Influence of MHD on Unsteady Helical Flows of Generalized Oldroyd-B Fluid between Two Infinite Coaxial Circular Cylinders

Sundos Bader , Ahmed M, Abdulhadi
Department of Mathematic, College of Science, University of Baghdad, Baghdad, Iraq
Email:badersundos@gmail.com, ahm6161@yahoo.com

ABSTRACT

Considering a fractional derivative model for unsteady magnetic hydrodynamic (MHD) helical flows of an Oldroyd-B fluid in concentric cylinders and circular cylinder are studied by using finite Hankel and Laplace transforms. The solution of velocity fields and the shear stresses of unsteady magnetic hydrodynamic (MHD) helical flows of an Oldroyd-B fluid in an annular pipe are obtained under series form in terms of Mittag-Leffler function, satisfy all imposed initial and boundary condition, Finally the influence of model parameters on the velocity and shear stress are analyzed by graphical illustrations.

I. Introduction

Recently, a considerable attention has been devoted to the problem of how to predict the behavior of non-Newtonian fluid. The main reason for this is probably that fluids (such as molten plastics, pulps, slurries, emulsions, etc.) which do not obey the assumption of Newtonian fluid that the stress tensor is directing proportional to the deformation tensor, are found in various engineering applications. An important class of non-Newtonian fluids is in the viscoelastic Oldroyd-B fluid which has been intensively studied, and it has been widely applied to flow problems of small relaxation and retardation times.

Rheological constitutive equations with fractional derivatives have been proved to be a valuable tool to handle viscoelastic properties. The fractional derivative models of the viscoelastic fluids are derived from the classical equations which are modified by replacing the time derivative of an integer order by precise non-integer order integrals or derivatives.

In recent years, the Oldroyd-B fluid has been acquired a special status amongst the many fluids of the rate type, as it includes as special case the classical Newtonian fluid and Maxwell fluid. As a result of their wide implications, a lot of papers regarding these fluids have been published in the last time.

The Oldroyd-B fluid model [11], which takes into account elastic and memory effects exhibited by most polymeric and biological liquids, has been used quite widely [4]. Existence, uniqueness and stability results for some shearing motions of such a fluid have been obtained in [13]. The exact solution for the flow of an Oldroyd-B fluid was established by Waters and Kings [14], Rajagopal and Bhatnager [12], Fetecau [3], and Fetecau [2], other analytical results were given by Georgiou [8] for small one-dimensional perturbations and for the limiting case of zero Reynolds number unsteady (unidirectional and rotating) transient flows of an Oldroyd-B fluid in an annular are obtained by Tong [7].

The general case of helical flow of Oldroyd-B fluid due to combine action of rotating cylinders (with constant angular velocities) and a constant axial pressure gradient has been considered by Wood [16]. The velocity fields and the associated tangential stresses corresponding to helical flows of Oldroyd-B fluids using forms of series in term of Bessel functions are given by Fetecau et al [1]. Recently, the velocity field, shear stress and vortex sheet of a generalized second–order fluid with fractional, fractional derivative using to the constitutive relationship models of Maxwell viscoelastic fluid and second order and some unsteady flows of a viscoelastic fluid and of second order fluids between two parallel plates are examined by Mingyo and wenchang [17].

Unidirectional flows of a viscoelastic fluid with the fractional Maxwell model helical flows of a generalized Oldroyd-B fluid with fractional calculus between two infinite coaxial circular cylinders are investigated by Dengke [6].

In this paper, we study the effect of MHD on the helical flows of a generalized Oldroyd-B fluid with fractional calculus between two infinite coaxial circular cylinders. The velocity fields and the resulting shear stresses are determined by means of Laplace and finite Hankel transform and are presented under integral and series forms in the Mittag-Leffler function.
II. Basic governing equations of Helical flow between concentric cylinders

We consider here an unsteady helical flow between two infinite coaxial cylinders located at \( r = R_1 \) and \( r = R_2 \) in the cylindrical coordinates \( (r, \theta, z) \), the helical velocity is given by
\[
\mathbf{V} = \mathbf{v}(r,t)e_\theta + w(r,t)e_z,
\]
where \( e_\theta \) and \( e_z \) are the unit vectors in the \( \theta \) and \( z \) directions, respectively. Since the velocity field is independent of \( \theta \) and \( z \), the constraint of incompressibility is automatically satisfied.

The constitutive equation of generalized Oldroyd-B (O Oldroyd-B) fluid has the form [1]
\[
\mathbf{S} + \frac{\partial \mathbf{S}}{\partial t} = \mu \left( 1 + \frac{\eta \partial^2}{\partial t^2} \right) \mathbf{A},
\]
where \( \mathbf{S} \) is the extra stress tensor, \( \mu \) is the dynamic viscosity, \( \lambda_1 \) and \( \lambda_2 \) are material time constants referred to, the characteristic relaxation and characteristic retardation times, respectively. It is assumed that \( \lambda_1 \geq \lambda_2 \geq 0 \). 

We introduce an unsteady helical flow between concentric cylinders. The fluid is incompressible then.

III. Momentum and continuity equation

We will write the formula of the momentum equation which governing the magnetohydrodynamic as follows:
\[
\rho \frac{D \mathbf{v}}{D t} = -\nabla p + \mathbf{S} + \mathbf{J} \times \mathbf{B}
\]

Where \( \rho \) is the density of the fluid, \( \mathbf{J} \) is the current density and \( \mathbf{B} = \{0, \beta_0, 0\} \) is the total magnetic field.

In the absence of body forces and a pressure gradient, the equation of motion reduce to the relevant equations
\[
\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial \mathbf{S}}{\partial t} = \frac{\partial \mathbf{S}_r}{\partial r} = \frac{\partial \mathbf{S}_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial \mathbf{S}_r}{\partial \theta} - \sigma \beta_0 \mathbf{v}
\]
\[
\frac{\partial \mathbf{v}}{\partial t} = \frac{\partial \mathbf{S}_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial \mathbf{S}_r}{\partial r} - \sigma \beta_0 \mathbf{v}
\]

Multiply above two equations by \((1 + \lambda_2^2 D_r^2)\), we get
\[
\rho (1 + \lambda_1 D_r^2) \frac{\partial \mathbf{v}}{\partial t} = \mu (1 + \lambda_2 D_r^2) \left( \frac{\partial^2 \mathbf{v}}{\partial r^2} + \frac{1}{r} \frac{\partial \mathbf{v}}{\partial \theta} \right) - \sigma \beta_0 \mathbf{v}
\]
\[
\rho (1 + \lambda_1 D_r^2) \frac{\partial \mathbf{v}}{\partial t} = \mu (1 + \lambda_2 D_r^2) \left( \frac{\partial^2 \mathbf{v}}{\partial r^2} + \frac{1}{r} \frac{\partial \mathbf{v}}{\partial \theta} \right) - \sigma \beta_0 \mathbf{v}
\]

Divide Eq(18) by \( \rho \) and divide Eq(19) by \( \rho r \), we get
\[
(1 + \lambda_2^2 D_r^2) \frac{\partial \mathbf{v}}{\partial t} = \nu (1 + \lambda_1^2 D_r^2) \left( \frac{\partial^2 \mathbf{v}}{\partial r^2} + \frac{1}{r} \frac{\partial \mathbf{v}}{\partial \theta} \right) - \sigma \beta_0 \mathbf{v}
\]
\[
(1 + \lambda_2^2 D_r^2) \frac{\partial \mathbf{v}}{\partial t} = \nu (1 + \lambda_1^2 D_r^2) \left( \frac{\partial^2 \mathbf{v}}{\partial r^2} + \frac{1}{r} \frac{\partial \mathbf{v}}{\partial \theta} \right) - \sigma \beta_0 \mathbf{v}
\]

Where \( \nu = \frac{\mu}{\rho} \) is the kinematic viscosity of the fluid. The boundary conditions are expressed by \( w(r_1, t) = u_1 \), \( w(r_2, t) = u_2 \), \( t > 0 \)

And
\[
v(r, t) = 0, \quad v(r_2, t) = \frac{\partial v}{\partial r}(r_2, t) = 0, \quad t > 0
\]

The initial conditions are expressed by \( w(r, 0) = w_0 \), \( \frac{\partial w}{\partial t}(r, 0) = \frac{\partial^2 w}{\partial t^2}(r, 0) = 0 \)

3.1. Calculation of the velocity field

Making the change of unknown function
\[
v(r, t) = \frac{u(r, t)}{r}
\]

Substitute the value of velocity \( v(r, t) \) in Eq. (21) with initial and boundary conditions, we get
\[
(1 + \lambda_2^2 D_r^2) \frac{\partial u}{\partial t} \frac{r}{r^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial u}{\partial r} - \frac{1}{r} \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial r^2} - \frac{1}{r^2} \frac{\partial^2 u}{\partial r^2}
\]
Multiply above equation by \( r \), we get
\[
(1 + \lambda t D^t) \frac{\partial}{\partial t} u = (1 + \lambda t D^t) \left( \frac{\partial^2 u}{\partial r^2} + \frac{u}{r \partial r} - \frac{u}{r^2} \right) - \frac{\sigma \beta^2}{\rho} (1 + \lambda t D^t) u
\] 
(28)

And
\( u(R_1, t) = R_1 \Omega_1, \quad u(R_2, t) = R_2 \Omega_2, \quad t > 0 \) 
(29)
\( u(r, 0) = \partial_u u(r, 0) = 0 \) 
(30)

To obtain the exact analytical solution of the above problems Eq.(20) and Eq.(28), and using initial conditions(24),(25) and (30), we first apply Laplace transform of fractional derivatives, with respect to \( r \), we get
\[
s(1 + \lambda t D^t) \tilde{\omega} = \left(1 + \lambda s D^s \right) \left( \frac{\partial^2 \tilde{\omega}}{\partial r^2} + \frac{\tilde{\omega}}{r \partial r} - \frac{\tilde{\omega}}{r^2} \right) - \frac{\sigma \beta^2}{\rho} \left(1 + \lambda s D^s \right) \tilde{\omega}
\]
(31)
\[
\tilde{\omega}(R_1, s) = \frac{\tilde{\Omega}_1}{s}, \quad \tilde{\omega}(R_2, s) = \frac{\tilde{\Omega}_2}{s}
\]
(32)
\[
s(1 + \lambda t D^t) \tilde{\omega} = \left(1 + \lambda s D^s \right) \left( \frac{\partial^2 \tilde{\omega}}{\partial r^2} + \frac{\tilde{\omega}}{r \partial r} - \frac{\tilde{\omega}}{r^2} \right) - \frac{\sigma \beta^2}{\rho} \left(1 + \lambda s D^s \right) \tilde{\omega}
\]
(33)
\[
\tilde{\omega}(R_1, s) = \frac{\tilde{\Omega}_1}{s}, \quad \tilde{\omega}(R_2, s) = \frac{\tilde{\Omega}_2}{s}
\]
(34)

The solutions in Laplace space for above problems are given
\[
\tilde{\omega}(r, s) = \frac{u_1 \psi_{10}(r, R_2, x(s)) - u_2 \psi_{10}(r, R_1, x(s))}{\psi_{10}(R_1, R_2, x(s))}
\]
(35)
\[
\tilde{\omega}(r, s) = \frac{\tilde{\Omega}_1}{s} \psi_{11}(r, R_2, x(s)) - \frac{\tilde{\Omega}_2}{s} \psi_{11}(r, R_1, x(s))
\]
(36)
\[
\tilde{\omega}(r, s) = \frac{\tilde{\Omega}_1}{s} \psi_{11}(r, R_2, x(s)) - \frac{\tilde{\Omega}_2}{s} \psi_{11}(r, R_1, x(s))
\]
(37)

Where \( x(s) = \frac{\sqrt{r s (1 + \lambda t D^t) + \lambda s D^s} - \frac{\sigma \beta^2}{\rho} (1 + \lambda s D^s)}{v (1 + \lambda s D^s)}, \) and \( \psi_{10}(a, b, y) = K_\nu(a) I_\nu(b) + (-1)^{m+1} i a I_\nu(b) \), \( K_m \) and \( I_n \) are the Bessel functions of the first and second kind, respectively.

The velocity field of helical flow between concentric cylinder is obtained by applying the Stehfest's method [9,10] of the numerical inversion of Laplace transform to Eqs.(35)and(36).

We use the finite Hankel transform with respect to \( r \) [5],defined as follows
\[
\tilde{\omega} = \int_0^\infty \tilde{\omega}(r, s) \psi_j(s_1 r) dr
\]
(38)
\[
\tilde{\omega} = \int_0^\infty \tilde{\omega}(r, s) \psi_j(s_2 r) dr
\]
(39)

And the inverse Hankel transform are
\[
\tilde{\omega}(r, s) = \pi \sum_{m=0}^\infty \frac{s^{m} \gamma_j(f_j(s_1 R_1))}{\Gamma(\nu+1)} \psi_j(s_1 r)
\]
(40)
\[
\tilde{\omega}(r, s) = \pi \sum_{m=0}^\infty \frac{s^{m} \gamma_j(f_j(s_2 R_1))}{\Gamma(\nu+1)} \psi_j(s_2 r)
\]
(41)

Where \( s_{1n} \) and \( s_{2n} \) are the positive roots of \( \psi_j(s_{1n} R_1) = 0 \) and \( \psi_j(s_{2n} R_1) = 0 \) respectively.
Then Eqs.(52) and (53), are become

\[ A_1(s_{12}, t) = 1 - \left( 1 + \lambda t^{s_{12}} + \frac{\sigma \beta_2^2}{\rho} \right) \left( s_{12} + \lambda t^{s_{12}} \right) \]

\[ \times \sum_{m=0}^\infty \sum_{b+c+d=2m} \frac{m!}{b!c!d!} (s_{12}^b u^c v^d t^d) \left( \frac{\sigma \beta_2^2}{\rho} \right)^d \frac{t^d}{(\lambda t^{s_{12}} + s_{12} + \lambda t^{s_{12}})^{m+1}} \times \left( s_{12} + \lambda t^{s_{12}} + s_{12} + \lambda t^{s_{12}} \right)^{m+1} \]  

(51)

Now, applying inverse Laplace transform to Eqs.(50) and (51), we obtain

\[ A_1(s_{12}, t) = 1 - \left( s_{12} + \lambda t^{s_{12}} + \frac{\sigma \beta_2^2}{\rho} \right) \left( s_{12} + \lambda t^{s_{12}} \right) \]

\[ \times \sum_{m=0}^\infty \sum_{b+c+d=2m} \frac{m!}{b!c!d!} (s_{12}^b u^c v^d t^d) \left( s_{12} + \lambda t^{s_{12}} \right)^d \times \left( s_{12} + \lambda t^{s_{12}} + s_{12} + \lambda t^{s_{12}} \right)^{m+1} \]  

(52)

And

\[ A_2(s_{12}, t) = 1 - \left( s_{12} + \lambda t^{s_{12}} + \frac{\sigma \beta_2^2}{\rho} \right) \left( s_{12} + \lambda t^{s_{12}} \right) \]

\[ \times \sum_{m=0}^\infty \sum_{b+c+d=2m} \frac{m!}{b!c!d!} (s_{12}^b u^c v^d t^d) \left( s_{12} + \lambda t^{s_{12}} \right)^d \times \left( s_{12} + \lambda t^{s_{12}} + s_{12} + \lambda t^{s_{12}} \right)^{m+1} \]  

(53)

We will use the following property of Mittag-Leffler function [5]

\[ L^{-1}\left( \frac{m^{s+1}}{s^{s+1}} \right) = e^{s \alpha} \Gamma(1 + \alpha) \left( R(\alpha) \right) \]  

\[ > |e^{s \alpha}| \]  

(54)

Then Eqs.(52) and (53), are become

\[ A_1(s_{12}, t) = 1 - \left( t + \lambda t^{s_{12}} + \frac{\sigma \beta_2^2}{\rho} \right) \left( t + \lambda t^{s_{12}} \right) \]

\[ \times \sum_{m=0}^\infty \sum_{b+c+d=2m} \frac{m!}{b!c!d!} (s_{12}^b u^c v^d t^d) \left( s_{12} + \lambda t^{s_{12}} \right)^d \times \left( s_{12} + \lambda t^{s_{12}} + s_{12} + \lambda t^{s_{12}} \right)^{m+1} \times \left( t + \lambda t^{s_{12}} + t + \lambda t^{s_{12}} \right) \]  

\[ \times \sum_{m=0}^\infty \sum_{b+c+d=2m} \frac{m!}{b!c!d!} (s_{12}^b u^c v^d t^d) \left( s_{12} + \lambda t^{s_{12}} \right)^d \times \left( s_{12} + \lambda t^{s_{12}} + s_{12} + \lambda t^{s_{12}} \right)^{m+1} \times \left( t + \lambda t^{s_{12}} + t + \lambda t^{s_{12}} \right) \]  

\[ + \frac{\sigma \beta_2^2}{\rho} \times \frac{t^d}{(1 + \frac{\sigma \beta_2^2}{\rho} t^d)} \]  

(55)

and

\[ A_2(s_{12}, t) = 1 - \left( t + \lambda t^{s_{12}} + \frac{\sigma \beta_2^2}{\rho} \right) \left( t + \lambda t^{s_{12}} \right) \]

\[ \times \sum_{m=0}^\infty \sum_{b+c+d=2m} \frac{m!}{b!c!d!} (s_{12}^b u^c v^d t^d) \left( s_{12} + \lambda t^{s_{12}} \right)^d \times \left( s_{12} + \lambda t^{s_{12}} + s_{12} + \lambda t^{s_{12}} \right)^{m+1} \times \left( t + \lambda t^{s_{12}} + t + \lambda t^{s_{12}} \right) \]  

\[ + \frac{\sigma \beta_2^2}{\rho} \times \frac{t^d}{(1 + \frac{\sigma \beta_2^2}{\rho} t^d)} \]  

(56)

Substitute Eqs.(55) and (56) into (46) and (48), we get

\[ w(r, t) = \pi \sum_{n=0}^{\infty} f_2(s_{12}, R_1) \psi_1(s_{12}, t) \left[ U_1 f_2(s_{12}, R_2) - U_1 f_1(s_{12}, R_2) \right] \times (1 - G_1(s_{12}, t)) \]  

(57)

Where

\[ G_1(s_{12}, t) = \sum_{m=0}^{\infty} (-1)^m \sum_{b+c+d=2m} \frac{m!}{b!c!d!} (s_{12}^b u^c v^d t^d) \left( \frac{\sigma \beta_2^2}{\rho} \right)^d \times \left( s_{12} + \lambda t^{s_{12}} + s_{12} + \lambda t^{s_{12}} \right)^{m+1} \]  

\[ \times t^{m+1} e^{(m+1) \frac{\epsilon}{\nu}(a-b)} \left( t \right) \]  

\[ + \lambda t^{s_{12}} e^{(m+1) \frac{\epsilon}{\nu}(a-b)} \left( t \right) \]  

\[ + \frac{\sigma \beta_2^2}{\rho} \times \frac{t^d}{(1 + \frac{\sigma \beta_2^2}{\rho} t^d)} \]  

(58)

And

\[ u(r, t) = \pi \sum_{n=0}^{\infty} f_2(s_{12}, R_1) \psi_2(s_{12}, t) \left[ R_1 \partial_t f_2(s_{12}, R_2) - R_1 \partial_t f_1(s_{12}, R_2) \right] \times (1 - G_1(s_{12}, t)) \]  

(59)

Where

\[ G_1(s_{12}, t) = \sum_{m=0}^{\infty} (-1)^m \sum_{b+c+d=2m} \frac{m!}{b!c!d!} (s_{12}^b u^c v^d t^d) \left( \frac{\sigma \beta_2^2}{\rho} \right)^d \times \left( s_{12} + \lambda t^{s_{12}} + s_{12} + \lambda t^{s_{12}} \right)^{m+1} \]  

\[ \times t^{m+1} e^{(m+1) \frac{\epsilon}{\nu}(a-b)} \left( t \right) \]  

\[ + \lambda t^{s_{12}} e^{(m+1) \frac{\epsilon}{\nu}(a-b)} \left( t \right) \]  

\[ + \frac{\sigma \beta_2^2}{\rho} \times \frac{t^d}{(1 + \frac{\sigma \beta_2^2}{\rho} t^d)} \]  

(60)

Finally, the inverse finite Hankel transform on \( W(r) \) and \( U(r) \)

\[ H(W(r)) = \int \frac{W(r)}{R_2} \left[ U_2 + \left( \frac{\ln \frac{R_2}{r}}{\pi} \right) \cdot \left( U_2 - U_1 \right) \right] \psi_1(s_{12}, r) dr \]  

\[ = 2 \left( U_1 f_2(s_{12}, R_2) - U_1 f_1(s_{12}, R_2) \right) \]  

(61)

Where

\[ W(r) = \left[ U_2 + \left( \frac{\ln \frac{R_2}{r}}{\pi} \right) \cdot \left( U_2 - U_1 \right) \right] \]  

We get

\[ w(r, t) = W(r) - \pi \sum_{n=0}^{\infty} f_2(s_{12}, R_1) \psi_1(s_{12}, t) \left[ U_1 f_2(s_{12}, R_2) - U_1 f_1(s_{12}, R_2) \right] \times (1 - G_1(s_{12}, t)) \]  

(62)
3.2. Calculation of the tangential tensions and normal tensions

Applying the Laplace transform to Eqs. (7) and (8), and using the initial conditions

\[ u(r,0) = \frac{\psi_0}{r}, \quad \frac{\partial u}{\partial r}(r,0) = 0 \]

we find that

\[ S_{\nu\nu} = \frac{\mu(1 + \chi^2 s^2 p^2)}{1 + \chi^2 s^2 p^2} \left( \frac{\partial \psi_0}{\partial r} \right) \]

\[ S_{\nu\nu} = \frac{\mu(1 + \chi^2 s^2 p^2)}{1 + \chi^2 s^2 p^2} \left( \frac{\partial \psi_0}{\partial r} \right) \]

Integrating the Eqs. (35) and (36) with respect to \( r \), we obtain

\[ \frac{\partial \psi_0}{\partial r} = \psi_0 \left( \frac{\partial \psi_0}{\partial r} \right) \]

\[ \frac{\partial \psi_0}{\partial r} = \psi_0 \left( \frac{\partial \psi_0}{\partial r} \right) \]

Substitute Eqs.(65) and (66) into Eqs.(63) and (64), respectively, we get

\[ S_{\nu\nu} = \frac{\mu(1 + \chi^2 s^2 p^2)}{1 + \chi^2 s^2 p^2} \left( \frac{\partial \psi_0}{\partial r} \right) \]

\[ S_{\nu\nu} = \frac{\mu(1 + \chi^2 s^2 p^2)}{1 + \chi^2 s^2 p^2} \left( \frac{\partial \psi_0}{\partial r} \right) \]

Differentiating the Eqs. (35) and (36) with respect to \( r \), we obtain

\[ \frac{\partial \psi_0}{\partial r} = \psi_0 \left( \frac{\partial \psi_0}{\partial r} \right) \]

\[ \frac{\partial \psi_0}{\partial r} = \psi_0 \left( \frac{\partial \psi_0}{\partial r} \right) \]

where \( \psi_0 \) is the initial deflection of the plate.
Then Eqs(75) and (76) are become

\[
S_n = \rho \left( \sum_{m=0}^{\infty} \frac{f_m(s_{1m}R_1)\psi(s_{1m}R_1)[U_{1m}J_0(s_{1m}R_1) - U_{1m}J_1(s_{1m}R_2)]}{s_{1m}J_0(s_{1m}R_1) - f_m(s_{1m}R_2)} \right) \times G_2(s_{1m}R_1) \tag{77}
\]

where

\[
G_2(s_{1m}t) = \left( -1 \right)^m \sum_{b=c+d=m}^{b+c=d=m} \left( \frac{\sigma b}{c} \right) \left( \frac{\lambda b}{c} \right)^{\frac{1}{2} + \frac{c+d}{2}} \left( \frac{\lambda b}{c} \right)^{\frac{1}{2} - \frac{m}{2} + \frac{c+d}{2}} \times \left[ \frac{\xi_{1m}}{c} \left( \frac{1}{\lambda b} \right) \right] \left( \frac{1}{\lambda b} \right) \tag{78}
\]

The inverse of Laplace transform to Eq.(79a),(79b) and (79c) can be expressed by

\[
S_n = \frac{1}{\lambda b} \int_0^t (t - \tau)^{a-1} E_{a,\lambda b}(\lambda b(t - \tau)) \, d\tau \tag{81a}
\]

The definition of the weissenberg number \( W_i \) is as

\[
W_i = \frac{S_n}{S_0} \tag{82}
\]

The shear stresses (the frictional force) i.e., the drag exerted per unit length of the inner or outer cylinders or of the cylinder of radius \( r = R_2 \), in the second case, can be calculated from Eqs.(73) and (74), we get

\[
S_1 = \left( \frac{\sigma}{c} \right) \left( \frac{\lambda}{c} \right)^{\frac{1}{2} + \frac{c+d}{2}} \left( \frac{\lambda}{c} \right)^{\frac{1}{2} - \frac{m}{2} + \frac{c+d}{2}} \times \left[ \frac{\xi_{1m}}{c} \left( \frac{1}{\lambda} \right) \right] \left( \frac{1}{\lambda} \right) \tag{83}
\]

Now, applying the inverse Laplace transform to Eqs.(83) and (84), we obtain

\[
S_0 = \frac{1}{\lambda b} \int_0^t (t - \tau)^{a-1} E_{a,\lambda b}(\lambda b(t - \tau)) \, d\tau \tag{84}
\]

The shear stresses (the frictional force) i.e., the drag exerted per unit length of the inner or outer cylinders or of the cylinder of radius \( r = R_2 \), in the second case, can be calculated from Eqs.(73) and (74), we get

\[
S_1 = \frac{1}{\lambda b} \int_0^t (t - \tau)^{a-1} E_{a,\lambda b}(\lambda b(t - \tau)) \, d\tau \tag{85}
\]
\[
\tau_2 = S_{\alpha_0 - 1} + \left( \frac{1}{n+1} \sum_{n=1}^{\infty} f_n(s_{3n}R_s) \right)
\]

\[
r = \rho \left( \frac{1}{n+1} \sum_{n=1}^{\infty} \left( \frac{1}{s_{3n}R_s} \right) \lambda \left( s_{3n}R_s \right) \right)
\]

\[
- \sigma \left( s_{3n}R_s \right) G_{s_{3n}R_s} + \left( s_{3n}R_s \right) \lambda \left( s_{3n}R_s \right)
\]

(86)

IV. Helical flow through circular cylinder

Taking the limit of Eqs (38) and (39), when \( R_1 \to 0 \) and \( R_2 \to R \), we find the Hankel transform

\[
\tilde{w} = \int_0^\infty f_3(s_3r^2) \tilde{w}(r,s) dr
\]

Where \( s_{3n} \) is positive root of \( f_3(s_{3n}R) = 0 \), and

\[
\tilde{u} = \int_0^\infty f_1(s_1r^2) \tilde{u}(r,s) dr
\]

Where \( s_{4n} \) is positive root of \( f_1(s_{4n}R) = 0 \), and the inverse Hankel transform are

\[
\tilde{w}(r,s) = \int_0^\infty \frac{1}{R} \sum_{n=1}^{\infty} \tilde{u}(s_{4n}R) f_1(s_{4n}R) dr
\]

and

\[
\tilde{u}(r,s) = \int_0^\infty \frac{1}{R} \sum_{n=1}^{\infty} \tilde{w}(s_{3n}R) f_3(s_{3n}R) dr
\]

(87)

(88)

Corresponding to the helical flow through an infinite circular cylinder, the boundary conditions must be changed by

\[
|w(0,t)| < \infty , \quad w(R,t) = U, \quad |u(0,t)| < \infty , \quad u(R,t) = R\tilde{u}
\]

(91)

Now apply Laplace transform to the boundary conditions, with respect to t, we get

\[
|w(0,s)| < \infty , w(R,s) = u(0,s) = \frac{|s|}{\beta}, \quad |u(0,s)| < \infty , \quad u(R,s) = \frac{|s|}{\beta}
\]

(92)

Now applying finite Hankel transform to Eqs. (31) and (33), we get

\[
\left[ R^2 \tilde{w}(s_3r^2) + \frac{s \beta_3}{\rho} (1 + \lambda^2 s^2) + s_3^2 \nu (1 + \lambda^2 s^2) \right] \tilde{w} = u + R^2 \tilde{u} = \frac{|s|}{\beta} \tilde{u}(s_3r^2) \]

(93)

Then

\[
\tilde{w} = \frac{s \beta_3}{\rho} (1 + \lambda^2 s^2) + s_3^2 \nu (1 + \lambda^2 s^2)
\]

(94)

\[
\left[ R^2 \tilde{u}(s_1r^2) + \frac{s \beta_1}{\rho} (1 + \lambda^2 s^2) + s_1^2 \nu (1 + \lambda^2 s^2) \right] \tilde{u} = \nu + R^2 \tilde{u} = \frac{|s|}{\beta} \tilde{u}(s_1r^2)
\]

(95)

Then

\[
\tilde{u} = \frac{R^2 \nu \tilde{u}(s_1r^2)}{s \beta_1 \nu (1 + \lambda^2 s^2) + s_1^2 \nu (1 + \lambda^2 s^2)}
\]

(96)

Substitute Eqs. (94) and (96) into Eq. (89) and (90) respectively and using the identity

\[
J_1(s_{3n}r) = J_{n-1}(s_{3n}r) = -J_{n+1}(s_{3n}r)
\]

we find that

\[
\tilde{w}(r,s) = \frac{2U}{R} \sum_{n=1}^{\infty} \frac{f_3(s_3r^2) \tilde{u}(s_3r^2)}{J_2(s_3r^2)}
\]

(97)

where

\[
\tilde{A}_1(s_{3n}R_s) = \frac{s_3^2 \nu (1 + \lambda^2 s^2)}{s \left( 1 + \lambda^2 s^2 \right) + \frac{s_3^2}{\rho} \nu (1 + \lambda^2 s^2) + \frac{s_3^2 \nu (1 + \lambda^2 s^2)}{s_3^2 \nu (1 + \lambda^2 s^2)}}
\]

(98)

and

\[
\tilde{u}(r,s) = 2U \sum_{n=1}^{\infty} \frac{f_3(s_3r^2) \tilde{A}_1(s_{3n}R_s)}{J_2(s_{3n}R_s)}
\]

(99)

where

\[
A_1(s_{3n}R_s) = \frac{s_3^2 \nu (1 + \lambda^2 s^2)}{s \left( 1 + \lambda^2 s^2 \right) + \frac{s_3^2}{\rho} \nu (1 + \lambda^2 s^2) + \frac{s_3^2 \nu (1 + \lambda^2 s^2)}{s_3^2 \nu (1 + \lambda^2 s^2)}}
\]

(100)

\[
\tilde{A}_1(s_{4n}R_s) = \frac{s_3^2 \nu (1 + \lambda^2 s^2)}{s \left( 1 + \lambda^2 s^2 \right) + \frac{s_3^2}{\rho} \nu (1 + \lambda^2 s^2) + \frac{s_3^2 \nu (1 + \lambda^2 s^2)}{s_3^2 \nu (1 + \lambda^2 s^2)}}
\]

(101)

Now, applying inverse Laplace transform to Eqs. (101) and (102), we obtain

\[
A_1(s_{3n}R_s) = 1 - \left( s \left( 1 + \lambda^2 s^2 \right) + \frac{s_3^2}{\rho} \nu (1 + \lambda^2 s^2) \right)
\]

(103)

\[
A_1(s_{4n}R_s) = 1 - \left( s \left( 1 + \lambda^2 s^2 \right) + \frac{s_3^2}{\rho} \nu (1 + \lambda^2 s^2) \right)
\]

(104)

Now, we will use the property (54) of Mittag-Leffler function [5], then Eqs. (103) and (104) are become

\[
A_1(s_{3n}R_s) = \frac{s \left( 1 + \lambda^2 s^2 \right) + \frac{s_3^2}{\rho} \nu (1 + \lambda^2 s^2)}{s \left( 1 + \lambda^2 s^2 \right) + \frac{s_3^2}{\rho} \nu (1 + \lambda^2 s^2) + \frac{s_3^2 \nu (1 + \lambda^2 s^2)}{s_3^2 \nu (1 + \lambda^2 s^2)}}
\]

(105)
Sundos Bader  

\[ \sum_{m=0}^{n} d^{m} e^{m(a-1)} = \frac{b^{m}}{t^{m+1}} \frac{d^{m}(t^{m+1})}{(t^{m+1})^{m+1}} \]

and

\[ A_{1}(s_{34}, t) = 1 - (t + \frac{2t}{t+1}) = \frac{b^{m}}{t^{m+1}} \frac{d^{m}(t^{m+1})}{(t^{m+1})^{m+1}} \]

Substitute Eqs.(105) and (106) into (97) and (99), we get

\[ w(r, t) = \sum_{m=0}^{n} \int_{s_{34}} \frac{f_{i}(s_{34}, t)}{R} (1 - G_{i}(s_{34}, t)) \]

where

\[ G_{i}(s_{34}, t) = \sum_{m=0}^{n} \int_{s_{34}} \frac{f_{i}(s_{34}, t)}{R} (1 - G_{i}(s_{34}, t)) \]

Finally, the inverse finite Hankel transform on \( W(r) \) and \( U(r) \), we get

\[ H(W(r)) = \int_{0}^{\infty} f_{i}(s_{34}, t) \text{ d}t = \frac{2U}{s_{34}} \]

\[ H(U(r)) = \int_{0}^{\infty} f_{i}(s_{34}, t) \text{ d}t = \frac{2U}{s_{34}} \]

4.1. Calculation of the tangential tensions and normal tensions

Differentiating the Eqs. (97) and (99) with respect to \( r \), and substitute into Eqs. (63) and (64) respectively, we obtain

\[ S_{r} = \mu(1 + \lambda_{3} s^{2}) \left( \frac{2U}{s_{34}} \int_{s_{34}} f_{i}(s_{34}, t) \text{ d}t \right) \]

where

\[ u(r, t) = \int_{s_{34}} \frac{f_{i}(s_{34}, t)}{R} (1 - G_{i}(s_{34}, t)) \]

Applying the Laplace transform to Eqs(113) and (114), we get

\[ s^{2} \int_{s_{34}} f_{i}(s_{34}, t) \text{ d}t \]

where

\[ \lambda_{3} = \frac{d^{2}}{c^{2}} \left( 1 + \lambda_{3} s^{2} \right) \]

\[ \frac{d^{2}}{c^{2}} \left( 1 + \lambda_{3} s^{2} \right) \]
\[ S_{r\theta} = \left(2\rho \Omega \sum_{n=0}^{\infty} \frac{[j_{1}(sr_{n})/r} - s_{\theta,0}(sr_{n})]}{r^{2}a_{n}a_{n}(sr_{n},R)} \right) \]
\[ \times L^{-1}\left\{ \frac{-r\beta_{n}^{2}y^{2} + \frac{\beta_{n} r^{2}}{2} + \frac{\beta_{n} (1 + \beta_{n}^{2} r^{2})}{2} + s_{\theta,0} \left(1 + \beta_{n}^{2} r^{2}\right)}{\sqrt{\beta_{n} + (1 + \beta_{n}^{2} r^{2})}} \right\} \]
\[ \times L^{-1}\left( \frac{1}{1 + \Delta_{r}^{2} r^{2}} \right) \]  

(116)

Then Eqs(115) and (116) become

\[ S_{r\theta} = \left( \frac{2\rho \Omega}{R} \sum_{n=0}^{\infty} j_{1}(sr_{n}) \times G_{2}(sr_{n},r) \right) \]

(117)

where

\[ G_{2}(sr_{n},r) = \left( -1 \right)^{m} \sum_{b=0,even}^{\infty} \sum_{d=0}^{\infty} \frac{\beta_{b}^{2} \left( \frac{s_{\theta,0}}{sr_{n}} \right)^{2}}{\beta_{b} \left( \frac{s_{\theta,0}}{sr_{n}} \right)^{2} \left( \frac{\beta_{b} r^{2}}{2} + \frac{\beta_{n} (1 + \beta_{b}^{2} r^{2})}{2} + s_{\theta,0} \left(1 + \beta_{b}^{2} r^{2}\right) \right)} \]

\[ \times \sum_{m=0}^{\infty} \frac{1}{\Delta_{r}^{2} r^{2} + 1} \right\} \]  

\[ \times G_{2}(sr_{n},r) \]

(118)

and

\[ S_{\theta\theta} = \left( \frac{2\rho \Omega}{R} \sum_{n=0}^{\infty} \frac{\left(1/(sr_{n})^2\right) - s_{\theta,0}(sr_{n})}{r^{2}a_{n}a_{n}(sr_{n},R)} \right) \]

\[ \times G_{2}(sr_{n},r) \]  

(119)

where

\[ G_{2}(sr_{n},r) = \left( -1 \right)^{m+1} \sum_{b=0,even}^{\infty} \frac{\beta_{b}^{2} \left( \frac{s_{\theta,0}}{sr_{n}} \right)^{2}}{\beta_{b} \left( \frac{s_{\theta,0}}{sr_{n}} \right)^{2} \left( \frac{\beta_{b} r^{2}}{2} + \frac{\beta_{n} (1 + \beta_{b}^{2} r^{2})}{2} + s_{\theta,0} \left(1 + \beta_{b}^{2} r^{2}\right) \right)} \]

\[ \times \sum_{m=0}^{\infty} \frac{1}{\Delta_{r}^{2} r^{2} + 1} \right\} \]  

\[ \times G_{2}(sr_{n},r) \]  

(120)

The inverse of Laplace transform to(120a),(120b) and (120c) can be expressed by

\[ S_{r\theta} = \left( \frac{1}{\Delta_{r}^{2} r^{2} + 1} \right) \]

\[ \left((t - t)^{n-1} E_{n,0}(\alpha_{n} t^{n}) \right) \]

(121a)

\[ S_{\theta\theta} = \left( \frac{1}{\Delta_{r}^{2} r^{2} + 1} \right) \]

\[ \left((t - t)^{n-1} E_{n,0}(\alpha_{n} t^{n}) \right) \]

(121b)

\[ S_{zz} = \left( \frac{1}{\Delta_{r}^{2} r^{2} + 1} \right) \]

\[ \left((t - t)^{n-1} E_{n,0}(\alpha_{n} t^{n}) \right) \]

(121c)

The definition of the weissenberg number \( W_{r} \) is as follows

\[ W_{r} = \frac{S_{zz} - S_{\theta\theta}}{S_{zz}} \]

(122)

The shear stresses (the frictional force ) i.e., the drag exerted per unit length of the inner or outer cylinders or of the cylinder of radius \( r = R_{2} \), in the second case can be calculated from Eqs.(117) and(118), we get

\[ r_{3} = S_{r\theta}|_{r=R_{2}} = \frac{2\rho \Omega}{R} \sum_{n=0}^{\infty} j_{1}(sr_{n}) \]

\[ \times G_{2}(sr_{n},r) \]

(123)

\[ r_{4} = S_{\theta\theta}|_{r=R_{2}} = 2\rho \Omega \sum_{n=0}^{\infty} \frac{1}{\Delta_{r}^{2} r^{2} + 1} \]

\[ \times G_{2}(sr_{n},r) \]

(124)

V. Limiting case

1. Making the limit of Eqs(61) and (63) when \( \alpha = 0, \beta = 0 \) and \( \frac{\varphi_{b} \beta_{n}^{2}}{\rho} = M = 0 \), we can attain the similar solution velocity distribution for unsteady helical flows of a generalized Oldroyd-B fluid, as obtained in [6], thus velocity fields reduces to
The velocity field $w(r,t)$ with the non-dimensional parameters $\alpha$, $\beta$, and $\nu$ is increasing with increase of fractional parameter $\alpha$. Fig(2) is prepared to show the effect of the variations of fractional parameter $\alpha$ on the velocity field $w$ and Laplace transform for fractional calculus. Moreover, some figures are plotted to show the behavior of various parameters involved in the expression of velocities $w_u$ and $w_v$ (Eqs(61 and 62)) , shear stresses $\tau_1$ and $\tau_2$ (Eqs(85 and 86)) respectively.

All the results in this section are plotted by using MATHEMATICA package.

Fig(1) is depicted to show the change of the velocity $w(r,t)$ with the non-integer fractional parameter $\alpha$ . The velocity is decreasing with increase of $\alpha$. Fig(2) is prepared to show the effect of the variations of fractional parameter $\alpha$ on the velocity field $w$ and Laplace transform for fractional calculus. Moreover, some figures are plotted to show the behavior of various parameters involved in the expression of velocities $w_u$ and $w_v$ (Eqs(61 and 62)) , shear stresses $\tau_1$ and $\tau_2$ (Eqs(85 and 86)) respectively.

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All the results in this section are plotted by using MATHEMATICA package.
the influence of time $t$ on the velocity field $w$. The velocity is increasing with increase the time $t$.

Figs(15-20) provide the graphically illustrations for the effects of (fractional parameters($\alpha$, $\beta$), relaxation $\lambda_1$, retardation $\lambda_2$, kinematic viscosity $v$, magnetic parameter $M$ and time $t$) on the shear stress $\tau_1$. The shear stress decreasing with increase the different parameters.

Figs(21-26) are established to show the behavior of the parameters (fractional parameters($\alpha$, $\beta$), relaxation $\lambda_1$, retardation $\lambda_2$, kinematic viscosity $v$, magnetic parameter $M$ and time $t$) on the shear stress $\tau_2$. The shear stress decreasing with increase the different parameters.

Fig1: the velocity $w$ for different value of fractional parameter $\alpha$ ($\lambda_1=15$, $\lambda_2=8$, $v=0.165$, $M=0$, $\beta=0$, $K_1=3.31114$, $t=2$, $R_1=1$, $R_2=2$, $u_1=2$, $u_2=2$)

Fig2: the velocity $w$ for different value of fractional parameter $\beta$ ($\lambda_1=15$, $\lambda_2=8$, $v=0.165$, $M=0$, $\alpha=0.4$, $K_1=3.31114$, $t=2$, $R_1=1$, $R_2=2$, $u_1=2$, $u_2=2$)

Fig3: the velocity $w$ for different value $\lambda_1$ ($\lambda_2=8$, $v=0.165$, $M=0.1$, $\alpha=0.4$, $K_1=3.31114$, $t=2$, $R_1=1$, $R_2=2$, $u_1=2$, $u_2=2$)

Fig4: the velocity $w$ for different value $\lambda_2$ ($\lambda_1=15$, $v=0.165$, $M=0.1$, $\beta=0$, $\alpha=0.4$, $K_1=3.31114$, $t=2$, $R_1=1$, $R_2=2$, $u_1=2$, $u_2=2$)

Fig5: the velocity $w$ for different value $v$ ($\lambda_1=15$, $\lambda_2=8$, $M=0.1$, $\beta=0$, $\alpha=0.4$, $K_1=3.31114$, $t=2$, $R_1=1$, $R_2=2$, $u_1=2$, $u_2=2$)
Fig 6: The velocity $w$ for different value $M$ \{ $\lambda_1$=15 , $\lambda_2$=8, $r$=0.165, $\alpha$=0.4 , $K_1$=3.31114 , $t$=2, $R_1$=1, $R_2$=2, $u_1$=2, $u_2$=2\}

Fig 7: The velocity $w$ for different value $t$ \{ $\lambda_1$=15 , $\lambda_2$=8, $M$=0.1, $r$=0.165, $\alpha$=0.4, $K_1$=3.31114 , $t$=2, $R_1$=1, $R_2$=2, $u_1$=2, $u_2$=2\}

The velocity of $u$

Fig 8: The velocity $u$ for different value of fractional parameter $\alpha$ \{ $\lambda_1$=15 , $\lambda_2$=8, $r$=0.165 , $M$=0.1, $\beta$=0.6 , $K_1$=3.31114 , $t$=2, $R_1$=1, $R_2$=2, $u_1$=2, $u_2$=2\}

Fig 9: The velocity $u$ for different value $\beta$ \{ $\lambda_1$=15 , $\lambda_2$=8, $r$=0.165 , $M$=0.1, $\alpha$=0.4, $K_1$=3.31114 , $t$=2, $R_1$=1, $R_2$=2, $u_1$=2, $u_2$=2\}

Fig 10: The velocity $u$ for different value $\lambda_1$ \{ $\lambda_2$=8, $r$=0.165 , $M$=0.1, $\alpha$=0.4, $K_1$=3.31114 , $t$=2, $R_1$=1, $R_2$=2, $u_1$=2, $u_2$=2\}

Fig 11: The velocity $u$ for different value $\lambda_2$ \{ $\lambda_1$=15 , $r$=0.165 , $M$=0.1, $\beta$=0.6 , $\alpha$=0.4, $K_1$=3.31114 , $t$=2, $R_1$=1, $R_2$=2, $u_1$=2, $u_2$=2\}

1.2 1.4 1.6 1.8 2.0
$r$ -3 -2 -1 1 2
1.2 1.4 1.6 1.8 2.0
$r$ -6 -4 -2 2
1.2 1.4 1.6 1.8 2.0
$r$ -8 -6 -4 -2 2
Fig 12: the velocity $u$ for different value $v$ ($\lambda_1=15$, $\lambda_2=8$, $M=0.1$, $\beta=0.6$, $\alpha=0.4$, $K_1=3.31114$, $t=1$, $R_1=1$, $R_2=2$, $u_1=2$, $u_2=2$)

Fig 13: the velocity $u$ for different value $M$ ($\lambda_1=15$, $\lambda_2=8$, $v=0.165$, $M=0.1$, $\beta=0.6$, $\alpha=0.4$, $K_1=3.31114$, $t=2$, $R_1=1$, $R_2=2$, $u_1=2$, $u_2=2$)

Fig 14: the velocity $u$ for different value $t$ ($\lambda_1=15$, $\lambda_2=8$, $v=0.165$, $M=0.1$, $\beta=0.6$, $\alpha=0.4$, $K_1=3.31114$, $t=2$, $R_1=1$, $R_2=2$, $u_1=2$, $u_2=2$)

Fig 15. the shear stress for different value of fractional parameter $\alpha$ ($\lambda_1=15$, $\lambda_2=8$, $v=0.165$, $M=0.1$, $\beta=0.6$, $K_1=3.31114$, $t=2$, $R_1=1$, $R_2=2$, $u_1=1$, $u_2=1$, $r=2$, $\rho=0.1$)

Fig 16. the shear stress for different value of fractional parameter $\beta$ ($\lambda_1=15$, $\lambda_2=8$, $v=0.165$, $M=0.1$, $\alpha=0.6$, $K_1=3.31114$, $t=2$, $R_1=1$, $R_2=2$, $u_1=1$, $u_2=1$, $r=2$, $\rho=0.1$)

Fig 17. the shear stress for different value of $\lambda_1$ ($\lambda_2=8$, $v=0.165$, $M=0.1$, $\beta=0.6$, $\alpha=0.6$, $K_1=3.31114$, $t=2$, $R_1=1$, $R_2=2$, $u_1=1$, $u_2=1$, $r=2$, $\rho=0.1$)

The shree stress
Fig. 18. The shear stress for different values of $\lambda_2$ ($\lambda_1=15$, $v=0.165$, $M=0.1$, $\beta=0.6$, $\alpha=0.6$, $K_1=3.31114$, $t=2$, $R_1=1$, $R_2=2$, $u_1=1$, $u_2=1$, $r=2$, $\rho=0.1$)

Fig. 19. The shear stress for different values of $v$ ($\lambda_1=15$, $\lambda_2=8$, $M=0.1$, $\beta=0.6$, $\alpha=0.6$, $K_1=3.31114$, $t=2$, $R_1=1$, $R_2=2$, $u_1=1$, $u_2=1$, $r=2$, $\rho=0.1$)

Fig. 20. The shear stress for different values of $M$ ($\lambda_1=15$, $\lambda_2=8$, $v=0.165$, $M=0.1$, $\beta=0.6$, $\alpha=0.6$, $K_1=3.31114$, $t=2$, $R_1=1$, $R_2=2$, $u_1=1$, $u_2=1$, $r=2$, $\rho=0.1$)

Fig. 21. The shear stress for different values of fractional parameter $\alpha$ ($\lambda_1=15$, $\lambda_2=8$, $v=0.165$, $M=0.1$, $\beta=0.6$, $K_1=3.31114$, $t=2$, $R_1=1$, $R_2=2$, $u_1=1$, $u_2=1$, $r=2$, $\rho=0.1$)

Fig. 22. The shear stress for different values of fractional parameter $\beta$ ($\lambda_1=15$, $\lambda_2=8$, $v=0.165$, $M=0.1$, $\alpha=0.6$, $K_1=3.31114$, $t=2$, $R_1=1$, $R_2=2$, $u_1=1$, $u_2=1$, $r=2$, $\rho=0.1$)

Fig. 23. The shear stress for different values of $\lambda_1$ ($\lambda_2=8$, $v=0.165$, $M=0.1$, $\beta=0.6$, $\alpha=0.6$, $K_1=3.31114$, $t=2$, $R_1=1$, $R_2=2$, $u_1=1$, $u_2=1$, $r=2$, $\rho=0.1$)
Fig. 24: The shear stress for different value of $\lambda_2$ [ $\lambda_1=15$, $v=0.165$, $M=0.1$, $\beta=0.6$, $\alpha=0.6$, $K=3.31114$, $u_1=2$, $R_1=1$, $R_2=2$, $u_2=1$, $r=2$, $\rho=0.1$]

Fig. 25: The shear stress for different value of $v$ [ $\lambda_1=15$, $\lambda_2=8$, $v=0.165$, $M=0.1$, $\beta=0.6$, $\alpha=0.6$, $K=3.31114$, $u_1=2$, $R_1=1$, $R_2=2$, $u_2=2$, $r=2$, $\rho=0.1$]

Fig. 26: The shear stress for different value of $M$ [ $\lambda_1=15$, $\lambda_2=8$, $v=0.165$, $\beta=0.6$, $\alpha=0.6$, $K=3.31114$, $u_1=2$, $R_1=1$, $R_2=2$, $u_2=2$, $r=2$, $\rho=0.1$]

References