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# Integral Points on The Hyperbola $3x^2 - 4y^2 = 3$

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**ABSTRACT**
This paper concerns with the problem of obtaining non-zero distinct integral points on the hyperbola.Two different sets of solutions satisfying the hyperbola under consideration are presented. Knowing a solution, a general formula for generating a sequence of solutions is presented.

***Keyword:*** Binary quadratic equations, integral points on the hyperbola

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## I. INTRODUCTION

It is well known that binary quadratic Diophantine equation both homogeneous and non homogeneous are rich in variety [1-4]. Particularly in [5-14], the binary quadratic non-homogeneous equations representing hyperbolas respectively are studied for their non-zero integral solutions. However, in [15] it is shown that the hyperbola represented by $3x^2 + xy = 14$ has only finite number of integral points. These results motivated us to search for other choices of hyperbolas having infinitely many non-zero integral solutions. It is towards this end, in this communication, we study the hyperbola given by $3x^2 - 4y^2 = 3$ for its non-trivial integral solutions. The recurrence relations satisfied by the solutions $x$ and y are given. Also a few interesting properties among the solutions are exhibited.

**I.Notations**

$$t_{m,n} = Polygonal\ number\ of\ rank\ n$$
$$with\ sides\ m = n\left[1 + \frac{(n-1)(m-2)}{2}\right].$$
$$p_n^m = Pyramidal\ number\ of\ rank\ n\ with$$
$$sides\ m = \frac{1}{6}n(n+1)[(m-2)n + (5-m)].$$
$$Obl_n = Oblong\ number\ of\ rank\ n = n(n+1).$$
$$PP_n = Pentagonal\ pyramidal\ number\ of$$
$$rank\ n = \frac{n^2(n+1)}{2}.$$

## II. METHOD OF ANALYSIS

To start with, the binary quadratic equation given by

$$3x^2 - 4y^2 = 3 \qquad (1)$$

represents a hyperbola.

Setting, $x = X + 4T, y = X + 3T$ (2)

in (1), it simplifies to the equation

$$X^2 = 12T^2 - 3 \qquad (3)$$

The smallest positive integer solution of $(T_0, X_0)$ of (3) is

$$T_0 = 1\,, X_0 = -3$$

To obtain, the other solutions of (3), consider the Pellian equation

$$X^2 = 12T^2 + 1$$

Whose general solution $(\tilde{T}_n, \tilde{X}_n)$ is given by

$$\tilde{X}_n + \sqrt{12}\tilde{T}_n = (7 + 2\sqrt{12})^{n+1}$$

Since irrational roots occur in pairs, we have

$$\tilde{X}_n - \sqrt{12}\tilde{T}_n = \left(7 - 2\sqrt{12}\right)^{n+1}, \quad n = 0,1,2, \dots$$

From the above two equations, we get

$$\tilde{X}_n = \frac{1}{2}\left[\left(7 + 2\sqrt{12}\right)^{n+1} + (7 - 2\sqrt{12})^{n+1}\right]$$
$$\tilde{T}_n = \frac{1}{2\sqrt{12}}\left[\left(7 + 2\sqrt{12}\right)^{n+1} - (7 - 2\sqrt{12})^{n+1}\right]$$
$$n = 0,1,2 \dots$$

Applying Brahmagupta Lemma between the solutions$(T_0, X_0)$ and $(\tilde{T}_n, \tilde{X}_n)$, the general solution $(T_{n+1}, X_{n+1})$ of (3) is found to be

$$T_{n+1} = \tilde{X}_n - 3\tilde{T}_n$$
$$X_{n+1} = -3\tilde{X}_n + 12\tilde{T}_n,$$
$$n = -1,0,1, \dots$$

Substituting these values in (2), the sequence of integral solutions of (1) can be written as

$$x_{n+1} = \tilde{X}_n$$
$$y_{n+1} = 3\tilde{T}_n, \quad n = -1,0,1, \dots$$

The values of x and y satisfies the recurrence relations

$$x_{n+3} - 14x_{n+2} + x_{n+1} = 0$$
$$y_{n+3} - 14y_{n+2} + y_{n+1} = 0$$

A few interesting properties among the solutions are presented below:

1. The $x$-values are odd and $y$-values are even.
2. $y_{n+1} \equiv 0 (mod\ 6)$, n=0,1,2,…
3. $x_{2n-1} \equiv 0 (mod\ 7)$, n=1,2,…
4. Each of the following expression represents a Nasty number:
   (i) $y_{n+2} - 13y_{n+1} - 12y_n$
   (ii)$x_{n+2} - 13x_{n+1} - 12x_n$
   (iii) $x_{n+3} - 15y_{n+2} - 13x_{n+1}$
   (iv)$y_{n+3} - 11y_{n+2} - 40y_{n+1} - 10y_n$
   (v) $y_{n+2} - 12y_{n+1} - 26y_n$
5. $y_{n+3} - 14y_{n+2} + 2x_{n+1}$ is a cubical integer.
6. $y_{n+3} - 10y_{n+2} - 54y_{n+1} - 8y_n \equiv 0\ (mod\ 6)$
7. $(obl_x)^2 (pp_x)^2 - 25(p_x^2)^2 \equiv 0 (mod\ 3)$
8. $6(\ p_x^5) - 4(t_{3,x}) \equiv 0 (mod\ 2)$
9. $(p_y^3) + 6(t_{3,y+1}) \equiv 0 (mod\ 3)$
10. Choose $r = s,\ s = x - y$ Treat r and s as the generators of the Pythagorean triangle$(\alpha, \beta, \gamma)$ where $\alpha = 2rs, \beta = 2r^2 - s^2, \gamma = r^2 + s^2$ Then this Pythagorean triangle is such that $\beta + 4\alpha - 3\gamma = 3$.
11. If we take the smallest positive integer solution $(T_0, X_0)$ of (3) is $T_0 = 1, X_0 = +3$ The result does not change.

It is worth mentioning that, instead of (2) one may also consider the linear transformations

$$x = X - 4T\ , \qquad y = X - 3T$$

For this case, the corresponding integral solutions of (1) are represented by

$$x_{n+1} = X_{n+1} - 4T_{n+1} = -7\tilde{X}_n + 24\tilde{T}_n$$
$$y_{n+1} = X_{n+1} - 3T_{n+1} = -6\tilde{X}_n + 21\tilde{T}_n\ ,$$
$$n = -1,0,1, \ldots$$

## III. GENERATION OF SOLUTIONS

Let $(x_0, y_0)$be any given solution of (1)

Assume $x_1 = x_0$+h, $y_1 = h - y_0$ (4)

to be the second solution of (1).

Substitution of (4) in (1) leads to

$$h = 6x_0 + 8y_0$$

Employing the value of h in (4), one obtains

$$x_1 = 7x_0 + 8y_0$$
$$y_1 = 6x_0 + 7y_0$$

Representing the above solution in matrix form, we have

$$(x_1, y_1)^t = \mathrm{A}\ (x_0, y_0)^t$$

Where t is the transpose and A is the second order matrix given by

$$\mathrm{A} = \begin{pmatrix} 7 & 8 \\ 6 & 7 \end{pmatrix}$$

Repeating the above process, we get the generalized form of the matrix

$$(x_n, y_n)^t = A^n (x_0, y_0)^t \qquad (5)$$

Wherein $A^n = \begin{pmatrix} \frac{1}{2}(\alpha^n + \beta^n) & \frac{1}{\sqrt{3}}(\alpha^n - \beta^n) \\ \frac{\sqrt{3}}{4}(\alpha^n - \beta^n) & \frac{1}{2}(\alpha^n + \beta^n) \end{pmatrix}$

which $\alpha^n \beta^n = 1$

Thus, substituting n= 1,2,3… inturn in (5), one can generate infinitely many integral solution satisfying (1).

## IV. CONCLUSION

To conclude, one may search for any other binary quadratic equations and their corresponding properties.

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