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Stability and stabilization of discrete-time systems with time-delay via Lyapunov-Krasovskii functional

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ABSTRACT
The stability and stabilization problems for discrete systems with time-delay are discussed .The stability and stabilization criterion are expressed in the form of linear matrix inequalities (LMI). An effective method allowing us transforming a bilinear matrix Inequality (BMI) to a linear matrix Inequality (LMI) is developed. Based on these conditions, a state feedback controller with gain is designed. An illustrative numerical example is provided to show the effectiveness of the proposed method and the reliability of the results.
***Keywords* -** Asymptotic stability; discrete-time systems; Lyapunov functional; Lyapunov-Krasovskii method; linear matrix inequalities (LMI); time-delay.

I. INTRODUCTION

Time delay is an important factor that may affect the performance of dynamical systems. It can even, in same situation, cause instability of a system that we would like to control if the presence of such time-delay during the design phases is not taken into account .For linear systems with time delay; we have seen an increasing interest during the last two decades. There are numerous results in the literature on time-delay systems [4,7,8,17].However; most results are focused on the continuous-time linear systems with time delay. Stability, stabilization and control problems for this system have been studied and numerous results are available in the literature such as [3,5,9]. However, for discrete-time linear systems with time-delay only few results have been reported in the literature. We believe that the main reason for this is that these systems can be transformed to equivalent systems without time-delay and then current results on stability, stabilization and control design can be applied.

Time delay is frequently encountered in many fields of engineering systems, such as manufacturing system, telecommunication, economic system, and chemical engineering system. It is generally regarded as a main source of stability and poor performance [12,14,19]. Therefore, the problems of stability analysis and stabilization of time-delay systems are important both in theory and practice, and are thus of interest to many researchers. Commonly, the approaches for solving time-delay systems can be classified into two types. Delay-dependent conditions [1,2,13], which include information on the size of delays, and delay-independent conditions, which are applicable to delays of arbitrary size. Since the stability of a system depends explicitly on the time-delay, a delay-independent condition is more conservative, especially for small delays, while a delay-dependent condition is usually less conservative.

Due to the development in the field of microelectronics analog controllers are yielding their places to digital computers. Indeed, and giving the importance of these control systems, we are using methods and numerical models to analyze and / or to control industrial processes.

Two types of representation are available to model a continuous or discrete dynamic system namely the external representation that uses input-output relations (transfer function) or the internal representation (matrix) of dynamic system which is based on the concept of state. To implement such a control structure and ensure the desired objectives, a modeling in the generally required discrete-time analog systems is needed.

Digital control of physical systems requires, usually the development of discrete models. Several modeling strategies, developed in the literature reflecting a meaningful description of dynamical systems to be studied led to mathematical tools leading generally to linear or non-linear models with or without delays whose behavior may be more or less close to the real system [10,11,15,18]. These models are described by relations between input variables and output variables that can be modified by inputs considered as secondary (disturbances) that always exist in practice.

The initial modeling of a discrete time-delays system often leads to writing a recurrent equation between different terms of the input and output sequences. This formulation of the recurrent equation is well suited for numerical calculation. This is the form in which these algorithms are digital control methods. The system is fully defined and the recurrent equation can be solved if the initial conditions are specified.

The analysis of the stability of delays systems has been conducted in the literature by numerous fundamental researches that depend on the type of systems considered and the scope. There are many study methods of the stability of linear discrete time-delay systems. These stability criteria can be classified into two main categories namely the frequency criterion using the notion of the characteristic equations and the time criterion based on Lyapunov theory.

This paper is organized as follows. In Section 2, the problem is stated and the objective of the paper is formulated. The problems of stability for the given system is examined and delay-dependent or independent sufficient condition is developed in section 3.We continue in section 4, to investigate the problem of stability and establish delay-dependent conditions. In addition, a design algorithm that stabilizes the resulting closed-loop system is provided. A numerical example is given in section 5 to illustrate the proposed theoretical results.

II. FORMULATION OF THE PROBLEM AND SOME PRELIMINARY

R	Real vector space.
$F=(f_{ij})\in R^{n*n}$	Real matrix.
F^T	Transpose of the matrix F.
$F>0$	Positive definite matrix.
$F\geq 0$	Positive semi-definite matrix.
$\lambda(F)$	Eigenvalue of the matrix F.
$\sigma(F)=\|\mathrm{F}\|$	Singular value of the matrix.
$\|\mathrm{F}\|=\sqrt{\lambda_{\max}(F^TF)}$	Euclidean norm of the matrix F.

Considering the dynamics of the discrete system with time-delays defined by the following equation:

$$x(k+1)=A_0x(k)+A_1x(k-q)+Bu(k) \qquad (1)$$

Where $x(k)\in R^n$ is the state at time k.

$\mathrm{x}(\theta)=\psi(\theta),\theta\in\{-q,-q+1,..,0\}$ represents the initial condition.

$A_i\in R^{n*n}$ are constant matrices of appropriate size.

$q=1,2,.....$ is a positive integer representing the time delay existing in the system.

Whether $\mathrm{V}:R^n\to R$ in such a way that $V(x)$ is bounded for all $\|x\|$ is bounded.

The aim of this paper is to establish sufficient conditions that guarantee the stability of the class of system (1). Based on stability conditions, the stabilization problem of this system (1) will be handled, too. The control law is given with a memory less state-feedback as: $u(k)=-L\,x(k)$, $L=(L_1\ L_2)$

Where L is the control gain to be computed.

III. STABILITY ANALYSIS

The stability of discrete time-delay systems has received much attention in the past several years [6, 7]. In the literature, there are some necessary and sufficient stability conditions for these systems. Based on these results, some necessary and sufficient stability conditions for discrete-time delay systems can be obtained. Roughly speaking, the stability of a system is its ability to resist any unknown small influences. Since in reality disturbances are always encountered, stability is an important property of any control system, delayed or non delayed.

In this section, LMIs-based conditions of delay-dependent or independent stability analysis will be considered for discrete-time systems with time-delays. The following result gives sufficient conditions to guarantee that the system (1) for $u(k)=0$, $k\geq 0$ is stable.

Fact 1: for any positive scalar α and for any two vectors x and y, we present the following inequality:

$$x^Ty+y^Tx\leq\alpha x^Tx+\alpha^{-1}y^Ty \qquad (2)$$

Note that:

$$V_\delta=\{x\in R^n:\|x\|<\delta\} \qquad (3)$$

Lemma 1: [16] the zero solution of the difference system is asymptotically stable if there exists a positive definite $V(x(k)):R^n\to R^+$ knowing that there is a $\rho>0$ as:

$$\Delta V(x(k))=V(x(k+1))-V(x(k))\leq-\rho\|x(k)\|^2 \qquad (4)$$

The above inequality is true throughout the linear resolution of the discrete system. If the above condition is valid for all $x(k)\in V_\delta$, the zero solution of the difference system is locally asymptotically stable.

Lemma 2: [16] for any constant symmetric matrix: $M\in R^{n*n}, M=M^T>0, \beta$ scalar as $\beta\in Z^+/\{0\}$, and the vector function $W:[0,\beta]\to R^n$, we have the following inequality:

$$\left(\sum_{i=0}^{\gamma-1}w(\mathrm{i})\right)^T\times M\times\left(\sum_{i=0}^{\gamma-1}w(\mathrm{i})\right)\leq\beta\sum_{i=0}^{\gamma-1}\left(w(\mathrm{i})^T\times M\times w(\mathrm{i})\right) \qquad (5)$$

A-**Delay-dependent stability:**

This group includes exact algebraic stability criteria depending on the delay and on the system constants and stability criteria which yield an upper bound of the admissible delay.

Using the stated theorem in the following and previously stated lemmas we can determine the asymptotic stability of the linear discrete system that is presented in equation (1).

Theorem 1:

The discrete time-delay system (1) is asymptotically stable for any delay $q > 0$, if there exist symmetric positive definite matrix $P = P^T > 0$, $G = G^T > 0$ and $W = W^T > 0$ satisfying the following matrix inequalities:

$$\psi_1 = \begin{pmatrix} (1,1) & 0 & 0 \\ 0 & (2,2) & 0 \\ 0 & 0 & (3,3) \end{pmatrix} < 0 \quad (6)$$

Such as:

$$(1,1) = A_0^T P A_0 + \alpha A_0^T P^2 A_0 + qG + W - P \quad (7)$$

$$(2,2) = A_1^T P A_1 + \alpha^{-1} A_1^T A_1 - W \quad (8)$$

$$(3,3) = -qG \quad (9)$$

Evidence: Consider the Lyapunov function defined as follows:

$$V(y(k)) = V_1(y(k)) + V_2(y(k)) + V_3(y(k)) \quad (10)$$

Where:

$$V_1(y(k)) = x^T(k) \times P \times x(k) \quad (11)$$

$$V_2(y(k)) = \sum_{i=k-q}^{k-1} (q - k + i) \times x^T(i) \times G \times x(i) \quad (12)$$

$$V_3(y(k)) = \sum_{i=k-q}^{k-1} x^T(i) \times W \times x(i) \quad (13)$$

$$y(k) = [x(k), x(k-q)] \quad (14)$$

With $P = P^T > 0$, $G = G^T > 0$ and $W = W^T > 0$ is symmetric positive definite solutions of (6) and $y(k) = [x(k), x(k-q)]$.

Then the difference of $V(y(k))$ along the path of the solution (4) is given by:

$$\Delta V(y(k)) = \Delta V_1(y(k)) + \Delta V_2(y(k)) + \Delta V_3(y(k)) \quad (15)$$

With:

$$\begin{aligned} \Delta V_1(y(k)) &= V_1(x(k+1)) - V_1(x(k)) \\ &= [A_0 x(k) + A_1 x(k-q)]^T P [A_0 x(k) + A_1 x(k-q)] - x^T(k) P x(k) \\ &= x^T(k)[A_0^T P A_0 - P]x(k) + x^T(k) A_0^T P A_1 x(k-q) + \\ &+ x^T(k-q) A_1^T P A_0 x(k) + x^T(k-q) A_1^T P A_1 x(k-q) \end{aligned} \quad (16)$$

$$\begin{aligned} \Delta V_2(y(k)) &= V_2(x(k+1)) - V_2(x(k)) \\ &= \Delta\left(\sum_{i=k-q}^{k-1} (q-k+i) x^T(i) G x(i) \right) \quad (17) \\ &= q x^T(k) G x(k) - \sum_{i=k-q}^{k-1} x^T(i) G x(i) \end{aligned}$$

$$\begin{aligned} \Delta V_3(y(k)) &= V_3(x(k+1)) - V_3(x(k)) \\ &= \Delta\left(\sum_{i=k-q}^{k-1} x^T(i)\, W\, x(i) \right) \quad (18) \\ &= x^T(k)\, W\, x(k) - x^T(k-q)\, W\, x(k-q) \end{aligned}$$

Applying the Fact 1 in equation (16), the following inequality is obtained:

$$\begin{aligned} & x^T(k) A_0^T P A_1 x(k-q) + x^T(k-q) A_1^T P A_0 x(k) \le \\ & \alpha x^T(k) A_0^T P^2 A_0 x(k) + \alpha^{-1} x^T(k-q) A_1^T A_1 x(k-q) \end{aligned} \quad (19)$$

Therefore:

$$\begin{aligned} \Delta V_1(y(k)) \le\ & x^T(k)\left[A_0^T P A_0 + \alpha A_0^T P^2 A_0 - P\right] x(k) + \\ & + x^T(k-q)\left[A_1^T P A_1 + \alpha^{-1} A_1^T A_1\right] x(k-q) \end{aligned} \quad (20)$$

Thus the expression (15) of $\Delta V(y(k))$ is rewritten as follows:

$$\begin{aligned} \Delta V(y(k)) \le\ & x^T(k)\left[A_0^T P A_0 + \alpha A_0^T P^2 A_0 - P\right] x(k) + \\ & + x^T(k-q)\left[A_1^T P A_1 + \alpha^{-1} A_1^T A_1\right] x(k-q) + q x^T(k) G x(k) - \\ & - \sum_{i=k-q}^{k-1} x^T(i) G x(i) + x^T(k)\, W\, x(k) - x^T(k-q)\, W\, x(k-q) \end{aligned} \quad (21)$$

Which is equivalent to

$$\begin{aligned} \Delta V(y(k)) \le\ & x^T(k)\left[A_0^T P A_0 + \alpha A_0^T P^2 A_0 + qG + W - P\right] x(k) + \\ & + x^T(k-q)\left[A_1^T P A_1 + \alpha^{-1} A_1^T A_1 - W\right] x(k-q) - \\ & - \sum_{i=k-q}^{k-1} x^T(i) G x(i) \end{aligned} \quad (22)$$

By using Lemma 2, we obtain the following inequality:

$$\left(\frac{1}{q} \sum_{i=k-q}^{k-1} x(i) \right)^T qG \left(\frac{1}{q} \sum_{i=k-q}^{k-1} x(i) \right) \le \sum_{i=k-q}^{k-1} x^T(i) G x(i) \quad (23)$$

It follows that:

$$\begin{aligned} \Delta V(y(k)) \le\ & x^T(k)\left[A_0^T P A_0 + \alpha A_0^T P^2 A_0 - P\right] x(k) + \\ & + x^T(k-q)\left[A_1^T P A_1 + \alpha^{-1} A_1^T A_1\right] x(k-q) + \\ & + q x^T(k) G x(k) - \sum_{i=k-q}^{k-1} x^T(i) G x(i) + \\ & + x^T(k)\, W\, x(k) - x^T(k-q)\, W\, x(k-q) \end{aligned} \quad (24)$$

From Fact 1 we get the following expression:

$$\Delta V(y(k))\leq x^T(k)\left[A_0^TPA_0+\alpha A_0^TP^2A_0+qG+W-P\right]x(k)+ \\ +x^T(k-q)\left[A_1^TPA_1+\alpha^{-1}A_1^TA_1-W\right]x(k-q)- \\ -\sum_{i=k-q}^{k-1}x^T(i)Gx(i) \quad (25)$$

Using Lemma 2, equation (25) will be rewritten as follows:

$$\Delta V(y(k))\leq\left[\begin{array}{c} x^T(k)\left[A_0^TPA_0+\alpha A_0^TP^2A_0+qG+W-P\right]x(k)+ \\ +x^T(k-q)\left[A_1^TPA_1+\alpha^{-1}A_1^TA_1-W\right]x(k-q)- \\ -\left(\frac{1}{q}\sum_{i=k-q}^{k-1}x(i)\right)^T qG\left(\frac{1}{q}\sum_{i=k-q}^{k-1}x(i)\right)\end{array}\right]=\Omega \quad (26)$$

$$\Omega=x^T(k)\left[A_o^TPA_o+\alpha A_o^TP^2A_o+qG+W-P\right]x(k)+ \\ +x^T(k-q)\left[A_1^TPA_1+\alpha^{-1}A_1^TA_1-W\right]x(k-q)- \\ -\left(\frac{1}{q}\sum_{i=k-q}^{k-1}x(i)\right)^T qG\left(\frac{1}{q}\sum_{i=k-q}^{k-1}x(i)\right) \\ =\left(x^T(k),x^T(k-q),\left(\frac{1}{q}\sum_{i=k-q}^{k-1}x(i)\right)^T\right)\times \quad (27) \\ \times\begin{pmatrix}(1,1)&0&0\\0&(2,2)&0\\0&0&(3,3)\end{pmatrix}\times\begin{pmatrix}x(k)\\x(k-q)\\\left(\frac{1}{q}\sum_{i=k-q}^{k-1}x(i)\right)\end{pmatrix} \\ =y^T(k)\times\psi_0\times y(k)$$

With:

$$y(k)=\begin{pmatrix}x(k)\\x(k-q)\\\left(\frac{1}{q}\sum_{i=k-q}^{k-1}x(i)\right)\end{pmatrix} \quad (28)$$

Where:

$$\Delta V(y(k))\leq y^T(k)\times\psi_0\times y(k) \quad (29)$$

Thus the condition (6) is satisfied, then $\Delta V(y(k))<0$,

$\forall$ x(k) $\neq 0$ which allowed us to conclude that the system defined in equation (5) is asymptotically stable.

Finally we conclude that $\Delta v(y(k))$ is negative definite; namely, there is a number $\rho>0$ such that $\Delta v(y(k))\leq-\rho\|y(k)\|^2$ and, consequently, the asymptotic stability of the system follows immediately from Lemma 1.

B- **Delay-independent stability:**

Delay-independent stability criteria are very useful, since in reality it is difficult to estimate the delays, especially if those delays are time-varying and/or state-dependent.

Theorem 2:

The discrete time-delay system (1) is asymptotically stable, if there exist symmetric positive definite matrix $N=N^T>0$ and $S=S^T>0$ such that following linear matrix inequality (LMI) hold:

$$\psi_2=\begin{pmatrix}N-S&0&A_0^TS\\0&-N&A_1^TS\\A_0^TS&A_1^TS&-S\end{pmatrix}<0. \quad (30)$$

Proof. Let the Lyapunov functional be:

$$V(x(k))=x^T(k)\mathrm{S}x(k)+\sum_{j=1}^{q}x^T(k-j)Nx(k-j) \quad (31)$$

$N=N^T>0$ and $S=S^T>0$.

The forward difference along the solutions of system (1) is:

$$\Delta V(y(k))=\left[A_0x(k)+A_1x(k-q)\right]^T S\left[A_0x(k)+A_1x(k-q)\right]- \\ -x^T(k)\mathrm{S}x(k)+x^T(k)Nx(k)-x^T(k-q)\mathrm{N}x(k-q) \\ =\begin{bmatrix}x(k)\\x(k-q)\end{bmatrix}^T\begin{bmatrix}A_0^TSA_0-S+N&A_0^TSA_1\\A_0^TSA_1&A_1^TSA_1-N\end{bmatrix}\begin{bmatrix}x(k)\\x(k-q)\end{bmatrix} \quad (32)$$

If the following equation is satisfied:

$$\begin{bmatrix}A_0^TSA_0-S+N&A_0^TSA_1\\A_0^TSA_1&A_1^TSA_1-N\end{bmatrix}<0 \quad (33)$$

Then

$$\begin{bmatrix}A_0^TSA_0-S+N&A_0^TSA_1\\A_0^TSA_1&A_1^TSA_1-N\end{bmatrix}=\begin{bmatrix}N-S&0\\0&-N\end{bmatrix}+\begin{bmatrix}A_0^TSA_0&A_0^TSA_1\\A_0^TSA_1&A_1^TSA_1\end{bmatrix} \\ =\begin{bmatrix}N-S&0\\0&-N\end{bmatrix}+\begin{bmatrix}A_0^T\\A_1^T\end{bmatrix}S\left[A_0\quad A_1\right]<0 \quad (34)$$

Using Schur complement [5], it is easy to see that the condition (34) is equivalent to:

$$\begin{pmatrix}N-S&0&A_0^TS\\0&-N&A_1^TS\\A_0^TS&A_1^TS&-S^{-1}\end{pmatrix}<0 \quad (35)$$

Note that the condition (35) is not LMI condition due to the existence of the term $-S^{-1}$. Pre and post multiply (36) with dig {I, I, S } we obtain LMI condition (31).

Thus the condition (31) is satisfied, then $\Delta V(y(k))<0$,

$\forall \; x(k)\neq 0$ Which allowed us to conclude that the system defined in equation (5) is asymptotically stable.

Finally we conclude that $\Delta v(y(k))$ is negative definite; namely, there is a number $\beta>0$ such that $\Delta v(y(k))\leq-\beta\|y(k)\|^2$ and, consequently, the asymptotic stability of the system follows immediately from Theorem 2.

IV. STABILIZABILITY

In this section we consider the stabilizability problem of linear system with mode dependent time delays. A state feedback controller design method will be given. If the time-delay in system (1) is known, the following state feedback controller is considered:

$$u(k) = -L\,x(k)\ , L = (L_1\ L_2) \qquad (36)$$

Definition system (1) is stabilizable, if for every initial state there exists a state-feedback controller (36) with gain $L = (L_1\ L_2)$ such that the resulting closed-loop of system is stable.

Replacing the control $u(k)$ by its expression given by equation (36) and substituting it into system (1), we get the following dynamics for the closed-loop system:

$$\begin{aligned} x(k+1) &= A_0 x(k) + A_1 x(k-q) + Bu(k) \\ &= A_0 x(k) + A_1 x(k-q) + B(-L\,x(k)) \\ &= (A_0 - BL)x(k) + A_1 x(k-q) \end{aligned} \qquad (37)$$

The aim of this important work is to design a memory less state-feedback controller which stabilizes the system (1), when the memory less state-feedback is substituted with plant dynamics (6). Note that stability analysis condition (6) is not convenient for us to design a memory less state-feedback.

The problem is to determine a stabilizing compensator L, which satisfies the following linear matrix inequality:

Theorem 3:

The discrete time-delay system (1) is asymptotically stable for any delay $q > 0$, if there exist symmetric positive definite matrix $P_1 = P_1^T > 0$, $G_1 = G_1^T > 0$ and $W_1 = W_1^T > 0$ satisfying the following matrix inequalities:

$$\psi_3 = \begin{pmatrix} (1,1) & 0 & 0 \\ 0 & (2,2) & 0 \\ 0 & 0 & (3,3) \end{pmatrix} < 0 \qquad (38)$$

Such as:

$$\begin{aligned} (1.1) = {} & (A_0 - BL)^T P_1 (A_0 - BL) + \varepsilon (A_0 - BL)^T P_1^2 (A_0 - BL) + \\ & + q\,G_1 + W_1 - P_1 \end{aligned} \qquad (39)$$

$$(2.2) = A_1^T P_1 A_1 - \varepsilon^{-1} A_1^T A_1 - W_1 \qquad (40)$$

$$(3.3) = -qG_1 \qquad (41)$$

We note that the inequality (39) is affine result, the product term P_1 and L form a Bilinear Matrix Inequality (BMI) (nonlinear). A_0 and B are given matrix, P_1 and L are two variables vectors. Finding a control law $u(k) = -L\,x(k)$ stabilizing the system (1) can be carried out as follows:

- Find P_1 and L such that inequality (6) is satisfied (feasibility problem)
- $X = BL$, from X pulling the value of compensator L.

A BMI problem can be reformulated as an LMI problem. Nevertheless, in some cases it is possible to introduce some transformation rules that can rewrite the BMI optimization problem into a problem of constrained optimization LMI equivalent such that:

- rebasing.
- variable change.
- elimination of variables.
- completion of the edges.
- Introduction of additional variables (bogus variables).

Several methods of resolution, with different variants are possible. We focus on one. To solve the problem presented in this work we are interested only in the method based on the change of variable. By appropriate changes of variables we can transform a BMI as a LMI. Through a series of transformations, we will show that we can obtain an equivalent LMI constraint after a change of appropriate variables.

$$\begin{aligned} (1.1) = {} & A_0^T P_1 A_0 - L^T B^T P_1 A_0 - A_0^T P_1 BL + L^T B^T P_1 BL + \\ & + \varepsilon (A_0^T P_1^2 A_0 - L^T B^T P_1^2 A_0 - A_0^T P_1^2 BL + L^T B^T P_1^2 BL) + \\ & + qG_1 + W_1 - P_1 \end{aligned} \qquad (42)$$

Let: $X = BL$, $X^T = L^T B^T$

We obtain a new bilinear matrix inequality variable, and even non-linear.

$$\begin{aligned} (1.1) = {} & A_0^T P_1 A_0 - X^T P_1 A_0 - A_0^T P_1 X + X^T P_1 X + \\ & + \varepsilon (A_0^T P_1^2 A_0 - X^T P_1^2 A_0 - A_0^T P_1^2 X + X^T P_1^2 X) + \\ & + qG_1 + W_1 - P_1 \end{aligned} \qquad (43)$$

We choose a new variable:

$Y = P_1 X$, $Y^T = X^T P_1$ the value of X is drawn $X = P_1^{-1} Y$.

The expression (43) will be rewritten as follows:

$$\begin{aligned} (1.1) = {} & A_0^T P_1 A_0 - Y^T A_0 - A_0^T Y + X^T Y + \\ & + \varepsilon (A_0^T P_1^2 A_0 - Y^T P_1 A_0 - A_0^T P_1 Y + Y^T Y) + \\ & + qG_1 + W_1 - P_1 \end{aligned} \qquad (44)$$

Posing: $Z = X^T Y$, $V = P_1 Y$, $V^T = Y^T P_1$, $Y = P_1^{-1} V$

Finally there leads to a linear matrix inequality (LMI) feasible for new variables which covers:

$Y = P_1^{-1} V$, $V = P_1 Y$, $Z = X^T Y$, which themselves cover:

$X = BL$ then we can write (6) as:

$$(1.1) = A_0^T P_1 A_0 - Y^T A_0 - A_0^T Y + Z + \\ + \varepsilon(A_0^T P_1^2 A_0 - V^T A_0 - A_0^T V + Y^T Y) + \\ + qG_1 + W_1 - P_1 \tag{45}$$

$$(2.2) = A_1^T P_1 A_1 + \varepsilon^{-1} A_1^T A_1 - W_1 \tag{46}$$

$$(3.3) = -q G_1 \tag{47}$$

From V is pulled Y, from Y is pulled X then pulls the expression of L.

V. NUMERICAL EXAMPLE

To illustrate the usefulness of the previous theoretical results, let us give the following numerical examples.

Consider the linear discrete time delay system autonomous defined by the following equation:

$$x(k+1) = \begin{pmatrix} 0.1 & 0.02 \\ 0.1 & -0.15 \end{pmatrix} x(k) + \begin{pmatrix} 0.1 & 0.01 \\ 0.2 & 0.2 \end{pmatrix} x(k-1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) \tag{37}$$

with:

$$A_0 = \begin{pmatrix} 0.1 & 0.02 \\ 0.1 & -0.15 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.1 & 0.01 \\ 0.2 & 0.2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

A-Applying Theorem 1 to the equation defined in system (1) and through the relationship (6) .Matrix P, W and G symmetric positive definite which satisfy the sufficient condition for stability is obtained:

$$P = \begin{pmatrix} 3.2162 & 0.0172 \\ 0.0172 & 3.1592 \end{pmatrix}, G = \begin{pmatrix} 1.0696 & -0.0055 \\ -0.0055 & 1.0555 \end{pmatrix} \text{and} W = \begin{pmatrix} 1.1628 & 0.0712 \\ 0.0712 & 1.1295 \end{pmatrix}$$

B-Applying Theorem 2 to the equation defined in system (1) and through the relationship (30).Matrix N and S symmetric positive definite which satisfy the sufficient condition for stability is obtained:

$$N = \begin{pmatrix} 0.1158 & 1.2007 \\ 1.2007 & 1.0573 \end{pmatrix} \text{and } S = \begin{pmatrix} 0.6163 & 1.6801 \\ 1.6801 & 1.0583 \end{pmatrix}$$

C-Let us now, see how we can use the design algorithm of theorem3, to determinate the controller gain $L = [L_1, L_2]$. For this purpose let us consider the following data:

$$A_0 = \begin{pmatrix} 0.1 & 0.02 \\ 0.1 & -0.15 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.1 & 0.01 \\ 0.2 & 0.2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Using this data, solving LMIs (38) gives the following feasible solution:

$$P_1 = \begin{pmatrix} 3.2162 & 0.0172 \\ 0.0172 & 3.1592 \end{pmatrix}, G_1 = \begin{pmatrix} 1.0696 & -0.0055 \\ -0.0055 & 1.0555 \end{pmatrix}, W_1 = \begin{pmatrix} 1.1628 & 0.0712 \\ 0.0712 & 1.1295 \end{pmatrix}$$

$$V = \begin{pmatrix} 0.1465 & -0.0072 \\ 13.6978 & -0.6734 \end{pmatrix} \Rightarrow \quad Y = P_1^{-1} V$$

$$Y = \begin{pmatrix} 0.0228 & -0.0011 \\ 4.2589 & -0.2094 \end{pmatrix} \Rightarrow \quad X = P_1^{-1} Y$$

$$X = \begin{pmatrix} 0 & 0 \\ 1.3242 & -0.0651 \end{pmatrix} \Rightarrow \quad X = BL$$

Which gives the following gain:

$$L = \begin{pmatrix} 1.3242 & -0.0651 \end{pmatrix}$$

VI. CONCLUSION

In this paper we have investigated the stability and stabilization of discrete time systems with time-delay. Moreover, we have got same equivalent stability conditions which are presented as LMI and thus easy to test, using the Lyapunov function approach .Furthermore; we have designed a feedback controller with gain based on one of these stability conditions. Finally, we have used a numerical example illustrating effectiveness of the proposed method.

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