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Bifurcation in triply diffusive couple stress fluid systems

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ABSTRACT
Bifurcation analysis of a triply diffusive couple stress fluid is investigated in terms of a simplified model consisting of seven nonlinear ordinary differential equations which reproduces results obtained by modified perturbation theory, to second order, for the full two-dimensional problem and also the linear stability analysis results are identical with those for the full problem. Condition for the occurrence of direct and Hopf bifurcations is obtained. Modified perturbation technique is used to analyze the stability of bifurcating equilibrium solution. It is found that subcritical bifurcation is possible depending on the choices of parametric values. The transient behavior of the Nusselt numbers is investigated by solving nonlinear autonomous ordinary differential equations using Runge-Kutta-Gill method.
Key words: Bifurcation , Couple stress fluid, , Triple diffusive convection, Nonlinear stability theory

I. Introduction

Many fluid dynamical systems occurring in nature and industrial applications involve three or more stratifying agencies having different molecular diffusivities. More complicated systems can be found in magmas and molten metals (Jakeman and Hurle [1]). This has prompted researchers to study convective instability in triple diffusive fluid systems both theoretically and experimentally (Griffiths [2], Turner [3], Pearlstein et al. [4], Terrones and Pearlstein [5], Moroz [6], Lopez et al. [7]). The possibilities of existing of some interesting situations which were not observed either in singly or doubly diffusive systems have been reported. The effects of cross-diffusion on the onset of convective instability in a horizontally unbounded triply cross-diffusive fluid layer have been investigated by Terrones [8]. Straughan and Walker [9] have analyzed various aspects of penetrative convection in a triply diffusive fluid layer, while multicomponent convection – diffusion with internal heating or cooling in a fluid layer has been considered by Straughan and Tracey [10].

The previous studies on triple diffusive convection are dealt with only Newtonian fluid theory. As propounded earlier, many fluid dynamical systems such as molten polymers, salt solutions, slurries, geothermally heated lakes, magmas and their laboratory models, synthesis of chemical compounds usually involve more than two diffusing components and can be well characterized by couple stress fluid theory rather than Newtonian theory. The couple-stress fluid theory represents the simplest generalization of the classical viscous fluid theory that allows for polar effects and whose microstructure is mechanically significant in fluids. For such a special kind of non-Newtonian fluids, the constitutive equations are given by Stokes [11] which allows the sustenance of couple stresses in addition to usual stresses. This fluid theory shows all the important features and effects of couple stresses and results in equations that are similar to Navier-Stokes equations. Recently, Shivakumara and Naveen Kumar [12] have investigated the effect of couple stresses on linear and weakly nonlinear stability of a triply diffusive fluid layer.

Nonetheless, a different approach is followed in the present paper to analyze bifurcation in a triply diffusive couple stress fluid systems. Instead of grappling with the full problem a simplified extended Lorenz model which reproduces qualitative features of the full system with remarkable fidelity is considered. This model problem, consisting of seven coupled nonlinear autonomous ordinary differential equations, are solved with sufficient accuracy by a combination of analytical and numerical techniques. Heat and mass transfer are calculated in terms of Nusselt numbers.

II. Mathematical Formulation

We consider an incompressible horizontal couple stress fluid layer of thickness d in which the density depends on three stratifying agencies namely, temperature T and solute concentrations C_1 and C_2 having different diffusivities. The density is assumed constant everywhere except in the body force and the off-diagonal contributions to the fluxes of the stratifying agencies are neglected. A Cartesian coordinate system (x, y, z) is used with the origin at the bottom of the fluid layer and the z-axis vertically upward. The gravity is acting vertically downwards with the constant acceleration, $\vec{g} = -g\hat{k}$ where $\hat{k}$ is the unit vector in the vertical direction. The lower boundary $z=0$ of the fluid layer is maintained at higher temperature $T_0+\Delta T$ and higher solute concentration $C_{i0}+\Delta C_i$ (i =1, 2), while the upper boundary $z=d$ is maintained at temperature T_0 and solute concentration C_{i0} (i =1,2). Following Shivakumara and Naveen Kumar [12], the governing equations in dimensionless form can then be shown to be:

$$\left(\frac{1}{Pr}\frac{\partial}{\partial t}-\nabla^2+\Lambda_c\nabla^4\right)\nabla^2\psi=-R_t\frac{\partial T}{\partial x}+R_{s1}\frac{\partial C_1}{\partial x}+R_{s2}\frac{\partial C_2}{\partial x}+\frac{1}{Pr}J\left(\psi,\nabla^2\psi\right) \quad (1)$$

$$\left(\frac{\partial}{\partial t}-\nabla^2\right)T=-\frac{\partial\psi}{\partial x}+J\left(\psi,T\right) \quad (2)$$

$$\left(\frac{\partial}{\partial t}-\tau_1\nabla^2\right)C_1=-\frac{\partial\psi}{\partial x}+J\left(\psi,C_1\right) \quad (3)$$

$$\left(\frac{\partial}{\partial t}-\tau_2\nabla^2\right)C_2=-\frac{\partial\psi}{\partial x}+J\left(\psi,C_2\right) \quad (4)$$

where $\psi(x,z,t)$ is a two-dimensional stream function, $R_t=\beta_t g d^3\Delta T/\nu\,\kappa_t$ is the thermal Rayleigh number, $R_{s1}=\beta_{c1}gd^3\Delta C_1/\nu\kappa_t$ and $R_{s2}=\beta_{c2}gd^3\Delta C_2/\nu\kappa_t$ are the solute Rayleigh numbers, $\Lambda_c=\mu_c/\mu d^2$ is the couple stress parameter, $Pr=\nu/\kappa_t$ is the Prandtl number, $\tau_1=\kappa_{c1}/\kappa_t$ and $\tau_2=\kappa_{c2}/\kappa_t$ are the ratios of diffusivities, $\nabla^2=\partial^2/\partial x^2+\partial^2/\partial z^2$ is the Laplacian operator and $J(\cdot\cdot,\cdot\cdot)$ stands for the Jacobian with respect to x and z. Here, μ_c is the couple stress viscosity, μ is the dynamic viscosity, ν is the kinematic viscosity, κ_t is the thermal diffusivity, κ_{c1} and κ_{c2} are the solute analogs of κ_t, β_t is the thermal volume expansion coefficient, β_{c1} and β_{c2} are the solute analogs of β_t.

The boundaries are considered to be stress-free and perfect conductors of heat and solute concentrations. Accordingly, the boundary conditions are:

$$\psi=\frac{\partial^2\psi}{\partial z^2}=\frac{\partial^4\psi}{\partial z^4}=T=C_1=C_2=0 \text{ at } z=0,1. \quad (5)$$

To study the above nonlinear boundary value problem, a minimal amplitude motion plus the distortion of temperature and species concentrations fields is constructed (Moroz [6]) as follows:

$$\psi=\frac{2\sqrt{2}\delta A(t)}{\alpha}\sin\left(\alpha x\right)\sin\left(\pi z\right) \quad (6)$$

$$T=\frac{2\sqrt{2}B(t)}{\alpha}\cos\left(\alpha x\right)\sin\left(\pi z\right)-\frac{C(t)}{\pi}\sin\left(2\pi z\right) \quad (7)$$

$$C_1=\frac{2\sqrt{2}D(t)}{\alpha}\cos\left(\alpha x\right)\sin\left(\pi z\right)-\frac{E(t)}{\pi}\sin\left(2\pi z\right) \quad (8)$$

$$C_2=\frac{2\sqrt{2}F(t)}{\alpha}\cos\left(\alpha x\right)\sin\left(\pi z\right)-\frac{G(t)}{\pi}\sin\left(2\pi z\right) \quad (9)$$

where $\delta^2=\pi^2+\alpha^2$ and α is the horizontal wave number. The problem now is to determine the amplitudes $A(t)$ to $G(t)$. We substitute Eqs. (6) - (9) into Eqs. (1) - (5) and consistently neglect all higher order terms to obtain the following system of nonlinear ordinary autonomous differential equations

$$\dot{A}=-Pr\left[\eta A+\left(R_tB-R_{s1}D-R_{s2}F\right)\frac{\alpha^2}{\delta^6}\right] \quad (10)$$

$$\dot{B}=A\left(C-1\right)-B \quad (11)$$

$$\dot{C}=\varpi\left(AB+C\right) \quad (12)$$

$$\dot{D}=A\left(E-1\right)-\tau_1 D \quad (13)$$

$$\dot{E}=\varpi\left(AD+\tau_1 E\right) \quad (14)$$

$$\dot{F}=A\left(G-1\right)-\tau_2 F \quad (15)$$

$$\dot{G}=\varpi\left(AF+\tau_2 G\right) \quad (16)$$

where, $\varpi=4\pi^2/\delta^2$, $\eta=1+\Lambda_c\delta^2$ and the dot above a quantity denotes the derivative with respect to t. The above system of equations possesses an important symmetry that they are invariant under the transformation

$$(A, B, C, D, E, F, G) = (-A, -B, C, -D, E, -F, G). \quad (17a)$$

Since the divergence of the flow in a seven dimensional phase space

$$\left[\frac{\partial\dot{A}}{\partial A}+\frac{\partial\dot{B}}{\partial B}+\frac{\partial\dot{C}}{\partial C}+\frac{\partial\dot{D}}{\partial D}+\frac{\partial\dot{E}}{\partial E}+\frac{\partial\dot{F}}{\partial F}+\frac{\partial\dot{G}}{\partial G}\right]=-\left[Pr\left(1+\Lambda_c\delta^2\right)+\left(1+\varpi\right)\left(1+\tau_1+\tau_2\right)\right] \quad (17b)$$

is always negative, the solutions are attracted to a set of measure zero in the phase space and this may be a fixed point, a limit cycle or a strange attractor.

III. Bifurcations from the static solution

Equations (10)-(16) admit the trivial solution $A = B = C = D = E = F = G = 0$ that corresponds to pure conduction of heat and solute concentrations with no fluid motion present. The linear stability properties of this static solution may be obtained from Eq. (10) upon neglecting all nonlinear terms and seeking the solutions of the form $\exp(\sigma t)$, where σ is the growth rate.

The direct bifurcation occurs at

$$R_t^d=\frac{R_{s1}}{\tau_1}+\frac{R_{s2}}{\tau_2}+\frac{\eta\delta^6}{\alpha^2} \quad (18)$$

and the Hopf bifurcation occurs at

$$R_t^H = \frac{\left(\omega^2 + \tau_1 \delta^4\right)}{\left(\omega^2 + \tau_1^2 \delta^4\right)} R_{s1} + \frac{\left(\omega^2 + \tau_2 \delta^4\right)}{\left(\omega^2 + \tau_2^2 \delta^4\right)} R_{s2} + \frac{\delta^2}{\alpha^2}\left\{\frac{-\omega^2}{Pr} + \delta^4 \eta\right\} \tag{19}$$

provided that

$$\Delta_1 \left(\omega^2\right)^2 + \Delta_2 \left(\omega^2\right) + \Delta_3 = 0 \tag{20}$$

where

$$\Delta_1 = \delta^2 \left(Pr + 1 + Pr \Lambda_c \delta^2\right)$$

$$\Delta_2 = \delta^6 \left(Pr + 1 + Pr \Lambda_c \delta^2\right)\left(\tau_1^2 + \tau_2^2\right) + Pr R_{s1} \alpha^2 \left(\tau_1 - 1\right) + Pr R_{s2} \alpha^2 \left(\tau_2 - 1\right)$$

$$\Delta_3 = \delta^{10} \left(Pr + 1 + Pr \Lambda_c \delta^2\right) \tau_1^2 \tau_2^2 + Pr R_{s1} \alpha^2 \delta^4 \tau_2^2 \left(\tau_1 - 1\right) + Pr R_{s2} \alpha^2 \delta^4 \tau_1^2 \left(\tau_2 - 1\right).$$

Equations (18) and (19) coincide with those of Shivakumara and Naveen Kumar [12] obtained from the full two-dimensional equations. Moreover, when Λ_c = 0, Eqs. (18) and (19) coincide with those of Pearlstein et al. [3]. It is thus observed that the model equations considered gives linear stability theory results which are identical with those for the full problem. Equation (20) suggests the possibility of having two different real positive values of ω^2 at the same α and for each one of these frequency values $(\omega^2 > 0)$ there is a corresponding real value of the thermal Rayleigh number on the Hopf bifurcation curve. From the Descartes' rule of signs, in order for Eq.(20) to have two positive roots, it is necessary that, $\Delta_2 < 0$ and $\Delta_3 > 0$ which is equivalent to satisfying one of the conditions $\tau_2 > \tau_1 > 1$ or $\tau_2 < \tau_1 < 1$. Thus Hopf bifurcation is possible even if the diffusivity ratios are greater than unity; a result of contrast compared to double diffusive systems.

IV. Subcritical/Supercritical bifurcation

The results presented in the previous section do not give any information about the stability of bifurcating finite amplitude solution. In this section, we discuss this aspect. The system is not amenable for analytical treatment, in general and has to be solved numerically. However, for a steady case Eqs.(10)-(16) can be solved in the closed form and such a study is useful because it predicts the possibility of the occurrence of subcritical instability. Equations (10)-(16) admit a non-trivial steady solution of the form

$$B = -\frac{A}{A^2 + 1}, \quad C = \frac{A^2}{A^2 + 1} = -AB,$$

$$D = -\frac{A\tau_1}{A^2 + \tau_1^2}, \quad E = \frac{A^2}{A^2 + \tau_1^2} = -\frac{A}{\tau_1} D \tag{21}$$

$$F = -\frac{A\tau_2}{A^2 + \tau_2^2}, \quad G = \frac{A^2}{A^2 + \tau_2^2} = -\frac{A}{\tau_2} F$$

and A satisfies the equation

$$R_t = \frac{\tau_1\left(A^2 + 1\right)}{A^2 + \tau_1^2} R_{s1} + \frac{\tau_2\left(A^2 + 1\right)}{A^2 + \tau_2^2} R_{s2} + \frac{\eta\left(A^2 + 1\right)\delta^6}{\alpha^2}. \tag{22}$$

This solution does not depend on the Prandtl number. Equation (23) is cubic in A^2 and given by

$$\begin{aligned} &\eta\delta^6\left(A^2\right)^3 + \left[\begin{array}{l}\eta\delta^6\left(\tau_2^2 + \tau_1^2 + 1\right) - R_t\alpha^2 \\ \quad + R_{s1}\alpha^2\tau_1 + R_{s2}\alpha^2\tau_2\end{array}\right]\left(A^2\right)^2 \\ &+ \left[\begin{array}{l}\eta\delta^6\left(\tau_1^2 + \tau_2^2 + \tau_1^2\tau_2^2\right) \\ \quad - R_t\alpha^2\left(\tau_1^2 + \tau_2^2\right) + R_{s1}\alpha^2\tau_1\left(\tau_2^2 + 1\right) \\ \quad + R_{s2}\alpha^2\tau_2\left(\tau_1^2 + 1\right)\end{array}\right]\left(A^2\right) \\ &+ \left[\begin{array}{l}\eta\delta^6\tau_1^2\tau_2^2 - R_t\alpha^2\tau_1^2\tau_2^2 \\ \quad + R_{s1}\alpha^2\tau_1\tau_2^2 + R_{s2}\alpha^2\tau_2\tau_1^2\end{array}\right] = 0. \end{aligned} \tag{23}$$

Since we are dealing with weakly nonlinear stability analysis, the amplitudes are assumed to be small. Accordingly, we can expand R_t in powers of A^2 ($A^2 << 1$) in the form

$$R_t = R_t^d + R_{t2}^d A^2 + ... \tag{24}$$

Substituting Eq.(24) into Eq.(23), and collecting the coefficients of different powers of A^2, we observe that at zeroth order in A^2 the linearly stability analysis result is retrieved and at first order in A^2 it is found that

$$R_{t2}^d = \frac{(\tau_1^2 - 1)}{\tau_1^3} R_{s1} + \frac{(\tau_2^2 - 1)}{\tau_2^3} R_{s2} + \frac{\eta\,\delta^6}{\alpha^2}. \tag{25}$$

This is the first non-trivial finite amplitude Rayleigh number and coincides with the one obtained from the full problem. The finite amplitude solution is said to be stable (i.e., supercritical) if $R_{t2}^d > 0$ and unstable (i.e., subcritical) if $R_{t2}^d < 0$ when $\omega^2 < 0$. In the absence of additional diffusing components (i.e., $R_{s1} = 0 = R_{s2}$), we find that $R^d_{t2} = R^d_t$, and hence subcritical instability is not possible.

V. Heat and Mass Transport

The vigor of convection can be measured in terms of either heat/mass flux. However, it is convenient to introduce normalized heat and mass fluxes through the Nusselt numbers. The thermal Nusselt number is defined as

$$Nu_t = -\left\langle \frac{\partial T_{total}}{\partial z} \right\rangle_{z=0} \tag{26}$$

where $T_{total} = 1 - z + T$ and the angular brackets denote the horizontal average. Substituting for T from Eq.(7) then Eq.(26) gives

$$Nu_t = 1 + 2C = 1 + \frac{2A^2}{A^2 + 1} \tag{27}$$

Similarly, the solute Nusselt numbers are defined and are given by

$$Nu_{s1} = 1 + 2E = 1 + \frac{2A^2}{A^2 + \tau_1^2} \tag{28}$$

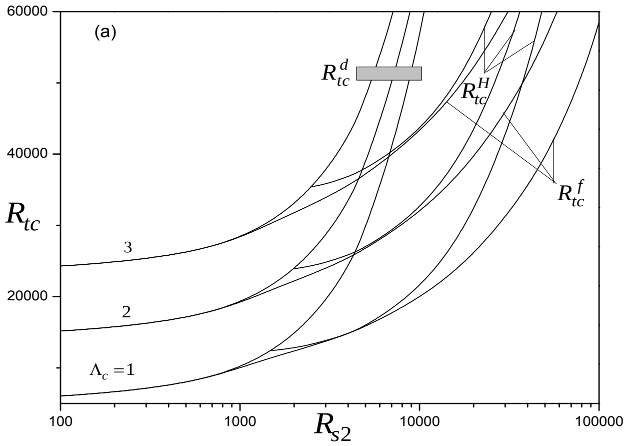
$$Nu_{s2} = 1 + 2G = 1 + \frac{2A^2}{A^2 + \tau_2^2} \tag{29}$$

In the absence of convection (i.e., $A = 0$), the heat/mass transfer is only by conduction and in that case $Nu_t = 1 = Nu_{s1} = Nu_{s2}$.

VI. Results and Discussion

The effect of couple stresses on two-dimensional triple diffusive convection is analyzed by constructing a system of autonomous nonlinear ordinary differential equations. Condition for the occurrence of direct, Hopf and finite amplitude bifurcations is obtained. The critical value of R^d_t and R^H_t computed numerically with respect to the wave number is denoted respectively by R^d_{tc} and R^H_{tc}. The critical value of finite amplitude Rayleigh number R^f_t is computed by finding the double minimum with respect to the amplitude A as well as α from Eq. (22) and is denoted by R^f_{tc}.

To know the occurrence of subcritical bifurcation, the critical Rayleigh numbers R^d_{tc}, R^H_{tc} and R^f_{tc} obtained as a function of R_{s2} for different values of Λ_c are compared in Figs. 1(a) and (b) for $R_{s1} = -1000$ (i.e., the component is destabilizing) and 1000 (i.e., the component is stabilizing), respectively. The results presented here are for $Pr = 10.2$, $\tau_1 = 0.22$ and $\tau_2 = 0.21$. From the figures it is observed that increasing R_{s2} and Λ_c is to increase the Rayleigh numbers and thus their effect is to delay the onset of triple diffusive convection. Also, Hopf bifurcation occurs when R_{s2} exceeds a threshold value which is higher when $R_{s1} = -1000$ for a fixed value of Λ_c, and also the threshold value increases with increasing Λ_c. Although the onset of convection is via Hopf bifurcation according to the linear theory, subcritical bifurcation is found to be possible at values of Rayleigh number lower than those of R^H_{tc} once R_{s2} exceeds certain value. It is further noted that the value of R_{s2} increases with increasing Λ_c and also when $R_{s1} = -1000$. Thus for certain choices of physical parameters, $R^f_{tc} < R^H_{tc} < R^d_{tc}$ indicating the possibility of occurring subcritical bifurcation.



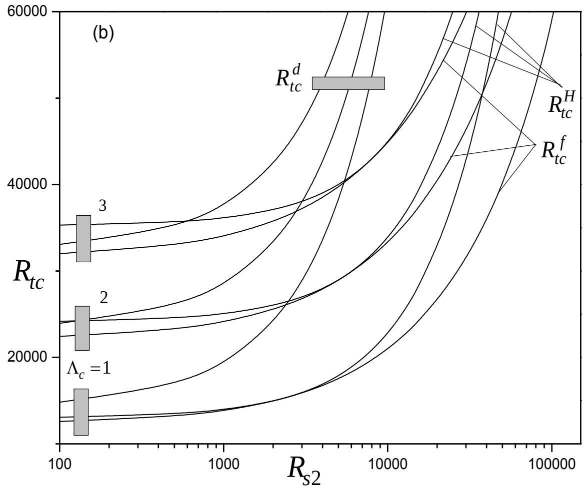
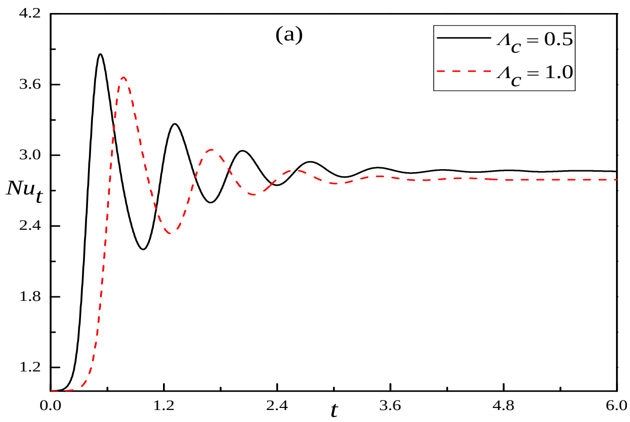
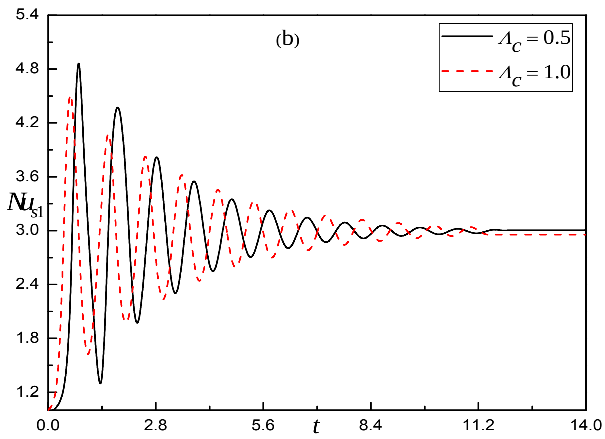


Fig. 1 Variation of R^d_{tc}, R^H_{tc} and R^f_{tc} with R_{s2} for (a) $R_{s1} = -1000$,(b) $R_{s1} = 1000$ when $Pr = 10.2$, $\tau_1 = 0.22$ and $\tau_2 = 0.21$.

The autonomous nonlinear system of differential equations is solved numerically using the Runge-Kutta-Gill method with appropriate initial conditions for different values of the governing parameters and the transient behavior of Nusselt numbers is demonstrated in Figs.2 (a, b, c) for $Pr = 10.2$, $R_{s1} = 1000$, $R_{s2} = -1000$, $\tau_1 = 0.22$, $\tau_2 = 0.21$ and for two values of $\Lambda_C = 0.5$ and 1.0. The Nusselt number oscillates initially and reaches a steady state value with further increase in time. The effect of increasing couple stress parameter is to suppress oscillations and to reduce the rate of heat and mass transfer. Thus the presence of couple stress is to inhibit the onset of convection. From the figures it is also evident that the solute Nusselt numbers oscillate with time more than the thermal Nusselt number. Moreover, the value of thermal Nusselt number is lower compared to solute Nusselt numbers.





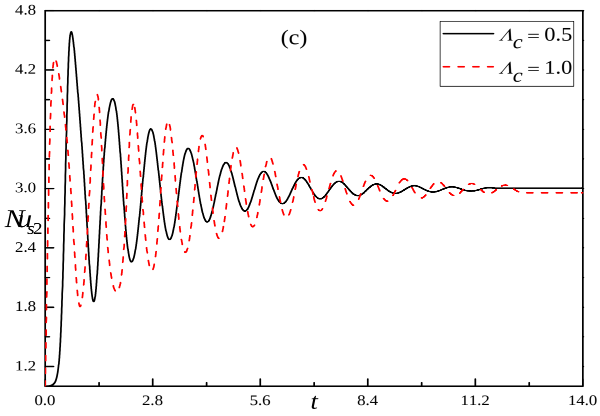


Fig. 2 Variation of (a) Nu_t, (b) Nu_{s1} and (c) Nu_{s2} with time for two values of Λ_c with $Pr = 10.2$, $R_{s1} = 1000$, $R_{s2} = -1000$, $\tau_1 = 0.22$, $\tau_2 = 0.21$.

VII. Conclusions

The results of the foregoing study may be summarized as follows:

(i) Hopf bifurcation is possible even if the diffusivity ratios are greater than unity; a result of contrast compared to doubly diffusive fluid systems. The presence of couple stress is to increase the threshold value of solute Rayleigh number for the existence of Hopf bifurcation.

(ii) Subcritical bifurcation is possible for certain choices of parametric values. Effect of increasing couple stress parameter is to delay the onset of direct, Hopf and finite amplitude convection.

(iii) Heat and mass transfer decrease with increasing couple stress parameter and increase when the diffusing components are destabilizing.

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