An Overview of Separation Axioms by Nearly Open Sets in Topology.

Dr. Thakur C. K. Raman, Vidyottama Kumari
Associate Professor & Head, Deptt. Of Mathematics, Jamshedpur Workers College, Jamshedpur, Jharkhand, INDIA.
Assistant Professor, Deptt. Of Mathematics, R.V.S College OF Engineering & Technology, Jamshedpur, Jharkhand, INDIA.

Abstract: The aim of this paper is to exhibit the research on separation axioms in terms of nearly open sets viz p-open, s-open, α-open & β-open sets. It contains the topological property carried by respective ϕ -Tₖ spaces (ϕ = p, s, α & β; k = 0, 1, 2) under the suitable nearly open mappings. This paper also projects ϕ -R₀ & ϕ -R₁ spaces where ϕ = p, s, α & β and related properties at a glance. In general, the ϕ -symmetry of a topological space for ϕ = p, s, α & β has been included with interesting examples & results.

Key Words: ϕ -Tₖ spaces, ϕ -R₀ & ϕ -R₁ spaces & ϕ -symmetry.

I. Introduction & Preliminaries:

The weak forms of open sets in a topological space as semi-pre open & b-open sets were introduced by D. Andrijevic through the mathematical papers[1,2]. The concepts of generalized closed sets with the introduction of semi-pre opens were studied by Levine [12] and Njasted [14] investigated α-open sets and Mashour et. al. [13] introduced pre-open sets. The class of such sets is named as the class of nearly open sets by Njasted[14].

After the works of Levine on semi-open sets, several mathematician turned their attention to the generalization of various concepts of topology by considering semi-open sets instead of open sets. When open sets are replaced by semi-open sets, new results were obtained. Consequently, many separation axioms have been formed and studied.

The study of topological invariants is the prime objective of the topology. Keeping this in mind several authors invented new separation axioms. The presented paper is the overview of the common facts of this trend at a glance for researchers.

Throughout this paper, spaces (X, T) and (Y, ω) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. The notions mentioned in [1,6], [2,12,14] were conceptualized using the closure operator (cl) & the interior operator( int ) in the following manner:

* Definition:

A subset A of a topological space (X,T) is called
I. a semi-pre-open [1] or β-open [6] set if A ⊆ cl(int(cl(A))) and a semi-pre closed or β-closed if int(cl(int(A))) ⊆ A.
II. a b-open[2] set if A ⊆ cl(int(A))∪ int(cl(A)) and a b-closed [8] if cl(int(A)) ∩ int(cl(A)) ⊆ A.
III. a semi-open [12] set if A ⊆ cl(int(A)) and semi-closed if int(cl(A)) ⊆ A.
IV. an α-open[14] set if A ⊆ int(cl(int(A))) and an α-closed set if cl(int(cl(A))) ⊆ A.
V. a pre-open [13] set if A ⊆ int(cl(A)) and pre-closed if cl(int(A)) ⊆ A.

The class of pre-open, semi-open, α-open, semi-pre-open and b-open subsets of a space (X,T) are usually denoted by PO(X,T), SO(X,T), T₀, SPO(X,T) & BO(X,T) respectively. Any undefined terminology used in this paper can be known from [4].

In 1996, D.Andrijevic made the fundamental observation:

* Proposition:

For every space (X,T), PO(X,T) ∪ SO(X,T) ⊆ BO(X,T) ⊆ SPO(X,T) holds but none of these implications can be reversed[10].
**Proposition**: Characterization [10]:

I. $S$ is semi-pre-open iff $S \subseteq \text{scl}(\text{scl} S)$.

II. $S$ is semi-open iff $S \subseteq \text{scl}(\text{int} S)$.

III. $S$ is pre-open iff $S \subseteq \text{pint}(\text{pint} S)$.

IV. $S$ is b-open iff $S \subseteq \text{pcl}(\text{pint} S)$, where $S \subseteq X \times (X,T)$ is a space.

**Remark**: The separation axioms of topological spaces are usually denoted with the capital letter “T” after the German word “Trennung” which means separation. Separation axioms are one among the most common important & interesting concepts in Topology and are used to coin more restricted classes of topological spaces. However, the structure and the properties of such spaces are not always that easy to comprehend.

§1. Separation axioms in terms of nearly open sets:

The separation axioms enable us to assert with precision whether a topological space has sufficient number of open sets as well as nearly open sets to serve the purpose that the larger the number of open sets as well as nearly open sets, the greater is the supply of the continuous or respective continuous functions because the concept of continuity or respective continuity is fundamental in analysis & topology and intimately linked with open or nearly open sets.

This section highlights the overview of the separation axioms in terms of nearly open sets viz p-open, s-open, α-open & β-open sets.

$\wp \cdot T_k$ Topological Spaces ($\wp = \wp, s, \alpha & \beta; k = 0,1,2$):

The literature survey on all $\wp \cdot T_k$ spaces ($\wp = \wp, s, \alpha & \beta; k = 0,1,2$) has been brought under a common frame work.

**Definition (1.1)**: A topological space $(X,T)$ is said to be:

(i) $\wp \cdot T_0$ space if for each pair of distinct points $x$ and $y$ of $X$, there exists a $\wp$-open set $A$ such that $x \in A$ but $y \notin B$ & that $y \in A$ but $x \notin B$.

Or

(ii) $\wp \cdot T_1$ space if for each pair of distinct points $x$ and $y$ of $X$, there exists a pair of $\wp$-open sets $A & B$ such that $x \in A$ but $y \in A$ & that $y \in B$ but $x \notin B$.

Or

(iii) $\wp \cdot T_2$ space if for each pair of distinct points $x$ and $y$ of $X$, there exists two disjoint $\wp$-open sets $A$ & $B$ in the manner that $A$ contains $x$ but not $y$ and $B$ contains $y$ but not $x$.

Or

(iv) $\wp \cdot T_3$ space if for any two distinct points $x$ and $y$ of $X$, there exist a $\wp$-open set $A & B$ such that $x \in A$ & $y \notin B$ and $A \cap B = \wp$.

**Remark (1.1)**: If a space $(X,T)$ is $\wp \cdot T_k$, then it is $\wp \cdot T_{k+1}$, $k = 1,2$. But the converse is not true.

**Example (1.1)**: Every $\wp \cdot T_0$ space is not necessarily a $\wp \cdot T_1$ space.

Let us consider the set $N$ of all natural numbers. Let $T = \{ \wp, N \& G_n = \{1,2,3,\ldots,n\}, n \in N \}$. Then $(N,T)$ is a topological space. Obviously, every $G_n$ is a $\wp$-open set where $\wp = \wp, s, \alpha & \beta$.

Clearly, the space $(N,T)$ is $\wp \cdot T_0$ space, because if we consider two distinct points $m$ and $n$ ($m < n$) then $G_m = \{1,2,3,\ldots,m\}$ is a $\wp$-open set containing $m$ but not containing $n$ and hence it is a $\wp \cdot T_0$ space.

But it is not a $\wp \cdot T_1$ space because if we choose $G_n = \{1,2,3,\ldots,n\}$, then $m \in G_m$ but $n \notin G_m$ and $n \notin G_n$ but $m \in G_n$ as $m < n$.

Hence, $(N,T)$ is not a $\wp \cdot T_1$ space, even though it is a $\wp \cdot T_0$ space.

**Example (1.2)**: Every $\wp \cdot T_1$ space is not necessarily a $\wp \cdot T_2$ space.

Let $T$ be the cofinite topology on an infinite set $X$, then $(X,T)$ is a cofinite topological space.
Obviously, every member of T is a $\vartheta$ -open set where $\vartheta = p, s, \alpha$ & $\beta$. Clearly, the space (X,T) is a $\vartheta$ -T$_1$ space because if we consider two distinct points x & y of X, then {x} & {y} are finite sets and hence X –{x}, X –{y} are members of T i.e. X- {x} & X- {y} are $\vartheta$ -open sets such that y $\in$ X- {x} & x $\in$ X- {y} but x $\notin$ X- {x} & y $\notin$ X- {y}.

But in this case the topological space(X,T) is not $\vartheta$ -T$_2$ space.

If possible, let (X,T) be a $\vartheta$ -T$_2$ space so that for distinct points x,y there exist $\vartheta$ -open sets G & H containing x & y respectively in the manner that G $\cap$ H = $\phi$. Consequently (G $\cap$ H)' = $\vartheta$ i.e. G' $\cup$ H' = X. Now, G & H being $\vartheta$ -open sets, therefore G' & H' are both finite by definition of co-finite topology and hence, there union X is also finite. But this contradicts the hypothesis that X is infinite which arises due to our assumption that (X,T) is a $\vartheta$ -T$_2$ space. Hence, (X,T) is not $\vartheta$ -T$_2$ space.

Example (1.3):

(i) We consider the topological space (X,T) where X = {a,b,c,d} And

$T = \{\varnothing, \{a\}, \{a,b\}, \{a,c\}, \{a,c,d\}\}$. Here closed sets are : $\varnothing, \{b\}, \{a,b\}, \{c,d\}, \{b,c\}, X$.

Simple computations show that

PO(X,T) = $\{\varnothing, \{a\}, \{a,b\}, \{a,c\}, \{a,c,d\}\}$. T = $\{\varnothing, \{a\}, \{a,b\}, \{a,c\}, \{a,c,d\}\}$. X

SO(X,T) = $\{\varnothing, \{a\}, \{a,b\}, \{a,c\}, \{a,c,d\}\}$. T X

$\alpha$(O)(X,T) = T & $\beta$(O)(X,T) = PO(X,T).

Thus (X,T) is p-T$_0$ & $\beta$-T$_0$ space but neither p-T$_1$ nor $\beta$-T$_1$ space where k = 1,2. Also , (X,T) is not a $\vartheta$ -T$_1$ space where $\vartheta = s$ & $\alpha$; k = 0,1,2.

(ii) Let the topological space (X,T) be given by X = {a,b,c,d} And

T = \{\varnothing, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\}. Here closed sets are : $\varnothing, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, X$.

Simple computations show that

PO(X,T) = $\{\varnothing, \{b\}, \{a,b\}, \{a,c\}, \{a,c,d\}\}$. T = $\{\varnothing, \{b\}, \{a,b\}, \{a,c\}, \{a,c,d\}\}$. X

SO(X,T) = $\{\varnothing, \{b\}, \{a,b\}, \{a,c\}, \{a,c,d\}\}$. T X

$\alpha$(O)(X,T) = PO(X,T) & $\beta$(O)(X,T) = SO(X,T).

Here (X,T) is s-T$_0$, s- T$_1$, s-T$_2$ space as well as $\beta$-T$_0$, $\beta$-T$_1$, $\beta$-T$_2$ space. But (X,T) is neither p-T$_k$ nor $\alpha$-T$_k$ space where k = 0,1,2.

(iii) Let the topological space (X,T) be illustrated as:

X = {a,b,c,d} and T = \{\varnothing, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\}. Here closed sets are : $\varnothing, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, X$.

Simple computations provide that

PO(X,T) = $\{\varnothing, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, X\}$. T = PO(X,T)

SO(X,T) = $\{\varnothing, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, X\}$. T X

$\alpha$(O)(X,T) = T & $\beta$(O)(X,T) = PO(X,T).

Here (X,T) is s-T$_2$ space as well as $\beta$-T$_2$ space. But it is not even p-T$_2$,$\alpha$-T$_2$ space or p-T$_1$, $\alpha$-T$_1$ space. Clearly, (X,T) is a $\vartheta$ -T$_0$ space where $\vartheta = p, s, \alpha & \beta$.

Observations:

(a) In the example (i), (X,T) is not a T$_0$ -space and also in the example (ii), (X,T) is not a T$_0$ -space. But in the example (iii) (X,T) is a T$_0$ –space.

(b) Since, every open set is p-open, s-open , $\alpha$ –open & $\beta$-open , hence

(bi) (X,T) is T$_0$ –space $\Rightarrow$ (X,T) is $\vartheta$ -T$_0$ –space.

(bii) (X,T) is T$_1$ –space $\Rightarrow$ (X,T) is $\vartheta$ -T$_1$ –space.

(bii) (X,T) is T$_2$ –space $\Rightarrow$ (X,T) is $\vartheta$ -T$_2$ –space, where $\vartheta = p, s, \alpha & \beta$.

However, the converse of these results may not be true.

(c) These facts establish that the concepts of $\vartheta$-T$_k$ spaces are different from the concepts of T$_k$ -spaces where k = 0,1,2 & $\vartheta = p, s, \alpha & \beta$.

Theorem (1.1):

A topological space (X,T) is a $\vartheta$ -T$_0$-space iff for each pair of distinct points x & y of X, the $\vartheta$ -cl(x)$\not=$ $\vartheta$ -cl(y) where $\vartheta = p, s, \alpha & \beta$.
Proof:

Necessity: Let \((X, T)\) be a \(\varnothing\)-\(T_0\)-space where \(\varnothing = p, s, a \& \beta\) and let \(x, y\) be any two distinct point of \(X\). Then we have to show that \(\varnothing - \text{cl}\{x\} \neq \varnothing - \text{cl}\{y\}\). Since the space is \(\varnothing\)-\(T_0\), there exists a \(\varnothing\)-open set \(G\) containing one of them, say \(x\), but not containing \(y\). Then \(X - G\) is a \(\varnothing\)-closed set which does not contain \(x\) but contains \(y\). By definition, \(\varnothing - \text{cl}\{y\}\) is the intersection of all \(\varnothing\)-closed set containing \(y\). It follows that \(\varnothing - \text{cl}\{y\}\subseteq X - G\). Hence \(x \notin X - \text{cl}\{y\}\) implies that \(x \notin \varnothing - \text{cl}\{y\}\). Thus \(x \notin \varnothing\text{cl}\{x\}\) but \(x \notin \varnothing - \text{cl}\{y\}\). It follows that \(\varnothing - \text{cl}\{x\} \neq \varnothing - \text{cl}\{y\}\).

Sufficiency:

Let \(x \neq y\Rightarrow \varnothing - \text{cl}\{x\} \neq \varnothing - \text{cl}\{y\}\) where \(x, y\) are points of \(X\). Since \(\varnothing - \text{cl}\{x\} \neq \varnothing - \text{cl}\{y\}\), there exists at least one point \(z\) of \(X\) which belongs to one of them, say \(\varnothing - \text{cl}\{x\}\) and does not belong to \(\varnothing - \text{cl}\{y\}\). We claim that \(x \notin \varnothing - \text{cl}\{y\}\). For if \(x \notin \varnothing - \text{cl}\{y\}\). Then \(\varnothing - \text{cl}\{x\}\subseteq \varnothing - \text{cl}\{y\}\) and so \(x \notin \varnothing - \text{cl}\{x\}\subseteq \varnothing - \text{cl}\{y\}\) which is a contradiction. Accordingly \(x \notin \varnothing - \text{cl}\{y\}\) and consequently \(x \in X - \varnothing - \text{cl}\{y\}\) which is \(\varnothing\)-open.

Hence, \(X - \varnothing - \text{cl}\{y\}\) is a \(\varnothing\)-open set containing \(x\) but not \(y\). It follows that \((X, T)\) is a \(\varnothing\)-\(T_0\)-space where \(\varnothing = p, s, a \& \beta\).

Theorem (1.2): A topological space \((X, T)\) is a \(\varnothing\)-\(T_1\)-space if and only if every singleton subset \([x]\) of \(X\) is \(\varnothing\)-closed where \(\varnothing = p, s, a \& \beta\).

Proof: The ‘if part‘: let every singleton subset \([x]\) of \(X\) be \(\varnothing\)-closed. We have to show that the space is \(\varnothing\)-\(T_1\). Let \(x, y\) be the two distinct point of \(X\). Then \((X - \varnothing\text{cl}\{x\})\) is a \(\varnothing\)-open set which contains \(y\) but does not contain \(x\). Similarly \(X - \varnothing\text{cl}\{y\}\) is a \(\varnothing\)-open set which contains \(x\) but does not contain \(y\). Hence, the space \((X, T)\) is a \(\varnothing\)-\(T_1\) where \(\varnothing = p, s, a \& \beta\).

The ‘only if‘ part: Let the space be \(\varnothing\)-\(T_1\), and let \(x\) be any point of \(X\). we want to show that \([x]\) is \(\varnothing\)-closed, that is, to show that \(X - [x]\) is \(\varnothing\)-open. Let \(y \in X - [x]\). Then \(y \neq x\). Since \(X\) is \(\varnothing\)-\(T_1\), there exists a \(\varnothing\)-open set \(G\) such that \(y \in G\), \(x \notin G\). It follows that \(y \in G\cap X - [x]\).

Hence, \(X - [x] = U\{G_j : y \in G_j\} = A\)-\(\varnothing\)-open set i.e. \([x]\) is a \(\varnothing\)-closed set where \(\varnothing = p, s, a \& \beta\).

Theorem (1.3): A space \((X, T)\) is a \(\varnothing\)-\(T_2\) space if for each point \(x \in X\), the intersection of all \(\varnothing\)-closed set containing \(x\) is the singleton set \([x]\), where \(\varnothing = p, s, a \& \beta\).

Proof: Necessity:

Suppose that \((X, T)\) is a \(\varnothing\)-\(T_2\) space where \(\varnothing = p, s, a \& \beta\). Then there exist a pair of \(\varnothing\)-open sets \(G \& H\) for each pair of distinct points \(x, y\) in \(X\) such that \(x \in G\), \(y \in H\) and \(G \cap H = \varnothing\). Now, \(G \cap H = \varnothing\Rightarrow G \subset H\). Hence \(x \in G \subset H\) so that \(H\) is a \(\varnothing\)-closed set containing \(x\), which does not contain \(y\) as \(y \notin H\). Therefore \(y\) cannot be contained in the intersection of all \(\varnothing\)-closed sets which contains \(x\). Since, \(y \neq x\) is arbitrary, it follows that the intersection of all \(\varnothing\)-closed sets containing \(x\) is the singleton set \([x]\). Consequently, \(\cap \{F : x \in F \cap \varnothing\text{cl}\{x\}\} = \varnothing\).

Sufficiency:

Suppose that \([x]\) is the intersection of all \(\varnothing\)-closed subsets of \((X, T)\) containing \(x\) where \(x\) is an arbitrary point of \(X\).

Let \(y\) be any other point of \(X\) which is different from \(x\). Obviously, by hypothesis \(y\) does not belong to the intersection of all \(\varnothing\)-closed subsets containing \(x\). So there must exist a \(\varnothing\)-closed set, say \(N\), containing \(x\) such that \(y \notin N\). Now, \(N\) being a \(\varnothing\)-closed nbhd of \(x\), there must exist a \(\varnothing\)-open set \(G\) such that \(x \in G \subset N\).

Thus, \(G \& N\) are \(\varnothing\)-open sets such that \(x \in G\), \(y \in N\) and \(G \cap N = \varnothing\). Consequently \((X, T)\) is a \(\varnothing\)-\(T_2\) space where \(\varnothing = p, s, a \& \beta\).

Hence, the theorem.

Remark (1.2): The following example is cited in the support of above three theorems.

Let \(X = \{a,b,c,d,e\}\), \(T = \{\varnothing, \{a\}, \{b\}, \{a,b\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}, \{b,c,d,e\}\}\), & \(T^* = \{\varnothing, \{a\}, \{c\}, \{a,c\}, \{a,b\}, \{b,c\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}, \{b,c,d,e\}\}\).

Then \(\varnothing\text{cl}(X) = \{\varnothing, \{a\}, \{b\}, \{a,b\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}, \{b,c,d,e\}\} = \{\varnothing, \{a\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}, \{a,b,c,d\}, \{b,c,d,e\}\} = P(X) - \{\{c\}, \{a,e\}\}\).
\[ \beta C(X,T) = P(X) - \{\{b,c,d\},\{a,b,c,d\}\}. \]
Here, \((X,T)\) is a \(\beta\)-T\(_2\) space, consequently, it is also a \(\beta\)-T\(_1\) and \(\beta\)-T\(_0\) space.

Obviously,
1. \(\beta\)-cl\([x]\) \(\neq\) \(\beta\)-cl\([y]\); \(\forall x,y \in X\) \& \(x \neq y\).
2. \(\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\) are \(\beta\)-closed sets i.e. every singleton set is \(\beta\)-closed.
3. \(\cap \{F \in \beta C(X,T) \text{ such that } x \in F\} = \{x\} \forall x \in X\).

**Definition (1.2): \(\vartheta\)-open mappings:**

A mapping \(f:(X,T)\rightarrow(Y,\sigma)\) from one space \((X,T)\) to another space \((Y,\sigma)\) is called \(\vartheta\) –open mapping (i.e. \(\vartheta = p,s,a,\beta\)) if \(f\) sends \(\vartheta\)-open sets (i.e. \(\vartheta = p,s,a,\beta\)-open) of \((X,T)\) into \(\vartheta\)-open sets (i.e. \(\vartheta = p,s,a,\beta\)-open) of \((Y,\sigma)\).

**Theorem (1.4): The property of a space being a \(\vartheta\) - T\(_0\) space is a topological property where \(\vartheta = p,s,a,\alpha,\beta\).**

**Proof:**

Let \(f:(X,T)\rightarrow(Y,\sigma)\) be a one-one onto \& \(\vartheta\)-open mapping from a \(\vartheta\) - T\(_0\) space \((X,T)\) to any other topological space \((Y,\sigma)\). It will be established that \((Y,\sigma)\) is also a \(\vartheta\) - T\(_0\) space where \(\vartheta = p,s,a,\beta\).

Let \(y_1\) \& \(y_2\) be any two distinct points of \(Y\) and as \(f\) is one-one \& onto, there must exist distinct points \(x_1\) \& \(x_2\) of \(X\) such that \(f(x_1) = y_1\) \& \(f(x_2) = y_2\).

\[ \text{(1)} \]

Since, \((X,T)\) is a \(\vartheta\) - T\(_0\) space so there exists a \(\vartheta\)-open set \(G\) in manner that \(x_1 \in G\) but \(x_2 \notin G\).

Again, \(f\), being \(\vartheta\)-open, provides that \(f(G)\) is a \(\sigma\)-\(\vartheta\)-open and containing \(f(x_1) = y_1\) and not containing \(f(x_2) = y_2\).

Thus, there exists a \(\sigma\)-\(\vartheta\)-open set \(f(G)\) which contains \(y_1\) and does not contain \(y_2\) and in turn \((Y,\sigma)\) is a \(\vartheta\) - T\(_0\) space.

Again, as the property of being \(\vartheta\) - T\(_0\) space is preserved under one-one \& onto \& \(\vartheta\)-open mapping, so it is a topological property.

Hence, the theorem.

**Theorem (1.5): The property of a space being a \(\vartheta\) - T\(_1\) space is a topological property where \(\vartheta = p,s,\alpha,\beta\).**

**Proof:**

Let \((X,T)\) be a \(\vartheta\) - T\(_1\) space and \((Y,\sigma)\) be any other topological space such that \(f(X,T)\rightarrow(Y,\sigma)\) is one-one onto \& \(\vartheta\)-open mapping from \((X,T)\) to \((Y,\sigma)\).

It is required to prove that \((Y,\sigma)\) is also a \(\vartheta\) - T\(_1\) space where \(\vartheta = p,s,\alpha,\beta\).

Let \(y_1\) \& \(y_2\) be any two distinct points of \(Y\) and as \(f\) is one-one \& onto, there must exist distinct points \(x_1\) \& \(x_2\) of \(X\) such that \(f(x_1) = y_1\) \& \(f(x_2) = y_2\).

\[ \text{(1)} \]

Again, \(f\), being \(\vartheta\)-open, provides that \(f(G)\) \& \(f(H)\) are \(\sigma\)-\(\vartheta\)-open sets such that \(f(x_1) = y_1\) \& \(f(x_2) = y_2\).

\[ \text{(2)} \]

Above relations show that \((Y,\sigma)\) is also a \(\vartheta\) - T\(_1\) space.

Again, as the property of being \(\vartheta\) - T\(_1\) space is preserved under one-one \& onto \& \(\vartheta\)-open mapping, so it is a topological property.

Hence, the theorem.

**Theorem (1.6): The property of a space being a \(\vartheta\) - T\(_2\) space is a topological property where \(\vartheta = p,s,\alpha,\beta\).**

**Proof:** Let \((X,T)\) be a \(\vartheta\) - T\(_2\) space and \((Y,\sigma)\) be any other topological space such that \(f(X,T)\rightarrow(Y,\sigma)\) is one-one onto \& \(\vartheta\)-open mapping from \((X,T)\) to \((Y,\sigma)\).

It is required to prove that \((Y,\sigma)\) is also a \(\vartheta\) - T\(_2\) space where \(\vartheta = p,s,\alpha,\beta\).

Let \(y_1\) \& \(y_2\) be any two distinct points of \(Y\) and as \(f\) is one-one \& onto, there must exist distinct points \(x_1\) \& \(x_2\) of \(X\) such that \(f(x_1) = y_1\) \& \(f(x_2) = y_2\).

\[ \text{(1)} \]

Since, \((X,T)\) is a \(\vartheta\) - T\(_2\) space so there exists a \(\vartheta\)-open set \(G\) \& \(H\) in manner such that \(x_1 \in G\), \(x_2 \notin G\) \& \(x_1 \notin H\).

\[ \text{(2)} \]

Above relations show that \((Y,\sigma)\) is also a \(\vartheta\) - T\(_2\) space.

Again, as the property of being \(\vartheta\) - T\(_2\) space is preserved under one-one \& onto \& \(\vartheta\)-open mapping, so it is a topological property.

Hence, the theorem.
Again, as the property of being $\mathcal{O} - T_3$ space is preserved under one-one, onto & $\mathcal{O}$-open mapping, so it is a topological property. 
Hence, the theorem.

§2. $\mathcal{O} - R_0$ spaces where $\mathcal{O} = p.s.a$ & $\beta$.

In this section, the notion of $\mathcal{O} - R_0$ spaces where $\mathcal{O}$ stands for p,s,a,$\beta$ is introduced and some basic properties are discussed. But before we take up it, we project the notion of the $\mathcal{O}$-kernel of a set A of a space $(X,T)$ and the $\mathcal{O}$-kernel of a point x of a space $(X,T)$ in the following manner:

**Definition (2.1):**

In a topological space $(X,T)$, if $A \subseteq X$, then the $\mathcal{O}$ - kernel of A, denoted by $\mathcal{O} - ker(A)$, is defined to be the set $\mathcal{O} - ker(A) = \cap \{ \emptyset \in \mathcal{O}O(X,T) | A \subseteq \emptyset \}$. 

**Definition (2.2):**

If x be a point of a topological space $(X,T)$, then the $\mathcal{O}$ - kernel of x, denoted by $\mathcal{O} - ker(x)$, is defined to be the set $\mathcal{O} - ker(x) = \cap \{ \emptyset \in \mathcal{O}O(X,T) | x \in \emptyset \}$. 

**Lemma (2.1):**

If A be a subset of of a topological space $(X,T)$, then $\mathcal{O} - ker(A) = \cap \{ x \in X | \mathcal{O} - cl(x) \cap A = \emptyset \}$. 

**Proof:** Let $x \in \mathcal{O} - ker(A)$ where $A \subseteq X$ & $(X,T)$ is a topological space. On the contrary, we assume that $\mathcal{O} - cl(x) \cap A = \emptyset$. Hence, $x \in X - (\mathcal{O} - cl(x))$ which is a $\mathcal{O}$-open set containing A. This is impossible as $x \notin \mathcal{O} - ker(A)$. Consequently, $\mathcal{O} - cl(x) \cap A \neq \emptyset$. 

Again let $\mathcal{O} - cl(x) \cap A \neq \emptyset$ exist and at the same time let $x \notin \mathcal{O} - ker(A)$. This means that there exists a $\mathcal{O}$-open set B containing A and $x \notin B$. 

Let $y \in \mathcal{O} - cl(x) \cap A$, therefore, B is a $\mathcal{O} - nbhd$ of y for which $x \notin B$. By this contradiction, we have $x \in \mathcal{O} - ker(A)$. 

Hence, $p - ker(A) = \cap \{ x \in X | \mathcal{O} - cl(x) \cap A \neq \emptyset \}$. 

**Definition (2.3):**

A topological space $(X,T)$ is said to be a $\mathcal{O} - R_0$ space if every $\mathcal{O} - open$ set contains the $\mathcal{O} - closure$ of each of its singletons, where $\mathcal{O} = p.s.a$ & $\beta$. 

The implications between $\mathcal{O} - R_0$ spaces are indicated by the following diagram:

$$R_0 \text{ space } \\ \Rightarrow \alpha-R_0 \text{ space } \\ \Rightarrow \downarrow \text{ s-R_0 space } \\ \Rightarrow \p - R_0 \text{ space } \\ \beta - R_0 \text{ space.}$$ 

We, however, know that a $R_0$-space is a topological space in which the closure of the singleton of every point of an open set is contained in that set. 

None of the above implications in the diagram is reversible, as illustrated by the following examples:

**Example (2.1):**

Let $X = \{a,b,c\}$, $T = \{\emptyset,\{a\},\{a,b\},\{a,b,c\},\{c,a\},X\}$. Then $PO(X,T) = \{\emptyset,\{a\},\{a,b\},\{a,b,c\},\{c,a\},X\}$. 

& $PC(X,T) = \{\emptyset,\{a\},\{a,b\},\{a,b,c\},\{c,a\},X\}$. 

Hence, $(X,T)$ is a $\p - R_0$ space. 

Again, $aO(X,T) = \{\emptyset,\{a\},\{a,b\},\{a,b,c\},\{c,a\},X\} = sO(X,T)$. 

& $aC(X,T) = \{\emptyset,\{a\},\{a,b\},\{a,b,c\},\{c,a\},X\} = sC(X,T)$. 

Since, $\alpha-cl(\{a\}) = X \not\subseteq \{a,b\} \in aO(X,T)$, hence, $(X,T)$ is not a $\alpha-R_0$ space. 

Similar is the reason for $(X,T)$ to be not a $s-R_0$ space.

**Example (2.2):**

Let $X = \{a,b,c\}$, $T = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$. 

Then $sO(X,T) = \{\emptyset,\{a\},\{b\},\{a,b\},\{b,c\},\{c,a\},X\} = \beta O(X,T)$. 

& $sC(X,T) = \{\emptyset,\{b\},\{a\},\{a,c\},\{b,c\},X\} = \beta C(X,T)$. 

Hence, $(X,T)$ is a $s-R_0$ space as well as $\beta-R_0$ space. 

Again, $PO(X,T) = \{\emptyset,\{a\},\{b\},\{a,b\},X\} = aO(X,T)$.

& $PC(X,T) = \{\emptyset,\{a\},\{b\},\{b,c\},X\} = aC(X,T)$. 


Since, pcl({a}) = {a,c} $\not\subset$ {a,b} $\in$ PO(X,T), hence, (X,T) is not a p-R_0 space.

Similar is the reason for (X,T) to be not an $\alpha$-R_0 space.

**Remark (2.1):**

1. The concepts of p-R_0 space and s-R_0 are independent. Example (2.1) shows that the space (X,T) is a p-R_0 but s-R_0 where as in example (2.2), the space (X,T) is s-R_0 but not p-R_0.

2. The notion of $\alpha$-R_0 does not imply the notion of R_0 as it is shown by the following example.

**Example (2.3):**

Let X be an infinite set and p $\in X$ be a fixed point. Let T = {G & G $\subseteq$ X $\setminus$ {p} & G is finite}. It can be observed that if G is an open set and x $\in$ G, then cl({x}) = X $\setminus$ G. So, (X,T) is not a R_0 space but as X is $\alpha$-T_1; so every {x} is $\alpha$-closed so $\alpha$-cl({x}) = {x} $\subset$ G, $\forall$ x $\in$ G & G $\in$ aO(X,T). Hence, (X,T) is an $\alpha$-R_0 space.

**Example (2.4):**

Let X = {a,b,c,d,e}. T = {φ, {a}, {b}, {a,b}, {a,c}, {a,d}, {a,b,c}, {a,b,d}, {a,c,d}, {a,b,c,d}, X}. & T’ = {φ, {a}, {e}, {b,e}, {a,b,e}, {a,c,e}, {a,d,e}, {a,b,c,e}, {a,b,e}, {b,c,d,e}, {b,c,d,e}, X}. Then PO(X,T) = {φ, {a}, {b}, {c}, {d}, {a,b}, {a,c}, {a,d}, {b,c}, {b,d}, {c,d}, {a,b,c}, {a,b,d}, {a,c,d}, {a,b,c,d}, X}. PO(X,T) = {φ, {a}, {b}, {c}, {d}, {e}, {a,b}, {a,c}, {a,d}, {b,c}, {b,d}, {c,d}, {a,b,c}, {a,b,d}, {a,c,d}, {a,b,c,d}, X}. Since, p-cl({b}) = {b,e} $\not\in$ {a,b,c,d} $\in$ PO(X,T), Hence, (X,T) is not a p-R_0 space.

Again, βO(X,T) = P(X) - {φ, {a}, {c}, {d}, {e}} & βC(X,T) = P(X) - {{b,c,d}, {a,b,c,d}, X}. Since, (X,T) is a β-T_1 space so every {x} is β-closed which means that β-cl({x}) = {x} $\subset$ G, $\forall$ x $\in$ G & G $\in$ aO(X,T). Consequently, (X,T) is a β-R_0 space.

Next, SO(X,T) = {φ, X, {a}, {b}, {a,b}, {b,c}, {a,b,c}, {a,b,d}, {a,c,d}, {a,b,c,d}, {a,b,c,d}, {a,b,c,d}, {a,c,d}, X}. SC(X,T) = {φ, X, {a}, {b}, {c}, {d}, {a,b}, {a,c}, {a,d}, {b,c}, {b,c}, {c,d}, {a,b,c}, {a,b,d}, {a,c,d}, {a,b,c,d}, X}. Here, (X,T) is a s-R_0 space.

Also, aO(X,T) = T & aC(X,T) = {φ, X, {a}, {b}, {c}, {d}, {a,b}, {a,c}, {a,d}, {b,c}, {b,c}, {c,d}, {a,b,c}, {a,b,d}, {a,c,d}, {a,b,c,d}, X}. Hence, (X,T) is not an $\alpha$-R_0 space.

We now, mention the following lemmas with proofs, useful in the sequel.

**Lemma (2.2):** In a topological space (X,T), for each pair of distinct points x, y $\in$ X, x $\in$ $\phi$ - cl(y) $\Rightarrow$ y $\in$ $\phi$ - ker(x), where $\phi$ = p, s, $\alpha$ & $\beta$.

**Proof:**

Suppose that y $\in$ $\phi$ - ker(x). Then there exists a $\phi$ - open set V containing x such that y $\in$ V. Therefore, we have x $\not\in$ $\phi$ - cl(y).

This means that y $\in$ $\phi$ - ker(x) $\Rightarrow$ x $\not\in$ $\phi$ - cl(y).

i.e. $\forall$ x $\in$ $\phi$ - cl(y) $\Rightarrow$ $\exists$ y $\in$ $\phi$ - ker(x)

i.e. $\forall$ x $\in$ $\phi$ - cl(y) $\Rightarrow$ y $\in$ $\phi$ - ker(x).

Similar is the argument for the proof of the converse i.e.

y $\in$ $\phi$ - ker(x) $\Rightarrow$ x $\in$ $\phi$ - cl(y).

Hence, the theorem.

**Lemma (2.3):** The following statement are equivalent for each pair of points x & y in a topological space (X,T):

(a) $\phi$ - ker({x}) $\neq$ $\phi$ - ker({y}).

(b) $\phi$ - cl({x}) $\neq$ $\phi$ - cl({y}). Where $\phi$ = p, s, $\alpha$ & $\beta$.

**Proof:** (a) $\Rightarrow$(b):

Suppose that $\phi$ - ker({x}) $\neq$ $\phi$ - ker({y}), then there exists a point z in X such that z $\in$ $\phi$ - ker({x}) and z $\notin$ $\phi$ - ker({y}). Since z $\in$ $\phi$ - ker({x}), hence x $\in$ $\phi$ - cl({x}). This means that

{z} $\cap$ x $\in$ $\phi$ - cl({z}).

By z $\notin$ $\phi$ - ker({y}), we have {y} $\not\subset$ $\phi$ - cl({z}) $= \phi$.

Since, x $\in$ $\phi$ - cl({z}), $\phi$ - cl({x}) $\subset$ $\phi$ - cl({z}) and {y} $\cap$ $\phi$ - cl({x}) = $\phi$.

Hence, $\phi$ - cl({x}) $\neq$ $\phi$ - cl({y}).
(b) \(\Rightarrow (a)\): Suppose that \(\varnothing - \text{cl}(\{x\}) \neq \varnothing - \text{cl}(\{y\})\). Then there exists a point \(z\) in \(X\) such that \(z \in \varnothing - \text{cl}(\{x\})\) and \(z \not\in \varnothing - \text{cl}(\{y\})\). There, there exists a \(\varnothing\) - open set containing \(z\) and therefore \(x\) but not \(y\) i.e. \(y \not\in \varnothing - \text{ker}(\{x\})\). Hence, \(\varnothing - \text{ker}(\{x\}) \neq \varnothing - \text{ker}(\{y\})\). Hence, the theorem.

**Theorem (2.1):**
A space \((X, T)\) is a \(\varnothing\)-R_{0} space if and only if for each pair \(x, y\) of distinct points in \(X\),
\[\varnothing - \text{cl}(\{x\}) \cap \varnothing - \text{cl}(\{y\}) = \varnothing \quad \text{or} \quad \{x, y\} \subset \varnothing - \text{cl}(\{x\}) \cap \varnothing - \text{cl}(\{y\}) \]
where \(\varnothing = p, s, a \& \beta\).

**Proof:** (b): Suppose that \(\varnothing - \text{cl}(\{x\}) \neq \varnothing - \text{cl}(\{y\})\). Then there exists a point \(z\) in \(X\) such that \(z \in \varnothing - \text{cl}(\{x\})\) and \(z \not\in \varnothing - \text{cl}(\{y\})\). Hence, \(\varnothing - \text{ker}(\{x\}) \neq \varnothing - \text{ker}(\{y\})\). Hence, the theorem.

**Sufficiency :**
Let \(U\) be a \(\varnothing\) - open set and \(x, y \in U\). Suppose that \(\varnothing - \text{cl}(\{x\}) \subset U\). There is a point \(y \in \varnothing - \text{cl}(\{x\}) \subset U\) such that \(y \subset U\) and \(\varnothing - \text{cl}(\{y\}) \cap U = \varnothing\).

Hence, for each pair of distinct points \(x, y\) of \(X\), we have \(\varnothing - \text{cl}(\{x\}) \cap \varnothing - \text{cl}(\{y\}) = \varnothing \) or \(\{x, y\} \subset \varnothing - \text{cl}(\{x\}) \cap \varnothing - \text{cl}(\{y\})\).

**Theorem (2.2):** For a topological space \((X, T)\), the following properties are equivalent:
(a) \((X, T)\) is a \(\varnothing - R_{0}\) space;
(b) \(\varnothing - \text{cl}(\{x\}) = \varnothing - \text{ker}(\{x\}), \forall \ x \in X\), where \(\varnothing = p, s, a \& \beta\).

**Proof:** (a) \(\Rightarrow (b)\):
Let \((X, T)\) be a \(\varnothing - R_{0}\) space.

By definition \((2.2)\), for any \(x \in X\), we have \(\varnothing - \text{ker}(\{x\}) = \cap \{\varnothing \in \varnothing O(X, T) | x \in \varnothing\} \).

And by definition \((2.3)\), each \(\varnothing\) - open set \(\theta\) containing \(x\) contains \(\varnothing - \text{cl}(\{x\})\).

Hence, \(\varnothing - \text{cl}(\{x\}) \subset \varnothing - \text{ker}(\{x\})\).

Let \(y \in \varnothing - \text{ker}(\{x\})\), then \(x \in \varnothing - \text{ker}(\{y\})\) by lemma \((2.2)\), and so \(\varnothing - \text{cl}(\{x\}) \subset \varnothing - \text{cl}(\{y\})\).

Therefore, \(y \in \varnothing - \text{cl}(\{x\})\). These mean that \(\varnothing - \text{ker}(\{x\}) \subset \varnothing - \text{cl}(\{x\})\).

Hence, \(\varnothing - \text{cl}(\{x\}) = \varnothing - \text{ker}(\{x\})\).

(b) \(\Rightarrow (a)\):
Suppose that for a topological space \((X, T), \varnothing - \text{cl}(\{x\}) = \varnothing - \text{ker}(\{x\}) \quad \forall \ x \in X\).

Let \(G\) be any \(\varnothing\) - open set in \((X, T)\), then for every \(p \in G, \varnothing - \text{ker}(\{p\}) = \cap \{G \in \varnothing O(X, T) | p \in G\}\). But \(\varnothing - \text{cl}(\{p\}) = \varnothing - \text{ker}(\{p\})\) by hypothesis. Hence, combining these two, we observe that for every \(p \in G \subseteq \varnothing O(X, T)\), \(\varnothing - \text{cl}(\{x\}) \in G\). Consequently, \((X, T)\) is a \(\varnothing - R_{0}\) space.

Hence, the theorem.

**Theorem (2.3):** For a topological space \((X, T)\), the following properties are equivalent:
(a) \((X, T)\) is a \(\varnothing - R_{0}\) space.
(b) If \(\varnothing\) is \(\varnothing\) - closed, then \(\varnothing = \varnothing - \text{ker}(F)\);
(c) If \(\varnothing\) is \(\varnothing\) - closed and \(x \in F\), then \(\varnothing - \text{ker}(\{x\}) \subset F\).
(d) If \(x \in X\), then \(\varnothing - \text{ker}(\{x\}) \subset \varnothing - \text{cl}(\{x\})\).

**Proof:** (a) \(\Rightarrow (b)\):
(a) Let \((X,T)\) be a \(\wp - R_0\) space \& \(F\), a \(\wp\)-closed set. Let \(x \notin F\), then \(F^c\) is a \(\wp\)-open set containing \(x\), so that 
\[
\wp - \text{cl}(\{x\}) \subset F^c
\]
as \((X,T)\) is a \(\wp - R_0\). This means that \(\wp - \text{cl}(\{x\}) \cap F = \emptyset\) and by lemma (2.1), \(x \notin \wp - \text{ker}(F)\).
Therefore, \(\wp - \text{ker}(F) = F\).
(b) \(\Rightarrow\) (c):
In general, \(A \subset B \Rightarrow \wp - \text{ker}(A) \subset \wp - \text{ker}(B)\), Hence, it follows that for \(x \in F\), \(\{x\} \subset F \Rightarrow \wp - \text{ker}(\{x\}) \subset \wp - \text{ker}(F) = F\) as \(F\) is \(\wp\)-closed.
(c) \(\Rightarrow\) (d):
Since, \(x \in \wp - \text{cl}(\{x\})\) and \(\wp - \text{cl}(\{x\})\) is \(\wp\)-closed, hence, using (c) we get 
\[
\wp - \text{ker}(\{x\}) \subset \wp - \text{cl}(\{x\}).
\]
(d) \(\Rightarrow\) (a):
Let \((X,T)\) be a topological space in which \(\wp - \text{ker}(\{x\}) \subset \wp - \text{cl}(\{x\})\) for every \(x \in X\).
Let \(y \in \wp - \text{cl}(\{x\})\), then \(x \in \wp - \text{ker}(\{y\})\), since, \(y \in \wp - \text{cl}(\{y\})\) and \(\wp - \text{cl}(\{y\})\) is \(\wp\)-closed, by hypothesis \(x \in \wp - \text{ker}(\{y\}) \subset \wp - \text{cl}(\{y\})\). Therefore \(y \in \wp - \text{cl}(\{x\}) \Rightarrow x \in \wp - \text{cl}(\{y\})\). Similarly, \(x \in \wp - \text{cl}(\{y\})\) implies \(y \in \wp - \text{cl}(\{x\})\). Thus \((X,T)\) is \(\wp-R_0\) space, using theorem (2.4).
Hence, the theorem.

**Theorem (2.4):** for a topological space \((X,T)\), the following properties are equivalent:
(a) \((X,T)\) is a \(\wp - R_0\) space;
(b) For any points \(x, y\) of \(X\), \(x \in \wp - \text{cl}(\{y\}) \iff y \in \wp - \text{cl}(\{x\})\).

**Proof:** (a) \(\Rightarrow\) (b):
Let \((X,T)\) be a \(\wp - R_0\) space. Let \(x \& y\) be any two points of \(X\). Assume that \(x \in \wp - \text{cl}(\{y\})\) and \(D\) is any \(\wp\)-open set such that \(y \in D\).
Now, by hypothesis, \(x \in D\). Therefore, every \(\wp\)-open set containing \(y\) contains \(x\). Hence, \(y \in \wp - \text{cl}(\{x\})\) i.e. \(x \in \wp - \text{cl}(\{y\}) \Rightarrow y \in \wp - \text{cl}(\{x\})\). The converse is obvious and \(x \in \wp - \text{cl}(\{y\}) \iff y \in \wp - \text{cl}(\{x\})\).
(b) \(\Rightarrow\) (a):
Let \(U\) be \(\wp\)-open set and \(x \in U\). If \(y \notin U\), then \(x \notin \wp - \text{cl}(\{y\})\) and hence, \(y \notin \wp - \text{cl}(\{x\})\). This implies that \(\wp - \text{cl}(\{x\}) \subset U\). Hence, \((X,T)\) is a \(\wp-R_0\) space.
Hence, the theorem.

§3. \(\wp - R_1\) spaces where \(\wp = p,s,\alpha & \beta\).
This section includes the notion of \(\wp - R_1\) spaces where \(\wp\) stands for \(p,s,\alpha & \beta\) and their basic properties.

**Definition (3.1):**
A topological space \((X,T)\) is said to be a \(\wp-R_1\) space if for each pair of distinct points \(x \& y\) of \(X\) with \(\wp - \text{cl}(\{x\}) \neq \wp - \text{cl}(\{y\})\), there exist disjoint pair of \(\wp\)-open sets \(U\) and \(V\) such that \(\wp - \text{cl}(\{x\}) \subset U\) \& \(\wp - \text{cl}(\{y\}) \subset V\), where \(\wp = p,s,\alpha & \beta\).

**Theorem (3.1):** If \((X,T)\) is a \(\wp-R_1\) space, then it is a \(\wp-R_0\) space.

**Proof:** Suppose that \((X,T)\) is a \(\wp-R_1\) space where \(\wp = p,s,\alpha & \beta\).
Let \(U\) be a \(\wp\)-open set and \(x \in U\), then for each point \(y \in U\), \(\wp - \text{cl}(\{x\}) \neq \wp - \text{cl}(\{y\})\).
Since, \((X,T)\) is a \(\wp-R_1\) space, there exist a pair of \(\wp\)-open sets \(U_x\) \& \(V_x\) such that \(\wp - \text{cl}(\{x\}) \subset U_x\) \& \(\wp - \text{cl}(\{y\}) \subset V_x\), \(U_x \cap V_x = \emptyset\).
Let \(A = \bigcup \{V_y:y \in U\}\). Then \(U \subset A, x \in A\) and \(A\) is a \(\wp\)-open set.
Therefore, \(\wp - \text{cl}(\{x\}) \subset A\subset U\) which means that \((X,T)\) is a \(\wp-R_0\) space.
Hence, the theorem.

**Example (3.1):**
If \(p\) be a fixed point of \((X,T)\) with \(T\) as the co-finite topology on \(X\) given as \(T = \{\{x,G\}_{x \in X - \{p\}} \& G^c\text{ is finite.}\}, then the space \((X,T)\) is \(\wp-R_0\) but it is not \(\wp-R_1\) where \(\wp = p,s,\alpha & \beta\).

**Theorem (3.2):**
A space \((X,T)\) is a \(\emptyset - R_1\) space iff for each pair of distinct points \(x \& y\) of \(X\) with \(\emptyset - \text{cl} \{\{x\}\} \neq \emptyset - \text{cl} \{\{y\}\}\), there exist disjoint pair of \(\emptyset - \)open sets \(U\) and \(V\) such that \(x \in U, y \in V \& U \cap V = \emptyset\).

**Necessity:**
Let \((X,T)\) be a \(\emptyset - R_1\) space. By definition (3.1), for each pair of distinct points \(x \& y\) of \(X\) with \(\emptyset - \text{cl} \{\{x\}\} \neq \emptyset - \text{cl} \{\{y\}\}\), there can always be obtained disjoint pair of \(\emptyset - \)open sets \(U\) and \(V\) such that \(\emptyset - \text{cl} \{\{x\}\} \subseteq U \& \emptyset - \text{cl} \{\{y\}\} \subseteq V\) where \(U \cap V = \emptyset\). We, however, know that \(p \in \emptyset - \text{cl} \{\{p\}\}, \forall p \in X\). Hence, \(x \in U, y \in V \& U \cap V = \emptyset\).

**Sufficiency:**
Let \(x, y \in X\) and \(x \neq y\) such that \(\emptyset - \text{cl} \{\{x\}\} \neq \emptyset - \text{cl} \{\{y\}\}\). Also let \(U \& V \) be disjoint \(\emptyset - \)open sets for which \(x \in U, y \in V\). Since, \(U \cap V = \emptyset\), hence, \(x \in \emptyset - \text{cl} \{\{x\}\} \subseteq U \& y \in \emptyset - \text{cl} \{\{y\}\} \subseteq V\). Consequently, \((X,T)\) is a \(\emptyset - R_1\) space.

**Corollary (3.1):**
Every \(\emptyset - T_2\) space is \(\emptyset - R_1\) space, but the converse is not true. However, we have the following result.

**Theorem (3.3):**
Every \(\emptyset - T_1 \& \emptyset - R_1\) space is \(\emptyset - T_2\) space.

**Proof:**
Let \((X,T)\) be a \(\emptyset - T_1\) as well as \(\emptyset - R_1\) space. Since, \((X,T)\) is a \(\emptyset - T_1\) space, hence, \(\emptyset - \text{cl} \{\{x\}\} = \{x\} \neq \{y\} = \emptyset - \text{cl} \{\{y\}\}\) for \(x, y \in X \& x \neq y\).

Now, theorem (3.2) provides that as \((X,T)\) is a \(\emptyset - R_1\) space and here, \(x, y \in X\) and \(x \neq y\) such that \(\emptyset - \text{cl} \{\{x\}\} \neq \emptyset - \text{cl} \{\{y\}\}\), so there exist \(\emptyset - \)open sets \(U \& V\) such that \(x \in U, y \in V \& U \cap V = \emptyset\). Consequently, \((X,T)\) is a \(\emptyset - T_2\) space.

Hence, the theorem.

**Theorem (3.4):**
For a topological space \((X,T)\), the following properties are equivalent:

(a) \((X,T)\) is a \(\emptyset - R_1\) space;
(b) For any two points \(x, y \in X\) with \(\emptyset - \text{cl} \{\{x\}\} \neq \emptyset - \text{cl} \{\{y\}\}\), there exist \(\emptyset - \)closed sets \(F_1 \& F_2\) such that \(x \in F_1, y \in F_2\) and \(F_1 \cup F_2 = X\), where \(\emptyset = p, s, \alpha \& \beta\).

**Proof:**
(a) \(\Rightarrow\) (b):
Suppose that \((X,T)\) is a \(\emptyset - R_1\) space. Let \(x, y \in X\) and \(x \neq y\) and with \(\emptyset - \text{cl} \{\{x\}\} \neq \emptyset - \text{cl} \{\{y\}\}\), by Theorem (3.2), there exist \(\emptyset - \)open sets \(U \& V\) such that \(x \in U, y \in V\). Then, \(F_1 = V^c\) is a \(\emptyset - \)closed set & \(F_2 = U^c\) is also \(\emptyset - \)closed set such that \(x \in F_1, y \in F_2\) and \(F_1 \cup F_2 = X\).

(b) \(\Rightarrow\) (a):
Let \(x, y \in X\) such that \(\emptyset - \text{cl} \{\{x\}\} \neq \emptyset - \text{cl} \{\{y\}\}\). This means that \(\emptyset - \text{cl} \{\{x\}\} \cap \emptyset - \text{cl} \{\{y\}\} = \emptyset\).

By the assumption (b), there exist \(\emptyset - \)closed sets \(F_1 \& F_2\) such that \(x \in F_1, y \in F_2\) and \(F_1 \cup F_2 = X\).

Therefore, \(x \in F_2^c = U = \emptyset\) - open set.

& \(y \in F_1^c = V = \emptyset\) - open set.

Also \(U \cap V = \emptyset\).

These facts indicate that \(x \in \emptyset - \text{cl} \{\{x\}\} \subseteq U \& y \in \emptyset - \text{cl} \{\{y\}\} \subseteq V\) such that \(U \cap V = \emptyset\). Consequently, \((X,T)\) is a \(\emptyset - R_1\) space.

Hence the theorem.

§4. \(\emptyset\) - symmetry of A space & \(\emptyset\) - generalized closed set:
We, now, define \(\emptyset\) - symmetry of a space & \((X,T)\) & \(\emptyset\) - generalized closed set (briefly \(\emptyset g\)-closed set) in a space \((X,T)\) as:

**Definition (4.1):** A space \((X,T)\) is said to be \(\emptyset\) - symmetric if for every pair of points \(x, y\) in \(X\), \(x \in \emptyset - \text{cl} \{\{y\}\}\) \(\Rightarrow\) \(y \in \emptyset - \text{cl} \{\{x\}\}\) where \(\emptyset = p, s, \alpha \& \beta\).

**Definition (4.2):** A subset \(A\) of a space \((X,T)\) is said to be a \(\emptyset\) - generalized closed set (briefly \(\emptyset g\)-closed set) if \(\emptyset - \text{cl} \{\{A\}\} \subseteq U\) whenever \(A \subseteq U \& U \) is \(\emptyset\) - open in \(X\)
Lemma (4.1): Every \( \varnothing \)-closed set is a \( \varnothing \varnothing \)-closed set but the converse is not true where \( \varnothing = p,s, \alpha \& \beta \).

Proof:
It follows from the fact that whenever \( A \) is \( \varnothing \)-closed set, we have \( \varnothing \mathcal{cl}(A) = A \) for \( \varnothing = p,s, \alpha \& \beta \), so the criteria \( \varnothing \mathcal{cl} \{\{A\}\} \subset U \) whenever \( A \subseteq U \& U \) is \( \varnothing \)-open exists & \( A \) turns to be a \( \varnothing \varnothing \)-closed set.

But the converse need not to be true as illustrated by the following example:

Let \( X = \{a,b,c,d\} \) and \( T = \{\varnothing,\{a\},\{a,b\},\{c,d\},\{a,c,d\},X\} \)

Here closed sets are : \( \varnothing,\{a\},\{a,b\},\{c,d\},\{a,c,d\},X \).

Then, \( T_s = \{\varnothing,\{a\},\{a,b\},\{c,d\},\{a,c,d\},X\}, \) & \( T_s^C = \varnothing,\{b\},\{a,b\},\{c,d\},\{b,c,d\},X \).

Now, \( T_s^C \) = the class of all \( \varnothing \)g-closed sets.

\( = \{\varnothing,\{b\},\{a\},\{a,b\},\{c\},\{b,c\},\{b,d\},\{a,b,c\},\{a,b,d\},\{b,c,d\},X\}. \)

Therefore, \( \{c\},\{d\},\{b,c\},\{b,d\},\{a,b,c\},\{a,b,d\} \) are \( \varnothing \)g-closed sets but not \( s \)-closed.

Also, \( T_s = T_s^C \) & \( T_s^C = T_s^C \) which show that \( \{c\},\{d\},\{b,c\},\{b,d\},\{a,b,c\},\{a,b,d\} \) are \( \varnothing \)g-closed sets but not \( \alpha \)-closed sets.

Similarly, the other cases can be dealt with.

Theorem (4.1): A space \((X,T)\) is \( \varnothing \)-symmetric if and only if \( \{x\} \) is \( \varnothing \varnothing \)-closed for each \( x \in X \), where \( \varnothing = p,s, \alpha \& \beta \).

Proof: Necessity: Let \((X,T)\) be \( \varnothing \)-symmetric, then for distinct points \( x,y \) of \( X \),

\( y \in \varnothing \mathcal{cl}\{\{x\}\} \Rightarrow x \in \varnothing \mathcal{cl}\{\{y\}\} \) where \( \varnothing = p,s, \alpha \& \beta \).

Let \( \{x\} \subseteq D \) where \( D \) is a \( \varnothing \)-open set in \((X,T)\). Let \( \varnothing \mathcal{cl}\{\{x\}\} \subset D \). This means that \( \varnothing \mathcal{cl}\{\{x\}\} \n D \). Let \( y \in \varnothing \mathcal{cl}\{\{x\}\} \). Now, \( x \in \varnothing \mathcal{cl}\{\{y\}\} \) & \( x \notin D \). But this is a contradiction. Hence, \( \varnothing \mathcal{cl}\{\{x\}\} \subseteq D \) whenever \( \{x\} \subseteq D \& D \) is \( \varnothing \)-open. Consequently, \( \{x\} \) is a \( \varnothing \varnothing \)-closed set.

Sufficiency:
Let in a space \((X,T)\), each \( \{x\} \) is a \( \varnothing \varnothing \)-closed set where \( x \in X \). Let \( x,y \in X \) & \( x \neq y \) such that \( x \in \varnothing \mathcal{cl}\{\{y\}\} \) but \( y \notin \varnothing \mathcal{cl}\{\{x\}\} \) .

This implies that \( y \in (\varnothing \mathcal{cl}\{\{x\}\}) \) \( \Rightarrow \{y\} \subseteq ((\varnothing \mathcal{cl}\{\{x\}\}) \) \( \Rightarrow \varnothing \mathcal{cl}\{\{y\}\} \subseteq (\varnothing \mathcal{cl}\{\{x\}\}) \) \( \Rightarrow \{y\} \subseteq (\varnothing \mathcal{cl}\{\{x\}\}) \) \( \Rightarrow \{y\} \subseteq \{x\} \) \( \Rightarrow \{y\} \subseteq \{x\} \) \( \Rightarrow \{y\} \subseteq \{x\} \) \( \Rightarrow \{y\} \subseteq \{x\} \) ; \( \forall x \neq y \). Therefore, the space \((X,T)\) is \( \varnothing \)-symmetric.

Hence the theorem.

Corollary (4.1): If a space \((X,T)\) is \( \varnothing \)-T_1 space, then it is \( \varnothing \)-symmetric, where \( \varnothing = p,s, \alpha \& \beta \).

Proof: In a \( \varnothing \)-T_1 space, singleton sets are \( \varnothing \)-closed by Theorem (1.2), and therefore \( \varnothing \varnothing \)-closed by Lemma (4.1). By Theorem (4.1), the space \((X,T)\) is \( \varnothing \)-symmetric, where \( \varnothing = p,s, \alpha \& \beta \).

Remark (4.1):
The converse of the corollary (4.1) is not necessarily true as shown in the following example:

Let \( X = \{a,b,c,d,e\} \). \( T = \{\varnothing,\{a\},\{b\},\{a,b\},\{c,d\},\{a,c,d\},\{a,b,c,d\},\{b,c,d\},\{a,c,d,e\},\{a,b,c,d\},\{a,c,d\},\{b,c,d\},\{a,b,c\},\{a,b\},\{a\},\{\}X\} \).

Then \( T_s^C = \{\varnothing,\{b\},\{a\},\{c\},\{b,c\},\{a,c\},\{b,c,d\},\{a,b,c,d\},\{a,c,d\},\{b,c,d\},\{a,b,c\},\{a,b\},\{a\},\{\}X\} \).

The space \((X,T)\) is not s-T_1 but s-symmetric.

Theorem (4.2): For a topological space \((X,T)\), the following properties are equivalent:
(a) \((X,T)\) is a \( \varnothing \)-symmetric & \( \varnothing \)-T_0 space;
(b) \((X,T)\) is \( \varnothing \)-T_1 space.
Proof: \((a) \Rightarrow (b)\):
Let \(x, y \in X\) and \(x \neq y\). Since, \((X,T)\) is a \(\mathcal{P} \rightarrow T_0\) space, hence we may assume that \(x \in G_1 \subset \{y\}^c\) for some \(\mathcal{P} \rightarrow\) open set \(G_1\). Then \(x \notin \mathcal{P} \rightarrow cl(\{y\})\). Consequently \(y \notin \mathcal{P} \rightarrow cl(\{x\})\). There exists a \(\mathcal{P} \rightarrow\) open set \(G_2\) such that \(y \in G_2 \subset \{x\}^c\). Therefore, \((X,T)\) is a \(\mathcal{P} \rightarrow T_1\) space.

\((b) \Rightarrow (a)\):
Corollary (4.1) depicts that \((X,T)\), being \(\mathcal{P} \rightarrow T_1\) space is \(\mathcal{P} \rightarrow\) symmetric .

Remark (1.1) provides that \((X,T)\) being \(\mathcal{P} \rightarrow T_1\) space is necessarily \(\mathcal{P} \rightarrow T_0\) space.

The above two facts together establish that \((b) \Rightarrow (a)\).

Hence the theorem.

**Corollary (4.2):** If \((X,T)\) is \(\mathcal{P} \rightarrow\) symmetric, then \((X,T)\) is \(\mathcal{P} \rightarrow T_0 \iff (X,T)\) is \(\mathcal{P} \rightarrow T_1\).

**Proof:** Here, \(\Rightarrow \) follows from Theorem (4.2) and \(\Leftarrow \) follows from Remark (1.1).

### II. Conclusion

An overview of separation axioms by nearly open sets focuses its attention on the literature of \(\mathcal{P} \rightarrow T_k (k=0,1,2 \& \mathcal{P} = p.s, \alpha \& \beta)\) spaces in the compact form in this paper.

The study of \(\mathcal{P} \rightarrow R_0 \& \mathcal{P} \rightarrow R_1\) spaces has been enunciated and the related properties are kept ready at a glance. The \(\mathcal{P} \rightarrow\) symmetry of a topological space along with example and basic results has been exhibited at one place.

The future scope of the overview is to compile the literature & research concern with \(\mathcal{P} \rightarrow T_{1/2}\) spaces\(\mathcal{P} = p.s, \alpha \& \beta\) and the related fundamental properties \& results are to be prepare as a ready reckoning at a glance.

### REFERENCES


