera calibration is a necessary step for extracting metric information from 2D images [8, 23]. From its point of view, a categorization for camera calibration methods is presented based on the dimension of the calibration objects. It is as follows [26]:

#### 4.2.1. 1D Object–Based Calibration

Calibration objects of this group are composed of a set of collinear points. In this case, a camera is able to be calibrated by observing a moving line around a fixed point, such as a string of balls hanging from the ceiling [8].

#### 4.2.2. 2D Object–Based Calibration

In this group, a planar pattern shown at a few different orientations is observed. In this case, the knowledge of the plane motion is not necessary. Since almost anyone can make such a calibration pattern by him/her-self, therefore the setup is easier for camera calibration.

**4.2.3. 3D Object–Based Calibration**

In this group, camera calibration is performed by observing a calibration object whose geometry in 3-D space is known with a very good precision. The calibration object usually consists of two or three planes orthogonal to each other. Sometimes, a plane undergoing a precisely known translation is also used, which equivalently provides 3D reference points. This group requires an elaborate setup and an expensive calibration device.

Based on the abovementioned categorization, self-calibration can be considered as 0D calibration method because it employs no calibration object and only the corresponding image points are needed for calibration. In this case, if images are taken by the same camera with fixed intrinsic parameters, correspondences between three images are adequate to retrieve both of the IOE and EOE which can reconstruct primary 3D structure. Although no calibration objects are necessary, many unknown parameters should be estimated, eventuating in a more complex mathematical problem [26].

**4.3. Other Categorizations**

Li (1999) presented another categorization for digital camera calibration methods based on the recovery situation of the calibration parameters [19]. He considered two main groups of individual and combined methods for camera calibration methods.

In individual methods, some calibration parameters describing lens distortion, principal point offset and sensor unflatness are determined separately from each other, based on either empirical or analytical methods using certain special devices such as a laser beam, collimators or a goniometer. Traditional laboratory methods belong to this group. The advantage of this group is the relative independence of the calibration parameters which makes the results more reliable. By contrast, the high time consumption and the necessity for special devices are two main disadvantages of this group. Moreover, the camera systems must be detached for the calibration and calibration is often separated from data evaluation. So, for some digital camera systems, the variation of their internal geometry may sometimes make the calibration results meaningless if the variation is unpredictable [19].

In contrast to first group, the combined methods are being widely used to determine the principal point offset, principal distance, lens distortion parameters and part of the sensor information (orthogonality and affinity) at the same time based on the relationship between a well-controlled test-field composed of an array of precisely coordinated targets and its distorted image. The general outline of this group is shown in figure 1. Calibration parameters can be determined based on single or multiple frame resectionifadequatelynumber of object-space control points. They can be simultaneously determined together with the EOE and object coordinates from the photogrammetric bundle adjustment. On-the-job and self-calibration can be considered as special cases of this group differing in the control necessity. Basically, this group is based on the mathematical modeling of the systematic errors of camera systems [19].

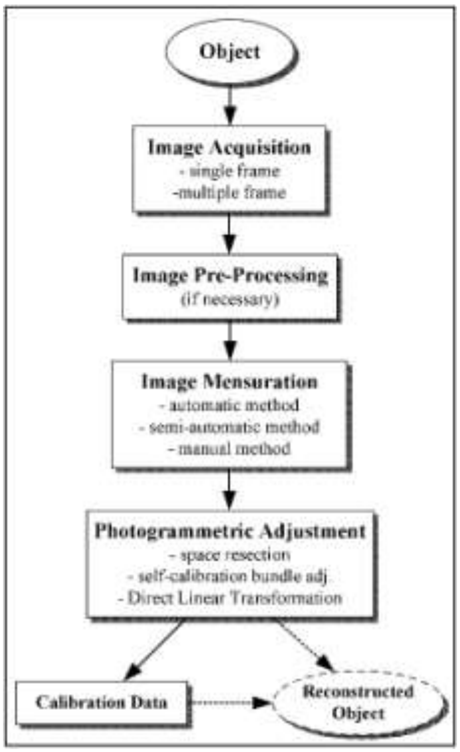



**Figure 1**. The outline of combined calibration methods [19].

In addition to the previous categorizations, another categorization is presented by Liu (1982). From his point of view, the calibration of a camera (either metric or non-metric type) can be divided into two main groups: the physical and the analytical one [22]. The physical methods are often called laboratory calibration. The use of optical instruments such as multi-collimators and goniometers are two most important methods of this group [24]. It is accomplished according to the physical properties of the camera component and it is more applicable for metric camera as compared to non-metric one [22].

In contrast to the first group, the analytical group employs object control points which provides for certain geometric conditions such that the camera parameters are determined within these conditions. This groups is able to be used for the calibration of any camera type. During recent years, many analytical programs have been developed for both metric and non-metric camera calibration. Field Calibration, stellar calibration, test-field calibration, plumb line method and self-calibration are the most important methods of this group [22].

In the following, we aim to briefly describe some of abovementioned camera calibration methods.

**4.3.1. Goniometer Method**

The principle of this method is to place a precise grid, called as a Reseau plate, on the image plane of the camera and to illuminate it from behind so that the images of the grid crosses were projected out into object space [2]. Lenses are usually calibrated at infinity focus using a collimator rotated about the front node of the lens. In order to locate the principal point, the auto-collimation procedure was employed. The principle of autocollimation was used for location of the principal point.

Hallert (1960) explained the goniometer principle as follows [2]: A precision grid having lines in a 10 mm spaced regular array is used. Then, it is illuminated and its etched pattern projects through the lens. Focused to infinity, a telescope is directed towards the camera lens. It is projected on the collimating mark of the telescope and adjusted into coincidence there. By rotating the telescope, the angles are measured (Figure 2). Therefore, by recording them to selected intersection points and having the grid spacing, estimating all of the camera IOEs is soluble.

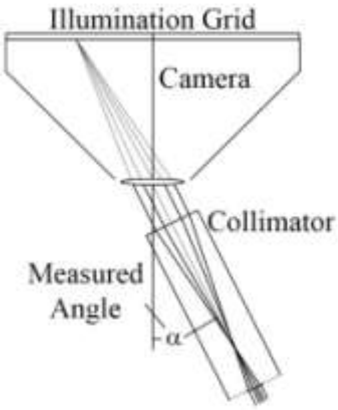



**Figure 2**. The moving collimator goniometer principle [2].

**4.3.2. Multi-Collimator Method**

The principle of this method is almost similar with the goniometer, except in a reverse sense. Collimators can be considered as telescopes with illuminated cross-hairs, focused at infinity and pointing at the lens of the camera from various directions. The bank of collimators shone their illuminated crosses through the lens and onto the image plane of the camera where they are recorded on film or a glass plate [24].

The locations of the crosses on the exposed plate are observed. Then, having the object space coordinates of all the collimators by precise surveying, the lens distortions is able to be calculated in a manner similar to the goniometer procedure [2]. The procedure of this method is shown in Figure 3 where each collimator produces an image at infinity of an illuminated cross-hair on the image plane.

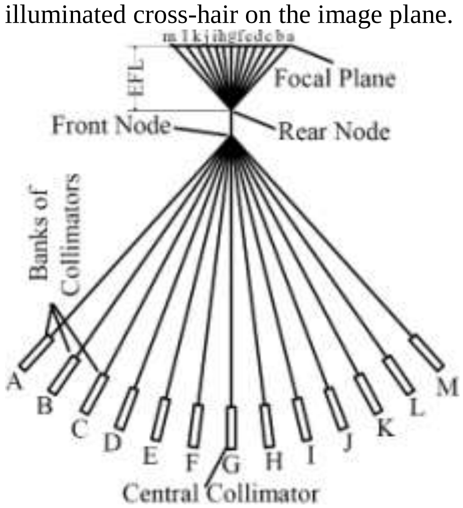



**Figure 3**. Multi-collimator calibration procedure [2].

**4.3.3. Field Calibration Method**

In this method, determined by the accurate surveying methods, several ground marks are taken by a camera [24]. Then, by comparing object coordinates and image coordinates of these marks, the calibration parameters can be computed.

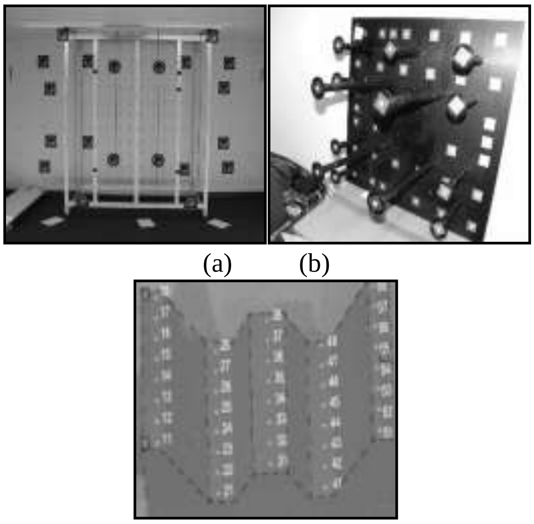
This method has several advantages: in the accuracy of these marks, which have been surveyed previously; the fact that the camera can be used in conditions similar to which it will operate; and calibration can take place at a similar time to use. Its disadvantage is the presence (for single camera calibration) or lack (for multi-camera calibration) of 3-D detail [2].

**4.3.4. Stellar Calibration Method**

In stellar method, several stars are taken by a camera and the angles related to the stars are measured and they are compared with their true values, then. In this method, the angular position of stars should be known to a high degree of accuracy and repeatability [24]. Shmid (1974) described the calibration of the Orbigon lens. In his study, the standard error in position of the stars is less than 0.4 seconds. Over 2420 star images are visible on each plate. Although, the necessity to identify each star and apply corrections for atmospheric refraction and diurnal aberration is a drawback of the method, the large number of observations causes to use the least squares estimation process. In addition, calibrated focal length, principal point and principal point of symmetry, radial and tangential distortion, and orientation of tangential distortion have been used in his study [2].

**4.3.5. Test-Field Calibration Method**

Also called as reference body calibration, this method is the use of images of a (3D) object having known and accurate geometric positions. The principle is as follows: Several images are taken from test field or reference body which 3D (or object) coordinates of target points are known. Then, image coordinates of all target points are measured. Finally, the distortion parameters, IOE and EOE are computed using a least squares estimation process. Several requirements should be considered for this method as follows: Test field should have a very accurate and stable position during calibration process. Grid network should also be stable. In addition, there should be the approximate values of the camera IOE and EOE [27, 28]. Test field can has a variety dimension (millimeter to meter). Figures 4a to 4c shows some different test fields for calibration.



(a) (b)

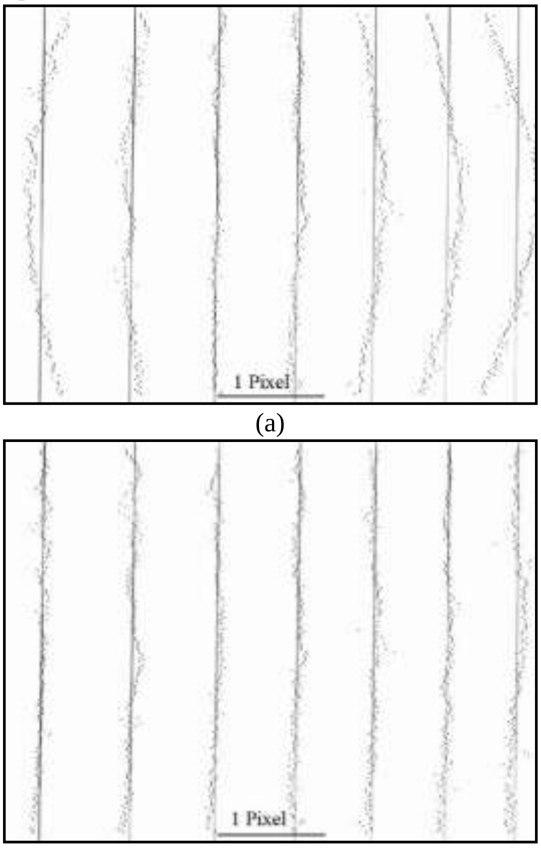
(c)

**Figure 4**. Different test fields for calibration with the dimensions of: (a) 1.5 × 2.5 × 4.0 meters prepared by surveying. (b) 30 × 50 × 50 centimeters prepared by high resolution photogrammetry. (c) 14 × 10 × 10 millimeters prepared by laser [28].

**4.3.6. Plumb-Line Method**

The plumb-line method was firstly used by Brown (1971) for calibration of lens distortions. This method is based on the presupposition that a straight line in object space will project as a straight line in image space, if lack of distortion [2] or in other words, the 2D image of a 3D line remains straight if the camera is a pinhole type (no lens distortion) [29]. However in reality, because of existing distortion (radial and decentering), the 2D image of a 3D line does not remain straight [30].

This method can be formulized based on the fitting of straight lines to digitized sets of observed x and y coordinates on the image plane. Deflecting a line from linearity is ascribed to the radial and decentering lens distortion (Figure 5a). In fact, it can be conceived that each digitized point consists of a "true" position plus the effects of radial and decentering distortion. To compute the effects of decentering distortion, it is necessary to digitize several sets of nearly horizontal and vertical lines [30].




(a)

(b)

**Figure 5.** (a) Before correction, best fit of plumb line data to a straight line. (b) After correction, best fit of corrected data to a straight line [2].

The parameters for radial and decentering distortion can be readily extracted by the analytical plumb-line technique. The principal distance and offsets of the principal point cannot be determined from this method [2]. To illustrate the efficiency of plumb-line method, consider that 50 points have been digitized for 10 horizontal lines and for 10 vertical lines, then 1000 items of data would be available to describe those 20 lines plus the parameters xp and yp (principal points position), k1, k2 and k3 (radial distortion parameters), p1, p2 and p3 (decentering distortion parameters).By employing the corrections to the image shown in Figure 5a, the image of Figure 5b is derived.

## V. MATHEMATICAL EQUATIONS FOR CAMERA CALIBRATION

### 5.1. Single–Image Calibration

Mathematical equations for (metric or non-metric) camera calibration using single-image method is space intersection based on collinearity condition. These equations are as follows [19]:

$$x - x_o + \Delta x = c\frac{X^*}{Z^*}$$
$$y - y_o + \Delta y = c\frac{Y^*}{Z^*} \quad (1)$$

where, x and y are image coordinates, xo, yo and c are the main IOE, Δx and Δy are additional parameters which model systematic effects and can be the functions of the main IOE, radial distortion (k1, k2 and k3) and decentering distortion (p1, p2 and p3) as follows [19]:

$$\Delta x = f_x(c, x_0, y_0, k_1, k_2, k_3, p_1, p_2, p_3)$$
$$\Delta y = f_y(c, x_0, y_0, k_1, k_2, k_3, p_1, p_2, p_3) \quad (2)$$

In addition,

$$X^* = m_{11}(X - X_O) + m_{12}(Y - Y_O) + m_{13}(Z - Z_O)$$
$$Y^* = m_{21}(X - X_O) + m_{22}(Y - Y_O) + m_{23}(Z - Z_O) \quad (3)$$
$$Z^* = m_{31}(X - X_O) + m_{32}(Y - Y_O) + m_{33}(Z - Z_O)$$

where, X, Y and Z are object coordinates of an optional point, XO, YO and ZO are object coordinates of image center and mij are elements of rotation matrix between object space and image space [19].

Having known control points, all the unknowns (IOEs, EOEs and additional parameters) can be simultaneously determined. It is possible using a least squares estimation process and helping additional control points. In this case, the observation equations are formed and then, the problem is solved using normal equations as follows:

$$V = B\delta + L \quad (4)$$
$$N\delta + W = 0 \quad (5)$$
$$\delta = -N^{-1}W \quad (6)$$

Eq. 4 is the basic observation equation where V is the residuals vector, B is the design matrix, L is the observations vector and δ is the unknowns vector containing IOEs, EOEs and all additional parameters. Eq. 5 and eq. 6 are the normal equations and normal solution corresponding with eq. 4.

Of course based on the function forms of Δx and Δy, systematic errors modeling methods can be divided into physical and algebric models or the combination of both so called hybrid models [19]. The first models are based on the known physical properties of the camera system such as radial and decentering lens distortions, scale change and non-orthogonality of image axes. In fact, the principle of this group is to model the cause of image deformations. So, it is also called as "causes models". The most widely used physical models has been presented by Fraser (1997) [19]. Table 1 presents some physical models for modeling systematic errors in digital camera system.

In contrast to the physical models, the algebraic models are constituted based on only geometric considerations, usually with orthogonal or near orthogonal components with their principle strength being low correlation among the parameters and being capable of compensating for unpredicted or unspecified effects. In fact, this group models the effects of image deformation resulting to call it "effect models", as well. The most general algebric models has been presented by Brown (1976) [19].

Two main drawbacks of the physical models are: 1) some systematic errors are not physically known and so they cannot be completely modeled by the abovementioned functions. 2) The over--parameterization and correlations among the additional parameters themselves and among them and other orientations parameters will sometimes decline the accuracy of the final photogrammetric results. By contrast, although the parameters of the algebric models are not physically interpretable, this group is more popular and usable than the physical group [19].

### 5.2. Multi–Image Self–Calibration

Also called as simultaneous multi-frame analytical calibration (SMAC), this method can be thought of as a general method for both of camera calibration task (determining IOEs, EOEs and additional parameters) and determining object coordinates of points [24]. Thus unlike the single-image calibration method, in multi-image self-calibration method, the unknown coordinates of object points are inserted into the linearized collinearity condition equations. Thus:

$$F_x = v_x + \frac{\partial F_x}{\partial X_1}\Delta X_1 + \frac{\partial F_x}{\partial X_2}\Delta X_2 + \frac{\partial F_x}{\partial X_3}\Delta X_3 + F_x^0 = 0$$
$$F_y = v_y + \frac{\partial F_y}{\partial X_1}\Delta X_1 + \frac{\partial F_y}{\partial X_2}\Delta X_2 + \frac{\partial F_y}{\partial X_3}\Delta X_3 + F_y^0 = 0 \quad (7)$$

where:

$X_1$:the unknown IOE vector and additional parameters ($x_o$, $y_o$, $f$, $k_1$, $k_2$, $k_3$, $p_1$, $p_2$, $p_3$);
$X_2$: the unknown EOE vector ($X_O$, $Y_O$, $Z_O$, $\omega$, $\varphi$, $\kappa$);
$X_3$: the unknown object coordinates vector ($X$, $Y$, $Z$);
$v_x$ and $v_y$: the residuals of $x$ and $y$ image coordinates;
$F_x^0$ and $F_y^0$: the functions values for the initial values and
$\partial F/\partial X$ :the partial deferential of each function in relation to the unknown parameters.

The matrix form of Taylor series expansion for eq. 7 is as follows [24]:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{114} & b_{115} \\ b_{21} & b_{22} & b_{23} & \dots & b_{214} & b_{215} \end{bmatrix} \cdot \begin{bmatrix} \Delta x_o \\ \Delta y_o \\ \Delta f \\ . \\ . \\ . \\ \Delta Y_O \\ \Delta Z_O \end{bmatrix} + \begin{bmatrix} b_{116} & b_{117} & b_{118} \\ b_{216} & b_{217} & b_{218} \end{bmatrix} \cdot \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} = \begin{bmatrix} -F_x^0 \\ -F_y^0 \end{bmatrix} \quad (8)$$

or

$$\mathrm{V}_{2mn\times 1} + \dot{\mathrm{B}}_{2mn\times(9+6m)}\dot{\delta}_{(9+6m)\times 1} + \ddot{\mathrm{B}}_{2mn\times 3n}\ddot{\delta}_{3n\times 1} = \varepsilon_{2mn\times 1} \quad (8)$$

where:

$m$ and $n$ are the number of images and the number of object points, respectively;

**V** and $\boldsymbol{\varepsilon}$ are the residuals and differences vectors, respectively;

$\dot{\mathrm{B}}$ and $\ddot{\mathrm{B}}$ are the design matrices (the partial differential in relation to the orientation and coordinates unknowns) and $\dot{\delta}$ and $\ddot{\delta}$ are the correction vectors of orientation and coordinates unknowns which should be repeatedly added to initial/previous unknowns values as follows:

$$\mathrm{X} = \tilde{\mathrm{X}} + \delta \quad (9)$$

where $\tilde{X}$ and $\delta$ are the initial values vector and the corrections vector of all unknown, respectively.

If there are the initial values of orientation and coordinate unknowns, then:

$$\mathrm{X} = \dot{\mathrm{X}} + v_{\mathrm{X}} \quad (10)$$

is a constraint equation where $\dot{X}$ and $v_{\mathrm{X}}$ are the initial values and the residual vectors of all unknowns. By comparing eq. 10 and eq. 11, eq. 12

$$\dot{\mathrm{X}} + v_{\mathrm{X}} = \tilde{\mathrm{X}} + \delta \quad \Rightarrow \quad \begin{cases} \dot{\mathrm{V}}_{(9+6m)\times 1} - \dot{\delta}_{(9+6m)\times 1} = \dot{\varepsilon}_{(9+6m)\times 1} & (1) \\ \ddot{\mathrm{V}}_{3n\times 1} - \ddot{\delta}_{3n\times 1} = \ddot{\varepsilon}_{3n\times 1} & (2) \end{cases} \quad (11)$$

is obtained [24] where (1) and (2) are the constraint equations for orientation and coordinates unknowns, respectively. Eq. 9 and eq. 12 can be combined as follows in matrix form [24]:

$$\overline{\mathrm{V}} + \overline{\mathrm{B}}\,\overline{\delta} = \overline{\varepsilon} \quad (12)$$

where

$$\overline{\mathrm{V}}_{N\times 1} = (\mathrm{V},\ \dot{\mathrm{V}}, \ddot{\mathrm{V}})^{\mathrm{T}} \quad , \quad \overline{\mathrm{B}}_{N\times l} = \begin{bmatrix} \dot{\mathrm{B}} & \ddot{\mathrm{B}} \\ -\mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{bmatrix}$$

$$\overline{\delta}_{U\times 1} = (\dot{\delta}, \ddot{\delta})^{\mathrm{T}} \quad , \quad \overline{\varepsilon}_{N\times 1} = (\varepsilon, \dot{\varepsilon}, \ddot{\varepsilon})^{\mathrm{T}}$$

where,$N$= 2$mn$ + 9 + 6$m$ + 3$n$ is the number of equations and $U$ = 9 + 6$m$ + 3$n$ is the number of unknowns.

Using least squares estimation process, the general form of the normal equations corresponding with eq. 13 is as follows:

$$(\overline{\mathrm{B}}^{\mathrm{T}}\overline{\mathrm{W}}\,\overline{\mathrm{B}})\overline{\delta} = \overline{\mathrm{B}}^{\mathrm{T}}\overline{\mathrm{W}}\,\overline{\varepsilon} \quad \equiv \quad \overline{\mathrm{N}}\,\delta = \overline{\mathrm{C}} \quad (13)$$

and the normal solution corresponding with eq. 14 is as:

$$\delta = -\overline{\mathrm{N}}^{-1}\overline{\mathrm{C}} \quad (14)$$

Of course, eq. 14 can be reformed as follows:

$$\begin{bmatrix} \dot{\mathrm{N}} + \dot{\mathrm{W}} & \hat{\mathrm{N}} \\ \hat{\mathrm{N}} & \ddot{\mathrm{N}} + \ddot{\mathrm{W}} \end{bmatrix} \cdot \begin{bmatrix} \dot{\delta} \\ \ddot{\delta} \end{bmatrix} = \begin{bmatrix} \dot{\mathrm{C}} - \dot{\mathrm{W}}\dot{\varepsilon} \\ \ddot{\mathrm{C}} - \ddot{\mathrm{W}}\ddot{\varepsilon} \end{bmatrix} \quad (15)$$

where [24]:

$$\begin{aligned} &\dot{\mathrm{N}}_{(9+6m)\times(9+6m)} = \dot{\mathrm{B}}^{\mathrm{T}}\mathrm{W}\dot{\mathrm{B}} \quad , \quad \dot{\mathrm{C}}_{(9+6m)\times 1} = \dot{\mathrm{B}}^{\mathrm{T}}\mathrm{W}\,\varepsilon \\ &\ddot{\mathrm{N}}_{3n\times 3n} = \ddot{\mathrm{B}}^{\mathrm{T}}\mathrm{W}\ddot{\mathrm{B}} \quad , \quad \ddot{\mathrm{C}}_{3n\times 1} = \ddot{\mathrm{B}}^{\mathrm{T}}\mathrm{W}\,\varepsilon \\ &\hat{\mathrm{N}}_{(9+6m)\times 3n} = \dot{\mathrm{B}}^{\mathrm{T}}\mathrm{W}\ddot{\mathrm{B}} \end{aligned} \quad (16)$$

where, $\dot{\mathrm{N}}$ and $\ddot{\mathrm{N}}$ are the normal matrices in relation to orientation unknowns and coordinates unknowns, respectively.

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**Table 1**. Some physical models for modeling systematic errors in digital camera [19].

| Project | Model Formats | Physical Meaning of Parameters |
|---|---|---|
| Bayer's Project | $\Delta x = \Delta x_0 - \frac{\bar{x}}{c}\Delta c - s_x\bar{x} + a\bar{y} + \bar{x}(k_1 r^2 + k_2 r^4 + k_3 r^6) + p_1(r^2 + 2\bar{x}^2) + 2p_2\overline{xy}$ <br> $\Delta y = \Delta y_0 - \frac{\bar{y}}{c}\Delta c + a\bar{x} + \bar{y}(k_1 r^2 + k_2 r^4 + k_3 r^6) + 2p_1\overline{xy} + p_2(r^2 + 2\bar{y}^2)$ | $k_1$, $k_2$, $k_3$: coefficients of radial lens distortion; <br> $p_1$, $p_2$: coefficients of decentering lens distortion; <br> $s_x$: scale factor in x direction; <br> $a$: shear factor |
| Burner's Project | $\Delta x = \Delta s_h\bar{x} + \Delta s_v\Delta\varphi\bar{y} + \bar{x}(k_1 r^2 + k_2 r^4 + k_3 r^6) + p_1(r^2 + 2\bar{x}^2) + 2p_2\overline{xy}$ <br> $\Delta y = \Delta s_v\bar{y} + \bar{y}(k_1 r^2 + k_2 r^4 + k_3 r^6) + 2p_1\overline{xy} + p_2(r^2 + 2\bar{y}^2)$ | $\Delta s_h$, $\Delta s_v$: different scale between the horizontal and vertical pixel spacing; <br> $\Delta\phi$: non-perpendicularity of the pixel axes; |
| Edmundson's Project | $\Delta x = \bar{x}(a_1 r^2 + a_2 r^4) + a_3(r^2 + 2\bar{x}^2) + 2a_4\overline{xy}$ <br> $\Delta y = \bar{y}(a_1 r^2 + a_2 r^4) + 2a_3\overline{xy} + a_4(r^2 + 2\bar{y}^2)$ | $a_1$, $a_2$: coefficients of radial lens distortion; <br> $a_3$, $a_4$: coefficients of decentering lens distortion; |

| | | |
|---|---|---|
| Fraser's Project | $\Delta x = -\frac{\bar{x}}{c}\Delta c + \bar{x}(k_1 r^2 + k_2 r^4 + k_3 r^6)$ $+ p_1(r^2 + 2\bar{x}^2) + 2p_2\overline{xy} + b_1\bar{x} + b_2\bar{y}$ $\Delta y = -\frac{\bar{y}}{c}\Delta c + \bar{y}(k_1 r^2 + k_2 r^4 + k_3 r^6)$ $+ 2p_1\overline{xy} + p_2(r^2 + 2\bar{y}^2)$ | $\Delta c$: variation of principle distance; $b_1$: parameter for differential scaling between the horizontal and vertical pixel spacing; $b_2$: parameter for non-orthogonality between the axes; |
| Heipke's Project | $\Delta x = A_1(r^2 - r_0^2)\bar{x} + A_2(r^4 - r_0^4)\bar{x}$ $+ B_1(\bar{y}^2 + 3\bar{x}^2) + 2B_2\overline{xy}$ $\Delta y = A_1(r^2 - r_0^2)\bar{y} + A_2(r^4 - r_0^4)\bar{y}$ $+ 2B_1\overline{xy} + B_2(\bar{x}^2 + 3\bar{y}^2)$ | $A_1$, $A_2$, $r_0$: parameters for radial lens distortion; $B_1$, $B_2$: parameters for decentering lens distortion; |
| Li and Faig's Project | $\Delta x = \bar{x}(k_1 r^2 + k_2 r^4 + k_3 r^6)$ $+ p_1(r^2 + 2\bar{x}^2) + 2p_2\overline{xy} + A\bar{y}$ $\Delta y = \bar{y}(k_1 r^2 + k_2 r^4 + k_3 r^6)$ $+ 2p_1\overline{xy} + p_2(r^2 + 2\bar{y}^2) + B\bar{y}$ | $k_1$, $k_2$, $k_3$: coefficients of radial lens distortion; $p_1$, $p_2$: coefficients of decentering lens distortion; $A$, $B$: coefficients of scale change and non-perpendicularity of coordinate axes; |
| Litchti and Chapman's Project | $\Delta x = \bar{x}(k_1 r^2 + k_2 r^4 + k_3 r^6) + p_1(r^2 + 2\bar{x}^2) + 2p_2\overline{xy}$ $\Delta y = \bar{y}(k_1 r^2 + k_2 r^4 + k_3 r^6) + 2p_1\overline{xy} + p_2(r^2 + 2\bar{y}^2)$ | As before |
| Peterson's Project | $\Delta x = \bar{x}(k_1 r^2 + k_2 r^4) + p_1(r^2 + 2\bar{x}^2) + 2p_2\overline{xy}$ $\Delta y = \bar{y}(k_1 r^2 + k_2 r^4) + 2p_1\overline{xy} + p_2(r^2 + 2\bar{y}^2)$ | As before |
| Wong's Project | $\Delta x = \bar{x}(k + l_1 r^2 + l_2 r^4)$ $+ [p_1(r^2 + 2\bar{x}^2) + 2p_2\overline{xy}](1 + p_3 r^2)$ $\Delta y = \bar{y}(l_1 r^2 + l_2 r^4)$ $+ [2p_1\overline{xy} + p_2(r^2 + 2\bar{y}^2)](1 + p_3 r^2)$ | $k$: scale correct for x-coordinates; $l_1$, $l_2$: coefficients of radial lens distortion; $p_1$, $p_2$, $p_3$: coefficients of decentering lens distortion; |

Where:

$\bar{x}, \bar{y}$ : image coordinates with respect to principle point $(x_o, y_o)$ i.e. $\bar{x} = x - x_o$, $\bar{y} = y - y_o$;

$r = \sqrt{\bar{x}^2 + \bar{y}^2}$ : radial distance from principle point to the image point under consideration;

$\Delta x_0, \Delta y_0$ and $\Delta c$ : variations of IOE