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On Steiner Dominating Sets and Steiner Domination Polynomials of Paths

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ABSTRACT

In this paper, we introduce a new concept of Steiner domination polynomial of a path P_n. The Steiner domination polynomial of P_n is the polynomial $S_\gamma(P_n, x) = \sum_{i=s_\gamma(P_n)}^{n} s_\gamma(P_n, i)x^i$, where $s_\gamma(P_n, i)$ is the number of Steiner dominating sets of P_n of size i and $s_\gamma(P_n)$ is the Steiner domination number of P_n. We obtain some properties of $S_\gamma(P_n, x)$ and its coefficients.

Key words: Steiner dominating set, Steiner domination number, Steiner domination polynomial.

I. Introduction

For a connected graph G and a set $W \subseteq V(G)$, a tree T contained in G is a Steiner tree with respect to W if T is a tree of minimum order with $W \subseteq V(T)$. The set $S(W)$ contains, of all vertices in G that lie on some Steiner tree with respect to W. If $S(W) = V(G)$, then W is called a Steiner set for G. The minimum cardinality among the Steiner sets of G is the Steiner number, $s(G)$. We denote the family of all Steiner sets of a connected graph G with cardinality i by $S(G, i)$.

Each extreme vertex of a graph G belongs to every Steiner set of G. In particular, each end-vertex of G belongs to every Steiner set of G.

A dominating set for a graph $G = (V, X)$ is a subset D of V such that every vertex not in D is adjacent to at least one member of D. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for G.

For a connected graph G, a set of vertices W in G is called a Steiner dominating set if W is both a Steiner set as well as a dominating set. We denote the family of Steiner dominating sets of a connected graph G with cardinality i by $S_\gamma(G, i)$. The minimum cardinality among the Steiner dominating sets of G is the Steiner domination number and is denoted by $s_\gamma(G)$. $\lfloor n \rfloor$ denotes the largest integer less than or equal to n. $[n]$ denotes the set of all positive integers less than or equal to n.

II. Steiner Domination Sets and Polynomial of a Path

Definition 2.1

Let P_n, $n \geq 2$, be a path of n vertices. Let $V(P_n) = \{1, 2, 3, \ldots, n\}$ and $E(P_n) = \{(1, 2), (2, 3), \ldots, (n\text{-}1, n)\}$. Let $S_\gamma(P_n, i)$ be the family of Steiner dominating sets of P_n with cardinality i and let $s_\gamma(P_n, i) = |S_\gamma(P_n, i)|$. Then the Steiner domination polynomial, $S_\gamma(P_n, x)$ of P_n is defined as

$S_\gamma(P_n, x) = \sum_{i=s_\gamma(P_n)}^{n} s_\gamma(P_n, i)x^i$, where $s_\gamma(P_n)$ is the Steiner domination number of P_n.

Lemma 2.2

For any path P_n, $n \geq 2$, with $|V(P_n)| = n$, the Steiner domination number is $s_\gamma(P_n) = \left\lfloor \frac{n+4}{3} \right\rfloor$.

Lemma 2.3

Let P_n, $n \geq 2$, be a path of n vertices. Then:

(i) $s_\gamma(P_n, i) = 0$ if $i < s_\gamma(P_n)$ or $i > n$ and

(ii) $s_\gamma(P_n, i) > 0$ if $\left\lfloor \frac{n+4}{3} \right\rfloor \leq i \leq n$.

Lemma 2.4

Let P_n, $n \geq 2$, be a path of n vertices. Then:

(i) $S_\gamma(P_n, x)$ has no constant term and first degree term.

(ii) $S_\gamma(P_n, x)$ is a strictly increasing function on $[0, \infty)$.

Lemma 2.5

Let P_n, $n \geq 2$, be a path of n vertices.

(i) If $S_\gamma(P_{n-1}, i-1) = \phi$ and $S_\gamma(P_{n-3}, i-1) = \phi$ then $S_\gamma(P_{n-2}, i-1) = \phi$.

(ii) If $S_\gamma(P_{n-1}, i-1) \neq \phi$ and $S_\gamma(P_{n-3}, i-1) \neq \phi$ then $S_\gamma(P_{n-2}, i-1) \neq \phi$.

(iii) If $S_\gamma(P_{n-1}, i-1) = S_\gamma(P_{n-2}, i-1) = S_\gamma(P_{n-3}, i-1) = \phi$, then $S_\gamma(P_n, i) = \phi$.

Proof:

(i) If $S_\gamma(P_{n-1}, i-1) = \phi$ and $S_\gamma(P_{n-3}, i-1) = \phi$ then $i-1 < \left\lfloor \frac{n-1+4}{3} \right\rfloor$ or $i-1 > n-1$ and $i-1 < \left\lfloor \frac{n-3+4}{3} \right\rfloor$ or $i-1 > n-3$.

$\Rightarrow i-1 < \left\lfloor \frac{n-3+4}{3} \right\rfloor$ or $i-1 > n-1$

$\Rightarrow i-1 < \left\lfloor \frac{n-2+4}{3} \right\rfloor$ or $i-1 > n-1 > n-2$.

Hence, $S_\gamma(P_{n-2}, i\text{-}1) = \phi$.

(ii) If $S_\gamma(P_{n-1}, i\text{-}1) \neq \phi$ and $S_\gamma(P_{n-3}, i\text{-}1) \neq \phi$ then $\left\lfloor \frac{n-1+4}{3} \right\rfloor \le i-1 \le n-1$ and

$\left\lfloor \frac{n-3+4}{3} \right\rfloor \le i-1 \le n-3$.

$\Rightarrow \left\lfloor \frac{n-1+4}{3} \right\rfloor \le i-1 \le n-3$

$\Rightarrow \left\lfloor \frac{n-2+4}{3} \right\rfloor \le \left\lfloor \frac{n-1+4}{3} \right\rfloor \le i-1 \le n-3 \le n-2$

$\Rightarrow S_\gamma(P_{n-2}, i\text{-}1) \neq \phi$.

(iii) If $S_\gamma(P_{n-1}, i\text{-}1) = S_\gamma(P_{n-2}, i\text{-}1) = \phi$ and $S_\gamma(P_{n-3}, i\text{-}1) = \phi$, then

$i-1 < \left\lfloor \frac{n-1+4}{3} \right\rfloor$ or $i-1 > n-1$;

$i-1 < \left\lfloor \frac{n-2+4}{3} \right\rfloor$ or $i - 1 > n-2$ and

$i-1 < \left\lfloor \frac{n-3+4}{3} \right\rfloor$ or $i-1 > n-3$.

$\Rightarrow i-1 < \left\lfloor \frac{n-3+4}{3} \right\rfloor$ or $i-1 > n-1$.

$\Rightarrow i < \left\lfloor \frac{n+4}{3} \right\rfloor$ or $i > n$.

$\Rightarrow S_\gamma(P_n, i) = \phi$. □

Lemma 2.6

Let P_n, $n \ge 2$, be a path of n vertices.

If $S_\gamma(P_n, i) \neq \phi$, then

(i) $S_\gamma(P_{n-1}, i-1) = \phi$ and $S_\gamma(P_{n-2}, i-1) = \phi$ if and only if $n = 3k+1$ and $i = k+1$ for any positive integer k.

(ii) $S_\gamma(P_{n-2}, i-1) = \phi$ and $S_\gamma(P_{n-3}, i-1) = \phi$ and $S_\gamma(P_{n-1}, i-1) \neq \phi$ if and only if $i = n$.

(iii) $S_\gamma(P_{n-1}, i-1) \neq \phi$ and $S_\gamma(P_{n-2}, i-1) \neq \phi$ and $S_\gamma(P_{n-3}, i-1) = \phi$ if and only if $i = n-1$.

(iv) $S_\gamma(P_{n-1}, i-1) = \phi$ and $S_\gamma(P_{n-2}, i-1) \neq \phi$ and $S_\gamma(P_{n-3}, i-1) \neq \phi$ if and only if n = $3k$ and $i = k+1$ for any positive integer k.

(v) $S_\gamma(P_{n-1}, i-1) \neq \phi$ and $S_\gamma(P_{n-2}, i-1) \neq \phi$ and $S_\gamma(P_{n-3}, i-1) \neq \phi$ if and only if $\left\lfloor \frac{n+6}{3} \right\rfloor \le i \le n-2$.

Proof:

Since $S_\gamma(P_n, i) \neq \phi$, we have $\left\lfloor \frac{n+4}{3} \right\rfloor \le i \le n$.

(i) If $S_\gamma(P_{n-1}, i-1) = \phi$ and $S_\gamma(P_{n-2}, i-1) = \phi$, then

$i-1 < \left\lfloor \frac{n-1+4}{3} \right\rfloor$ or $i-1 > n-1$ and

$i-1 < \left\lfloor \frac{n-2+4}{3} \right\rfloor$ or $i-1 > n-2$.

$\Rightarrow i-1 < \left\lfloor \frac{n-2+4}{3} \right\rfloor$ or $i-1 > n-1$.

But, $i-1 > n-1$ is not possible, since $i \le n$.

Hence, $i-1 < \left\lfloor \frac{n-2+4}{3} \right\rfloor$. Also $\left\lfloor \frac{n+4}{3} \right\rfloor \le i$

Therefore, $\left\lfloor \frac{n+4}{3} \right\rfloor \le i < \left\lfloor \frac{n+2}{3} \right\rfloor + 1$.

This is possible only when $n = 3k+1$, where k is any positive integer.

Also, $i = \left\lfloor \frac{n+4}{3} \right\rfloor = \left\lfloor \frac{3k+1+4}{3} \right\rfloor = k+1$.

Conversely, let $i = k+1$ and $n = 3k+1$.

Then, $i-1 = \frac{n-1}{3}$.

Hence, $i-1 < \left\lfloor \frac{n-1+4}{3} \right\rfloor$

Therefore, $S_\gamma(P_{n-1}, i\text{-}1) = \phi$.

Also, $i-1 = \frac{n-1}{3}$. $\Rightarrow i-1 < \left\lfloor \frac{n-2+4}{3} \right\rfloor$.

Hence, $S_\gamma(P_{n-2}, i-1) = \phi$.

Therefore, $S_\gamma(P_{n-1}, i-1) = \phi$ and $S_\gamma(P_{n-2}, i-1) = \phi$ if and only if $n = 3k+1$ and $i = k+1$ for any positive integer k.

(ii) If $S_\gamma(P_{n-1}, i-1) \neq \phi$, then $\left\lfloor \frac{n-1+4}{3} \right\rfloor \le i-1 \le n-1$. Hence, $\left\lfloor \frac{n+6}{3} \right\rfloor \le i \le n$.

Also, if $S_\gamma(P_{n-2}, i-1) = \phi$ and $S_\gamma(P_{n-3}, i-1)\ 1) = \phi$, then

$i-1 < \left\lfloor \frac{n-2+4}{3} \right\rfloor$ or $i-1 > n-2$ and

$i-1 < \left\lfloor \frac{n-3+4}{3} \right\rfloor$ or $i-1 > n-3$.

$\Rightarrow i-1 < \left\lfloor \frac{n-3+4}{3} \right\rfloor$ or $i-1 > n-2$

$\Rightarrow i < \left\lfloor \frac{n+4}{3} \right\rfloor$ or $i > n-1$.

But, $i < \left\lfloor \frac{n+4}{3} \right\rfloor$ is not possible, because $\left\lfloor \frac{n+6}{3} \right\rfloor \le$ i.

Hence, $i > n - 1$. Also, $i \le n$.
Therefore, $i = n$.
Conversely, let $i = n$.
Then, $i - 1 = n - 1 > n - 2 > n - 3$.
Hence, $S_\gamma(P_{n-2}, i-1) = \phi$ and $S_\gamma(P_{n-3}, i-1) = \phi$.
Also, $i = n \Rightarrow S_\gamma(P_{n-1}, i-1) \ne \phi$, since $| S_\gamma(P_{n-1}, i-1) | = 1$.

(iii) If $S_\gamma(P_{n-1}, i-1) \ne \phi$ and $S_\gamma(P_{n-2}, i-1) \ne \phi$, then $\left\lfloor \frac{n-1+4}{3} \right\rfloor \le i-1 \le n-1$ and $\left\lfloor \frac{n-2+4}{3} \right\rfloor \le i-1 \le n-2$

$\Rightarrow \left\lfloor \frac{n-1+4}{3} \right\rfloor \le i-1 \le n-2$

$\Rightarrow \left\lfloor \frac{n+3}{3} \right\rfloor \le i-1 \le n-2$ ……… (1)

Also, if $S_\gamma(P_{n-3}, i-1) = \phi$, then $i-1 < \left\lfloor \frac{n-3+4}{3} \right\rfloor$ or $i-1 > n-3$.

But, $i-1 < \left\lfloor \frac{n+1}{3} \right\rfloor$ is not possible, because $i-1 \ge \left\lfloor \frac{n+3}{3} \right\rfloor$ (by (1)).

Therefore, $i - 1 > n - 3$.
Also, by (1), $i - 1 \le n - 2$
Hence, $i - 1 = n - 2$
$\Rightarrow i = n - 1$
Conversely, let $i = n - 1$.
Then $i - 1 = n - 2 > n - 3$.
$\Rightarrow S_\gamma(P_{n-3}, i-1) = \phi$,
Also, if $i - 1 = n - 2$, then
$S_\gamma(P_{n-2}, i-1) = S_\gamma(P_{n-2}, n-2) \ne \phi$, since $| S_\gamma(P_{n-2}, n-2) | = 1$.
Again, if $i - 1 = n - 2$, then
$S_\gamma(P_{n-1}, i-1) = S_\gamma(P_{n-1}, n-2) \ne \phi$.

(iv) If $S_\gamma(P_{n-2}, i-1) \ne \phi$ and $S_\gamma(P_{n-3}, i-1) \ne \phi$, then

$\left\lfloor \frac{n-2+4}{3} \right\rfloor \le i-1 \le n-2$ and

$\left\lfloor \frac{n-3+4}{3} \right\rfloor \le i-1 \le n-3$

$\Rightarrow \left\lfloor \frac{n-2+4}{3} \right\rfloor \le i-1 \le n-3$

$\Rightarrow \left\lfloor \frac{n+2}{3} \right\rfloor \le i-1 \le n-3$ ……… (2)

Again, if $S_\gamma(P_{n-1}, i-1) = \phi$, then

$i - 1 < \left\lfloor \frac{n-1+4}{3} \right\rfloor$ or $i-1 > n-1$

But, $i - 1 > n - 1$ is not possible, since $i \le n$

Therefore, $i - 1 < \left\lfloor \frac{n+3}{3} \right\rfloor$

Hence, $\left\lfloor \frac{n+2}{3} \right\rfloor \le i-1 < \left\lfloor \frac{n+3}{3} \right\rfloor$.

$\Rightarrow \left\lfloor \frac{n+5}{3} \right\rfloor \le i < \left\lfloor \frac{n+6}{3} \right\rfloor$.

This is possible only when $n = 3k$, for any positive integer k and in this case

$i = \left\lfloor \frac{n+5}{3} \right\rfloor$

Therefore, $i = \left\lfloor \frac{n+5}{3} \right\rfloor = \left\lfloor \frac{3k+5}{3} \right\rfloor$

$\Rightarrow i = k + 1$.
Conversely, let $n = 3k$ and $i = k + 1$
Then, $i = \frac{n}{3} + 1 \Rightarrow i - 1 = \frac{n}{3}$.

$\Rightarrow \left\lfloor \frac{n-3+4}{3} \right\rfloor \le \left\lfloor \frac{n-2+4}{3} \right\rfloor \le i-1 \le n-3 \le n-2$

$\Rightarrow S_\gamma(P_{n-2}, i-1) \ne \phi$ and $S_\gamma(P_{n-3}, i-1) \ne \phi$.

(v) $S_\gamma(P_{n-1}, i-1) \ne \phi$; $S_\gamma(P_{n-2}, i-1) \ne \phi$ and $S_\gamma(P_{n-3}, i-1) \ne \phi$

$\Leftrightarrow \left\lfloor \frac{n-1+4}{3} \right\rfloor \le i-1 \le n-1$;

$\left\lfloor \frac{n-2+4}{3} \right\rfloor \le i-1 \le n-2$ and

$\left\lfloor \frac{n-3+4}{3} \right\rfloor \le i-1 \le n-3$.

$\Leftrightarrow \left\lfloor \frac{n-1+4}{3} \right\rfloor \le i-1 \le n-3$

$\Leftrightarrow \left\lfloor \frac{n+3}{3} \right\rfloor \le i-1 \le n-3$

$\Leftrightarrow \left\lfloor \frac{n+6}{3} \right\rfloor \le i \le n-2$. □

Theorem 2.7

Let P_n, $n \ge 2$, be a path of n vertices.

(i) For every $n \ge 1$, $S_\gamma(P_{3n+1}, n+1) = \{\{1, 4, 7, 10, \ldots, 3n-2, 3n+1\}\}$.

(ii) If $S_\gamma(P_{n-2}, i-1) = \phi$ and $S_\gamma(P_{n-3}, i-1) = \phi$ and $S_\gamma(P_{n-1}, i-1) \ne \phi$, then $S_\gamma(P_n, i) = S_\gamma(P_n, n) = \{\{1, 2, 3, 4, \ldots, n\}\}$, for all $n \ge 2$.

(iii) If $S_\gamma(P_{n-1}, i-1) \ne \phi$ and $S_\gamma(P_{n-2}, i-1) \ne \phi$ and $S_\gamma(P_{n-3}, i-1) = \phi$, then $S_\gamma(P_n, i) = S_\gamma(P_n, i-1) = \{[n]\text{-}\{x\}\}/ x \in [n]$ and $x \ne 1, x \ne n\}$, for all $n \ge 3$.

(iv) If $S_\gamma(P_{n-1}, i-1) \ne \phi$ and $S_\gamma(P_{n-2}, i-1) \ne \phi$ and $S_\gamma(P_{n-3}, i-1) \ne \phi$, then
$S_\gamma(P_n, i) = \{X \cup \{n\}/ X \in S_\gamma(P_{n-1}, i-1)\} \cup$

$\{Y \cup \{n\} / Y \in S_\gamma(P_{n-2}, i-1)\} \cup$
$\{Z \cup \{n\} / Z \in S_\gamma(P_{n-3}, i-1)\}$.

Proof:

(i) For any n ≥ 1, $S_\gamma(P_{3n+1}, n+1)$ has the only one Steiner dominating set $\{\{1, 4, 7, 10, \ldots, 3n-2, 3n+1\}\}$.

(ii) If $S_\gamma(P_{n-2}, i-1) = \phi$ and $S_\gamma(P_{n-3}, i-1) = \phi$ and $S_\gamma(P_{n-1}, i-1) \neq \phi$, then by lemma 2.6, $i = n$. Therefore,
$S_\gamma(P_n, i) = S_\gamma(P_n, n) = \{\{1, 2, 3, 4, \ldots, n\}\}$.

(iii) If $S_\gamma(P_{n-1}, i-1) \neq \phi$ and $S_\gamma(P_{n-2}, i-1) \neq \phi$ and $S_\gamma(P_{n-3}, i-1) = \phi$, then by lemma 2.6, $i = n-1$. Therefore, $S_\gamma(P_n, i) = S_\gamma(P_n, n-1) = \{[n]-\{x\}\} / x \in [n]$ and $x \neq 1, x \neq n\}$.

(iv) The set $S_\gamma(P_n, i)$ can be constructed from the sets $S_\gamma(P_{n-1}, i-1)$, $S_\gamma(P_{n-2}, i-1)$ and $S_\gamma(P_{n-3}, i-1)$ as follows:
Let X be a Steiner set of P_{n-1} with cardinality $i-1$. All the elements of $S_\gamma(P_{n-1}, i-1)$ end with 1 and $n-1$. Therefore, every X of $S_\gamma(P_{n-1}, i-1)$ together with n belongs to $S_\gamma(P_n, i)$.
Let Y be a Steiner set of P_{n-2} with cardinality $i-1$. All the elements of $S_\gamma(P_{n-2}, i-1)$ end with 1 and $n-2$. Therefore, every Y of $S_\gamma(P_{n-2}, i-1)$ together with n belongs to $S_\gamma(P_n, i)$.
Let Z be a Steiner set of P_{n-3} with cardinality $i-1$. All the elements of $S_\gamma(P_{n-3}, i-1)$ end with 1 and $n-3$. Therefore, every Z of $S_\gamma(P_{n-3}, i-1)$ together with n belongs to $S_\gamma(P_n, i)$.
Hence, $\{X \cup \{n\} / X \in S_\gamma(P_{n-1}, i-1)\} \cup$
$\{Y \cup \{n\} / Y \in S_\gamma(P_{n-2}, i-1)\} \cup$
$\{Z \cup \{n\} / Z \in S_\gamma(P_{n-3}, i-1)$
$\subseteq S_\gamma(P_n, i)$ ……………..(1)
Suppose, $X \in S_\gamma(P_n, i)$. Then $n \in X$. Also, $(n-1)$ or $(n-2)$ or $(n-3) \in X$. If $(n-1) \in X$ then $X = P \cup \{n\}$, where $P \in S_\gamma(P_{n-1}, i-1)$. If $(n-2) \in X$ but $(n-1) \notin X$, then $X = Q \cup \{n\}$, where $Q \in S_\gamma(P_{n-2}, i-1)$. If $(n-3) \in X$ but $(n-1)$ and $(n-2) \notin X$, then $X = R \cup \{n\}$, where $R \in S_\gamma(P_{n-3}, i-1)$.
Hence,
$S_\gamma(P_n, i) \subseteq \{X \cup \{n\}/X \in S_\gamma(P_{n-1}, i-1)\} \cup \{Y \cup \{n\} / Y \in S_\gamma(P_{n-2}, i-1)\} \cup$
$\{Z \cup \{n\} / Z \in S_\gamma(P_{n-3}, i-1)\}$ .... (2)
From (1) and (2), we have,
$S_\gamma(P_n, i) = \{X \cup \{n\} / X \in S_\gamma(P_{n-1}, i-1)\} \cup \{Y \cup \{n\} / Y \in S_\gamma(P_{n-2}, i-1)\} \cup$
$\{Z \cup \{n\} / Z \in S_\gamma(P_{n-3}, i-1)\}$. □

Theorem 2.8

Let P_n, $n \geq 5$, be a path of n vertices. Let $S_\gamma(P_n, i)$ be the family of Steiner dominating sets of cardinality i and $|S_\gamma(P_n, i)| = s_\gamma(P_n, i)$, then

$$s_\gamma(P_n, i) = s_\gamma(P_{n-1}, i-1) + s_\gamma(P_{n-2}, i-1) + s_\gamma(P_{n-3}, i-1).$$

Proof:

There are four cases.

Case (i):

If $S_\gamma(P_{n-1}, i-1) = \phi$ and $S_\gamma(P_{n-3}, i-1) = \phi$ then $S_\gamma(P_{n-2}, i-1) = \phi$, by part (i) of lemma 2.5.
Hence, $S_\gamma(P_n, i) = \phi$, by part (iii) of lemma 2.5.
Therefore, $s_\gamma(P_n, i) = 0 = s_\gamma(P_{n-1}, i-1) + s_\gamma(P_{n-2}, i-1) + s_\gamma(P_{n-3}, i-1)$.

Case (ii):

If $S_\gamma(P_{n-2}, i-1) = \phi$ and $S_\gamma(P_{n-3}, i-1) = \phi$ and $S_\gamma(P_{n-1}, i-1) \neq \phi$, then $i = n$, by lemma 2.6. Therefore, $S_\gamma(P_n, i) = S_\gamma(P_n, n) = \{\{1, 2, 3, \ldots, n\}\}$, by part (ii) of theorem 2.7.
Also, $S_\gamma(P_{n-1}, i-1) = S_\gamma(P_{n-1}, n-1) = \{\{1, 2, 3, \ldots, n-1\}\}$
Therefore, $s(P_n, i) = s_\gamma(P_{n-1}, i-1)$
Hence, $s_\gamma(P_n, i) = s_\gamma(P_{n-1}, i-1) + s_\gamma(P_{n-2}, i-1) + s_\gamma(P_{n-3}, i-1)$.

Case (iii):

If $S_\gamma(P_{n-1}, i-1) \neq \phi$ and $S_\gamma(P_{n-2}, i-1) \neq \phi$ and $S_\gamma(P_{n-3}, i-1) = \phi$, then $i = n-1$, by lemma 2.6.
Therefore,
$S_\gamma(P_n, i) = S_\gamma(P_n, n-1) = \{[n]-\{x\}\} / x \in [n]$ and $x \neq 1, x \neq n\}$, by theorem 2.7.
Hence, $s_\gamma(P_n, i) = n-2$.
Similarly, $s_\gamma(P_{n-1}, i-1) = s_\gamma(P_{n-1}, n-2) = n-3$.
Also, $S_\gamma(P_{n-2}, i-1) = S_\gamma(P_{n-2}, n-2) = \{\{1, 2, 3, \ldots, n-2\}\}$.
Therefore, $s_\gamma(P_{n-2}, i-1) = 1$.
Again, $s_\gamma(P_{n-3}, i-1) = s_\gamma(P_{n-3}, n-2) = 0$, since $i-1 > n-3$
Hence, $s_\gamma(P_n, i) = s_\gamma(P_{n-1}, i-1) + s_\gamma(P_{n-2}, i-1) + s_\gamma(P_{n-3}, i-1)$.

Case (iv):

If $S_\gamma(P_{n-1}, i-1) \neq \phi$ and $S_\gamma(P_{n-2}, i-1) \neq \phi$ and $S_\gamma(P_{n-3}, i-1) \neq \phi$, then
$S_\gamma(P_n, i) = \{X \cup \{n\} / X \in S_\gamma(P_{n-1}, i-1)\} \cup \{Y \cup \{n\} / Y \in S_\gamma(P_{n-2}, i-1)\} \cup$
$\{Z \cup \{n\} / Z \in S_\gamma(P_{n-3}, i-1)\}$, by theorem 2.7,
Hence, $s_\gamma(P_n, i) = s_\gamma(P_{n-1}, i-1) + s_\gamma(P_{n-2}, i-1) + s_\gamma(P_{n-3}, i-1)$.
Therefore, in all cases,

$$s_\gamma(P_n, i) = s_\gamma(P_{n-1}, i-1) + s_\gamma(P_{n-2}, i-1) + s_\gamma(P_{n-3}, i-1).$$ □

Theorem 2.9

Let P_n, $n \geq 5$, be a path of n vertices. Then the Steiner domination polynomial is

$$S_\gamma(P_n, x) = x\,[S_\gamma(P_{n-1}, x) + S_\gamma(P_{n-2}, x) + S_\gamma(P_{n-3}, x)],$$

with the initial values
$S_\gamma(P_2, x) = x^2$, $S_\gamma(P_3, x) = x^2 + x^3$ and
$S_\gamma(P_4, x) = x^2 + 2x^3 + x^4$.

Proof:

By theorem 2.8, we have

$$s_\gamma(P_n, i) = s_\gamma(P_{n-1}, i-1) + s_\gamma(P_{n-2}, i-1) + s_\gamma(P_{n-3}, i-1)$$

When $i = 5$,

$$s_\gamma(P_n, 5) = s_\gamma(P_{n-1}, 4) + s_\gamma(P_{n-2}, 4) + s_\gamma(P_{n-3}, 4)$$

$$\Rightarrow x^5 s_\gamma(P_n, 5) = x^5 s_\gamma(P_{n-1}, 4) + x^5 s_\gamma(P_{n-2}, 4) + x^5 s_\gamma(P_{n-3}, 4)$$

when $i = 6$,

$$s_\gamma(P_n, 6) = s_\gamma(P_{n-1}, 5) + s_\gamma(P_{n-2}, 5) + s_\gamma(P_{n-3}, 5)$$

$$\Rightarrow x^6 s_\gamma(P_n, 6) = x^6 s_\gamma(P_{n-1}, 5) + x^6 s_\gamma(P_{n-2}, 5) + x^6 s_\gamma(P_{n-3}, 5)$$

When $i = 7$,

$$s_\gamma(P_n, 7) = s_\gamma(P_{n-1}, 6) + s_\gamma(P_{n-2}, 6) + s_\gamma(P_{n-3}, 6)$$

$$\Rightarrow x^7 s_\gamma(P_n, 7) = x^7 s_\gamma(P_{n-1}, 6) + x^7 s_\gamma(P_{n-2}, 6) + x^7 s_\gamma(P_{n-3}, 6)$$

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When $i = n-1$,

$$s_\gamma(P_n, n-1) = s_\gamma(P_{n-1}, n-2) + s_\gamma(P_{n-2}, n-2) + s_\gamma(P_{n-3}, n-2)$$

$$\Rightarrow x^{n-1} s_\gamma(P_n, n-1) = x^{n-1} s_\gamma(P_{n-1}, n-2) + x^{n-1} s_\gamma(P_{n-2}, n-2) + x^{n-1} s_\gamma(P_{n-3}, n-2)$$

When $i = n$

$$s_\gamma(P_n, n) = s_\gamma(P_{n-1}, n-1) + s_\gamma(P_{n-2}, n-1) + s_\gamma(P_{n-3}, n-1)$$

$$\Rightarrow x^n s_\gamma(P_n, i) = x^n s_\gamma(P_{n-1}, n-1) + x^n s_\gamma(P_{n-2}, n-1) + x^n s_\gamma(P_{n-3}, n-1)$$

Hence,

$$x^5 s_\gamma(P_n, 5) + x^6 s_\gamma(P_n, 6) + x^7 s_\gamma(P_n, 7) + \dots + x^{n-1} s_\gamma(P_n, n-1) + x^n s_\gamma(P_n, n)$$
$$= x^5 s_\gamma(P_{n-1}, 4) + x^5 s_\gamma(P_{n-2}, 4) + x^5 s_\gamma(P_{n-3}, 4) + x^6 s_\gamma(P_{n-1}, 5) + x^6 s_\gamma(P_{n-2}, 5) + x^6 s_\gamma(P_{n-3}, 5)$$
$$+ x^7 s_\gamma(P_{n-1}, 6) + x^7 s_\gamma(P_{n-2}, 6) + x^7 s_\gamma(P_{n-3}, 6) + \dots + x^{n-1} s_\gamma(P_{n-1}, n-2)$$
$$+ x^{n-1} s_\gamma(P_{n-2}, n-2) + x^{n-1} s_\gamma(P_{n-3}, n-2) + x^n s_\gamma(P_{n-1}, n-1) + x^n s_\gamma(P_{n-2}, n-1) + x^n s_\gamma(P_{n-3}, n-1).$$

$$x^5 s_\gamma(P_n, 5) + x^6 s_\gamma(P_n, 6) + x^7 s_\gamma(P_n, 7) + \dots + x^{n-1} s_\gamma(P_n, n-1) + x^n s_\gamma(P_n, n)$$
$$= x^5 s_\gamma(P_{n-1}, 4) + x^6 s_\gamma(P_{n-1}, 5) + x^7 s_\gamma(P_{n-1}, 6) + \dots + x^{n-1} s_\gamma(P_{n-1}, n-2)$$
$$+ x^n s_\gamma(P_{n-1}, n-1) + x^5 s_\gamma(P_{n-2}, 4) + x^6 s_\gamma(P_{n-2}, 5) + x^7 s_\gamma(P_{n-2}, 6) + \dots$$
$$+ x^{n-1} s_\gamma(P_{n-2}, n-2) + x^n s_\gamma(P_{n-2}, n-1) + x^5 s_\gamma(P_{n-3}, 4) + x^6 s_\gamma(P_{n-3}, 5)$$
$$+ x^7 s_\gamma(P_{n-3}, 6) + \dots + x^{n-1} s_\gamma(P_{n-3}, n-2) + x^n s_\gamma(P_{n-3}, n-1).$$

$$x^5 s_\gamma(P_n, 5) + x^6 s_\gamma(P_n, 6) + x^7 s_\gamma(P_n, 7) + \dots + x^{n-1} s_\gamma(P_n, n-1) + x^n s_\gamma(P_n, n)$$
$$= x^5 s_\gamma(P_{n-1}, 4) + x^6 s_\gamma(P_{n-1}, 5) + x^7 s_\gamma(P_{n-1}, 6) + \dots + x^{n-1} s_\gamma(P_{n-1}, n-2)$$
$$+ x^n s_\gamma(P_{n-1}, n-1) + x^5 s_\gamma(P_{n-2}, 4) + x^6 s_\gamma(P_{n-2}, 5) + x^7 s_\gamma(P_{n-2}, 6) + \dots$$
$$+ x^{n-2} s_\gamma(P_{n-2}, n-3) + x^{n-1} s_\gamma(P_{n-2}, n-2) + x^5 s_\gamma(P_{n-3}, 4) + x^6 s_\gamma(P_{n-3}, 5) + x^7 s_\gamma(P_{n-3}, 6)$$
$$+ \dots + x^{n-3} s_\gamma(P_{n-3}, n-2) + x^{n-2} s_\gamma(P_{n-3}, n-3).$$

[Since, $s_\gamma(P_{n-2}, n-1) = 0$, $s_\gamma(P_{n-3}, n-2) = s_\gamma(P_{n-3}, n-1) = 0$]

$$\sum_{i=5}^{n} s_\gamma(P_n, i)x^i = x\sum_{i=5}^{n-1} s_\gamma(P_{n-1}, i)x^i + x\sum_{i=5}^{n-2} s_\gamma(P_{n-2}, i)x^i + x\sum_{i=5}^{n-3} s_\gamma(P_{n-3}, i)x^i$$

ie, $S_\gamma(P_n, x) = x\,[S_\gamma(P_{n-1}, x) + S_\gamma(P_{n-2}, x) + S_\gamma(P_{n-3}, x)]$. Hence the theorem. □

Using theorem 2.9, we get $s_\gamma(P_n, n)$ for $2 \le n \le 15$ as shown in the Table 1.

Table 1: $s_\gamma(P_n, i)$ is the number of Steiner dominating sets of P_n with cardinality i.

n \ i	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	1													
3	1	1												
4	1	2	1											
5	0	3	3	1										
6	0	2	6	4	1									
7	0	1	7	10	5	1								
8	0	0	6	16	15	6	1							
9	0	0	3	19	30	21	7	1						
10	0	0	1	16	45	50	28	8	1					
11	0	0	0	10	51	90	77	36	9	1				
12	0	0	0	4	45	126	161	112	45	10	1			
13	0	0	0	1	30	141	266	266	156	55	11	1		
14	0	0	0	0	15	126	357	504	414	210	66	12	1	
15	0	0	0	0	5	90	393	784	882	615	275	78	13	1

Theorem 2.10

The following properties for the coefficients of $S_\gamma(P_n, x)$ hold:

i) For every $n \geq 2$, $s_\gamma(P_n, n) = 1$,

ii) For every $n \geq 3$, $s_\gamma(P_n, n-1) = n-2$,

iii) For every $n \geq 4$, $s_\gamma(P_n, n-2) = \frac{(n-2)(n-3)}{2}$

iv) For every $n \geq 5$, $s_\gamma(P_n, n-3) = \frac{n(n-4)(n-5)}{6}$

v) For every $n \geq 1$, $s_\gamma(P_{3n+1}, n+1) = 1$,

vi) For every $n \geq 1$, $s_\gamma(P_{3n}, n+1) = n$,

vii) For every $n \geq 1$, $s_\gamma(P_{3n-1}, n+1) = \frac{n(n+1)}{2}$,

viii) For every $n \geq 1$, $s_\gamma(P_{3n+1}, n+2) = \frac{n(n+1)(n+5)}{6}$

ix) $\sum_n s_\gamma(P_n, i) = 3^{i-1}$.

Proof:

i) For any connected graph G with n vertices, the whole set $\{[n]\}$ is a Steiner dominating set. Hence, $s_\gamma(G, n) = 1$.
Therefore, $s_\gamma(P_n, n) = 1$, if $n \geq 2$.

ii) We prove by induction on n.
The result is true for $n = 3$, since $s_\gamma(P_3, 2) = 1$.
Assume that the result is true for all natural numbers less than n. Now, we prove it for n.
By theorem 2.8,

$$s_\gamma(P_n, n-1) = s_\gamma(P_{n-1}, n-2) + s_\gamma(P_{n-2}, n-2) + s_\gamma(P_{n-3}, n-2)$$
$$= (n-1) - 2 + 1 \quad [\text{since } s_\gamma(P_{n-2}, n-2) = 1]$$
$$= n-2$$

Therefore, the result is true for all n.

iii) We prove by induction on n.
The result is true for $n = 4$,
since $s_\gamma(P_4, 2) = 1 = \frac{2\times 1}{2}$.
Assume that the result is true for all natural numbers less than n. Now, we prove it for n.
By theorem 2.8,

$$s_\gamma(P_n, n-2) = s_\gamma(P_{n-1}, n-3) + s_\gamma(P_{n-2}, n-3) + s_\gamma(P_{n-3}, n-3)$$
$$= \frac{(n-3)n-4)}{2} + n - 4 + 1 \quad [\text{ by part (i) and (ii)}]$$
$$= \frac{(n-3)n-4)}{2} + n - 3$$
$$= \frac{(n-3)(n-4)+2(n-3)}{2}$$
$$= \frac{(n-3)(n-4+2)}{2}$$
$$= \frac{(n-2)(n-3)}{2}$$

Therefore, the result is true for all n.

iv) The result is true for $n = 5$ and $n = 6$.
When $n = 5$,

$$s_\gamma(P_n, n-3) = s_\gamma(P_5, 2) = 0 = \frac{5\times 1\times 0}{6}$$

When $n = 6$,

$$s_\gamma(P_n, n-3) = s_\gamma(P_6, 3) = 2 = \frac{6\times 2\times 1}{6}$$

Assume that the result is true for all natural numbers less than n. Now, we prove it for n.
By theorem 2.8,

$$s_\gamma(P_n, n-3) = s_\gamma(P_{n-1}, n-4) + s_\gamma(P_{n-2}, n-4) + s_\gamma(P_{n-3}, n-4)$$
$$= \frac{(n-1)(n-5)(n-6)}{6} + \frac{(n-4)(n-5)}{2} + n - 5 \quad [\text{by part (ii) and (iii)}]$$
$$= \frac{(n-1)(n-5)(n-6)+3(n-4)(n-5)+6(n-5)}{6}$$
$$= \frac{(n-5)[(n-1)(n-6)+3(n-4)+6]}{6}$$
$$= \frac{(n-5)[n^2-7n+6+3n-12+6]}{6}$$
$$= \frac{(n-5)[n^2-4n]}{6}$$
$$= \frac{n(n-4)(n-5)}{6}$$

Therefore, the result is true for all n.

v) For every $n \geq 1$, $S_\gamma(P_{3n+1}, n+1)$ has only one Steiner dominating set
$\{1, 4, 7,\ldots, 3k-2, 3k+1\}$.
Therefore, $s_\gamma(P_{3n+1}, n+1) = 1$.

vi) We prove by induction on n.
The result is true for $n = 1$, since $s_\gamma(P_3, 2) = 1$ [by part (ii)]
Assume that the result is true for all natural numbers of the form $3i$, where $i < n$. Now, we prove it for $3n$.
By theorem 2.8,

$$s_\gamma(P_{3n}, n+1) = s_\gamma(P_{3n-1}, n) + s_\gamma(P_{3n-2}, n) + s_\gamma(P_{3n-3}, n) \quad \ldots\ldots\ldots (1)$$

We have, $s_\gamma(P_{3n-1}) = \left\lfloor \frac{3n-1+4}{3} \right\rfloor = n+1$

Therefore, $S_\gamma(P_{3n-1}, n) = \phi$ since n is less than the Steiner domination number of P_{3n-1}.
$\Rightarrow s_\gamma(P_{3n-1}, n) = 0$
Also, $s_\gamma(P_{3n-2}, n) = 1$ [by part (v)]
From (1),

$$s_\gamma(P_{3n}, n+1) = 0 + 1 + n - 1 = n$$

Therefore, $s_\gamma(P_{3n}, n+1) = n$

vii) We prove by induction on n.

The result is true for $n = 1$, since $s_\gamma(P_2, 2) = 1$ [by part (i)]
Assume that the result is true for all natural numbers of the form $3i - 1$ where $i < n$. Now, we prove it for $3n - 1$.
By theorem 2.8,

$$s_\gamma(P_{3n-1}, n+1) = s_\gamma(P_{3n-2}, n) + s_\gamma(P_{3n-3}, n) + s_\gamma(P_{3n-4}, n).$$

$$= 1 + n - 1 + \frac{(n-1)n}{2}$$

$$= \frac{2n+n^2-n}{2}$$

$$= \frac{n^2+n}{2}$$

$$= \frac{n(n+1)}{2}$$

Therefore, the result is true for all n.

viii) We prove by induction on n.
The result is true for $n = 1$,
since $s_\gamma(P_4, 3) = 2$. [by part (ii)]
Assume that the result is true for all natural numbers of the form $3i + 1$ where $i < n$.
Now, we prove it for n.
By theorem 2.8,

$$s_\gamma(P_n, i) = s_\gamma(P_{n-1}, i-1) + s_\gamma(P_{n-2}, i-1) + s_\gamma(P_{n-3}, i-1).$$

Hence,

$$s_\gamma(P_{3n+1}, n+2) = s_\gamma(P_{3n}, n+1) + s_\gamma(P_{3n-1}, n+1) + s_\gamma(P_{3n-2}, n+1).$$

$$= n + \frac{n(n+1)}{2} + \frac{(n-1)n(n+4)}{6}$$

[by part (vi) and (vii)]

$$= \frac{6n+3n(n+1)+(n-1)n(n+4)}{6}$$

$$= \frac{n[6+3n+3+n^2+3n-4]}{6}$$

$$= \frac{n[n^2+6n+5]}{6}$$

$$= \frac{n(n+1)(n+5)}{6}$$

Therefore, the result is true for all n.

ix) We prove by induction on i.
By theorem 2.8,

$$s_\gamma(P_n, i) = s_\gamma(P_{n-1}, i-1) + s_\gamma(P_{n-2}, i-1) + s_\gamma(P_{n-3}, i-1).$$

Also we have, $s_\gamma(P_n, i) = 0$, if $i < s_\gamma(P_n)$ or $i > n$

When $i = 2$,

$$\text{L.H.S} = \sum_n s_\gamma(P_n, 2)$$

$$= s_\gamma(P_1, 2) + s_\gamma(P_2, 2) + s_\gamma(P_3, 2) + s_\gamma(P_4, 2) + s_\gamma(P_5, 2)$$

$$= 0 + 1 + 1 + 1 + 0, \quad \{\text{since } s_\gamma(P_1, 2) = 0,\ 2 > 1 \text{ and } s_\gamma(P_5, 2) = 0,\ 2 < s_\gamma(P_5)\}$$

$$= 3.$$

$$\text{R.H.S} = 3^{i-1} = 3^{2-1} = 3.$$

Therefore, the result is true for $i = 2$.
Assume that the result is true for all natural numbers less than i.
Now,

$$\sum_n s_\gamma(P_n, i) = \sum_n [s_\gamma(P_{n-1}, i-1) + s_\gamma(P_{n-2}, i-1) + s_\gamma(P_{n-3}, i-1)]$$

$$= \sum_n s_\gamma(P_{n-1}, i-1) + \sum_n s_\gamma(P_{n-2}, i-1) + \sum_n s_\gamma(P_{n-3}, i-1)$$

$$= 3^{i-2} + 3^{i-2} + 3^{i-2}$$

$$= 3 \times 3^{i-2}$$

$$= 3^{i-1}$$

Therefore, the result is true for all i.
Hence the theorem. □

III. Conclusion

We have provided table for the coefficients of the polynomial upto $n = 15$. Also, we provided a recursive relation to find the polynomial for any n, using the initial values. It is interesting to note that the characters of the coefficients are established. If we provide an algorithm for finding the coefficients, we can directly find the polynomial for any n. We are searching for some applications in physical problems.

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