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CR- Submanifolds of Nearly Hyperbolic Sasakian Manifold Endowed With A Quarter Symmetric Non-Metric Connection

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Abstract
We consider a nearly hyperbolic Sasakian manifold endowed with a quarter symmetric non-metric connection and study CR- submanifolds of nearly hyperbolic Sasakian manifold endowed with a quarter symmetric non-metric connection. We also obtain parallel distributions and discuss inerrability conditions of distributions on CR-submanifolds of nearly hyperbolic Sasakian manifold with quarter symmetric non-metric connection.
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I. Introduction

CR-submanifolds of a Kaehler manifold as generalization of invariant and anti-invariant submanifolds was introduced and studied by A. Bejancu in ([1], [2]). Since then, several papers on Kaehler manifolds were published. CR-submanifolds of Sasakian manifold was studied by C.J. Hsu in [3] and M. Kobayashi in [4]. Later, several geometers (see, [5], [6], [7], [8], [9], [10], [11], [12], [13]) enriched the study of CR-submanifolds of almost contact manifolds. The almost hyperbolic (f, g, η, ξ)-structure was defined and studied by Upadhyay and Dube in [14]. Dube and Bhatt studied CR-submanifolds of trans-hyperbolic Sasakian manifold in [15]. On the other hand, S. Golab introduced the idea of semi-symmetric and quarter symmetric connections in [16]. CR-submanifolds of LP-Sasakian manifold with quarter symmetric non-metric connection were studied by the first author and S.K. Lovejoy Das in [17]. CR-submanifolds of a nearly hyperbolic Sasakian manifold admitting a semi-symmetric semi-metric connection were studied by the first author, M.D. Siddiqi and S. Rizvi in [18]. In this paper, we study some properties of CR-submanifolds of a nearly hyperbolic Sasakian manifold with a quarter symmetric non-metric connection.

II. Preliminaries

Let $\bar{M}$ be an n-dimensional almost contact metric manifold with the almost contact metric structure (ϕ, ξ, η, g), where a tensor ϕ of type (1,1), a vector field ξ called structure vector field and η the dual 1-form of ξ satisfying the followings:

(2.1) $\phi^2 X = X + \eta(X)\xi, \quad g(X, \xi) = \eta(X),$

(2.2) $\eta(X) = -1, \quad \phi(\xi) = 0, \quad \eta o \phi = 0,$

(2.3) $g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y)$

for any X, Y tangent to M [19]. In this case

(2.4) $g(\phi X, Y) = -g(\phi Y, X).$

An almost hyperbolic contact metric structure-(ϕ, ξ, η, g) on $\bar{M}$ is called hyperbolic Sasakian [19] if and only if

(2.5) $(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$

(2.6) $\nabla_X \xi = \phi X$

for all vectors X, Y tangent to $\bar{M}$ and a Riemannian metric g and Riemannian connection ∇.

Further, an almost hyperbolic contact metric manifold $\bar{M}$ on (ϕ, ξ, η, g) is called nearly hyperbolic Sasakian [19] if

(2.7) $(\nabla_X \phi)Y + (\nabla_Y \phi)X = 2g(X, Y)\xi - \eta(Y)X - \eta(X)Y.$

Now, let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let TM and $T^\perp M$ be the Lie algebras of vector fields tangential to M and normal to M respectively and ∇^* be induced Levi-Civita connection [20] on M, then the Gauss and Weingarten formulas are given respectively by

(2.8) $\nabla_X Y = \nabla_X^* Y + h(X, Y),$

(2.9) $\nabla_X N = -A_N X + \nabla_X^\perp N$

for any $X, Y \in TM$ and $N \in T^\perp M$, where $\nabla^\perp$ is the connection on the normal bundle $T^\perp M$, h is the second fundamental form and A_N is the Weingarten map associated with N as

(2.10) $g(A_N X, Y) = g(h(X, Y), N).$

For any $x \in M$ and $X \in T_x M$, we write

(2.11) $X = PX + QX,$

where $PX \in D$ and $QX \in D^\perp$.

Similarly for N normal to M, we have

(2.12) $\phi N = BN + CN,$

where BN (resp. CN) is the tangential component (resp., normal component) of ϕN.

Owing due presence of 1-form η, we define a quarter symmetric non-metric connection [16] by

(2.13) $\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X,$

$(\bar{\nabla}_X g)(Y, Z) = \eta(Y) g(\phi X, Z) - \eta(Z) g(\phi X, Y)$

for any $X, Y \in TM$, ∇ is the induced connection on M.

Using (2.13) and (2.7), we get

(2.14) $(\overline{\nabla}_X\phi)Y+(\overline{\nabla}_Y\phi)X=2g(X,Y)\xi-2\eta(X)Y-2\eta(Y)X-2\eta(X)\eta(Y)\xi.$

An almost hyperbolic contact manifold is called nearly hyperbolic Sasakian [19] manifold with quarter symmetric non-metric connection if it satisfies (2.14).

Also, from (2.6) and (2.13), we get

(2.15) $\overline{\nabla}_X\xi=2\phi X.$

Gauss and Weingarten formula for quarter symmetric non-metric connection are given respectively by

(2.16) $\overline{\nabla}_XY=\nabla_XY+h(X,Y),$

(2.17) $\overline{\nabla}_XN=-A_NX+\nabla_X^{\perp}N.$

Definition 2.1. An m-dimensional Riemannian submanifold M of a nearly hyperbolic Sasakian manifold $\overline{M}$ is called a CR-submanifold [20] of $\overline{M}$, if there exists a differentiable distribution $D: x\longrightarrow D_x$ on M satisfying the following conditions:

i. D is invariant, that is $\phi D_x\subset D_x$ for each $x\in M$.

ii. The complementary orthogonal distribution $D^{\perp}$ of D is anti-invariant, that is $\phi D_x^{\perp}\subset T_x^{\perp}M$. If $\dim D_x^{\perp}=0$ (resp., $\dim D_x=0$), then the CR-submanifold is called an invariant (resp., anti-invariant) submanifold. The distribution D (resp., $D^{\perp}$) is called the horizontal (resp., vertical) distribution. Also, the pair $(D,D^{\perp})$ is called $\xi-$horizontal (resp., vertical) if $\xi_X\in D_X$ (resp., $\xi_X\in D_X^{\perp}$).

III. Some Basic Lemmas

Lemma 3.1. Let M be a CR-submanifold of a nearly hyperbolic Sasakian manifold $\overline{M}$ with quarter symmetric non-metric connection. Then

(3.1) $2g(X,Y)P\xi-2\eta(X)PY-2\eta(Y)PX-2\eta(X)\eta(Y)P\xi+\phi P(\nabla_XY)+\phi P(\nabla_YX)=P\nabla_X(\phi PY)+P\nabla_Y(\phi PX)-PA_{\phi QX}Y-PA_{\phi QY}X,$

(3.2) $2g(X,Y)Q\xi-2\eta(X)QY-2\eta(Y)QX-2\eta(X)\eta(Y)Q\xi+2Bh(X,Y)=Q\nabla_X(\phi PY)+Q\nabla_Y(\phi PX)-QA_{\phi QX}Y-QA_{\phi QY}X,$

(3.3) $\phi Q\nabla_XY+Q\nabla_YX+2Ch(X,Y)=h(X,\phi PY)+h(Y,\phi PX)+\nabla_X^{\perp}(\phi QY)+\nabla_Y^{\perp}(\phi QX)$

for $X,Y\in TM$.

Proof. From (2.11), we have

$$\phi Y=\phi PY+\phi QY.$$

Differentiating covariant and using (2.16) and (2.17), we get

$$\overline{\nabla}_X(\phi Y)=(\overline{\nabla}_X\phi)Y+\phi\nabla_XY+\phi h(X,Y).$$

Also,

$$\overline{\nabla}_X(\phi PY+\phi QY)=\nabla_X(\phi PY)+h(X,\phi PY)-A_{\phi QY}X+\nabla_X^{\perp}(\phi QY).$$

Thus, we have

$$(\overline{\nabla}_X\phi)Y+\phi(\nabla_XY)+\phi h(X,Y)=\nabla_X(\phi PY)+h(X,\phi PY)-A_{\phi QY}X+\nabla_X^{\perp}(\phi QY).$$

Interchanging X and Y, we get

$$(\overline{\nabla}_Y\phi)X+\phi(\nabla_YX)+\phi h(Y,X)=\nabla_Y(\phi PX)h(Y,\phi PX)+h(Y,\phi PX)-A_{\phi QX}Y+\nabla_Y^{\perp}(\phi QX).$$

Adding above two equations, we have

$$(\overline{\nabla}_X\phi)Y+(\overline{\nabla}_Y\phi)X+\phi(\nabla_XY)+\phi(\nabla_YX)+\varphi h(X,Y)=\nabla_X(\phi PY)+\nabla_Y(\phi PX)+h(X,\phi PY)+h(Y,\phi PX)-A_{\phi QY}X-A_{\phi QX}Y+\nabla_X^{\perp}(\phi QY)+\nabla_Y^{\perp}(\phi QX).$$

Using (2.14) in above equation, we get

(3.4) $2g(X,Y)-2\eta(X)Y-2\eta(Y)X-2\eta(X)\eta(Y)\boldsymbol{\xi}+\phi(\nabla_XY)+\phi(\nabla_YX)+2\phi h(Y,X)=\nabla_X(\phi PY)+\nabla_Y(\phi PX)+h(X,\phi PY)+h(Y,\phi PX)-A_{\phi QY}X-A_{\phi QX}Y+\nabla_X^{\perp}(\phi QY)+\nabla_Y^{\perp}(\phi QX).$

Equations (3.1)-(3.3) follow by the comparison of the tangential, vertical and normal components of (3.4).

Hence the lemma is proved. □

Lemma 3.2. Let M be a CR-submanifold of a nearly hyperbolic Sasakian manifold $\overline{M}$ with quarter symmetric non-metric connection. Then

(3.5) $2(\overline{\nabla}_X\phi)Y=2g(X,Y)\xi-\eta(Y)\phi X-\eta(X)\phi Y-\eta(Y)X-\eta(X)Y+2\eta(X)\eta(Y)\xi-\nabla_X\phi Y\ h(X,\phi Y)-\nabla_Y\phi X-h(Y,\phi X)-\phi[X,Y],$

(3.6) $2(\overline{\nabla}_Y\phi)X=2g(X,Y)\xi-\eta(Y)\phi X-\eta(X)\phi Y-\eta(Y)X-\eta(X)Y+2\eta(X)\eta(Y)\xi-\nabla_X\phi Y\ h(X,\phi Y)+\nabla_Y\phi X+h(Y,\phi X)+\phi[X,Y]$

for any $X,Y\in D$.

Proof. From Gauss formula (2.16), we get

(3.7) $\overline{\nabla}_X\phi Y-\overline{\nabla}_Y\phi X=\nabla_X\phi Y+h(X,\phi Y)-\nabla_Y\phi X-h(Y,\phi X).$

Also, we have

(3.8) $\overline{\nabla}_X\phi Y-\overline{\nabla}_Y\phi X=(\overline{\nabla}_X\phi)Y-(\overline{\nabla}_Y\phi)X+\phi[X,Y].$

From (3.7) and (3.8), we get

(3.9) $(\overline{\nabla}_X\phi)Y-(\overline{\nabla}_Y\phi)X=\nabla_X\phi Y+h(X,\phi Y)-\nabla_Y\phi X-h(Y,\phi X)-\phi[X,Y].$

Adding (2.14) and (3.9), We have

$$2(\overline{\nabla}_X\phi)Y=2g(X,Y)\xi-2\eta(Y)X-2\eta(X)Y-2\eta(X)\eta(Y)\xi+\nabla_X\phi Y+h(X,\phi Y)-\nabla_Y\phi X-h(Y,\phi X)-\phi[X,Y].$$

Subtracting (3.9) from (2.14), we get

$$2(\overline{\nabla}_Y\phi)X=2g(X,Y)\xi-2\eta(Y)X-2\eta(X)Y-2\eta(X)\eta(Y)\xi-\nabla_X\phi Y-h(X,\phi Y)+\nabla_Y\phi X+h(Y,\phi X)+\phi[X,Y]$$

Hence the lemma is proved. □

Corollary 3.3. If M be a $\xi-$vertical CR-submanifold of a nearly hyperbolic Sasakian manifold $\overline{M}$ with quarter symmetric non-metric connection. Then

$$2(\overline{\nabla}_X\phi)Y=2g(X,Y)\xi+\nabla_X\phi Y+h(X,\phi Y)-\nabla_Y\phi X-h(Y,\phi X)-\phi[X,Y]$$

and

$$2(\overline{\nabla}_Y\phi)X=2g(X,Y)\xi-\nabla_X\phi Y-h(X,\phi Y)+\nabla_Y\phi X+h(Y,\phi X)+\phi[X,Y]$$

for any $X,Y\in D$.

Lemma 3.4. Let M be a CR-submanifold of a nearly hyperbolic Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection, then

(3.10) $2(\bar{\nabla}_Z\phi)Y = 2g(Z,Y)\boldsymbol{\xi} - \eta(Y)\phi Z - \eta(Z)\phi Y - \eta(Y)Z - \eta(Z)Y + 2\eta(Z)\eta(Y)\xi + A_{\phi Z}Y - A_{\phi Y}Z + \nabla_Z^{\perp}\phi Y - \nabla_{\phi Y}^{\perp}Z - \phi[Y,Z],$

(3.11) $2(\bar{\nabla}_Y\phi)Z = 2g(Y,Z)\boldsymbol{\xi} - \eta(Y)\phi Z - \eta(Z)\phi Y - \eta(Y)Z - \eta(Z)Y + 2\eta(Z)\eta(Y)\xi - A_{\phi Z}Y + A_{\phi Y}Z - \nabla_Z^{\perp}\phi Y + \nabla_Y^{\perp}\phi Z - \phi[Z,Y]$

for any $Y,Z \in D^{\perp}$.

Proof. Let $Y,Z \in D^{\perp}$, then from Weingarten formula (2.17), we have

(3.12) $\bar{\nabla}_Z\phi Y - \bar{\nabla}_Y\phi Z = -A_{\phi Y}Z + A_{\phi Z}Y + \nabla_Z^{\perp}\phi Y - \nabla_Y^{\perp}\phi Z.$

Also, we have

(3.13) $\bar{\nabla}_Z\phi Y - \bar{\nabla}_Y\phi Z = -(\bar{\nabla}_Y\phi)Z + (\bar{\nabla}_Z\phi)Y + \phi[Z,Y].$

From (3.12) and (3.13), we get

(3.14) $(\bar{\nabla}_Z\phi)Y - (\bar{\nabla}_Y\phi)Z = -A_{\phi Y}Z + A_{\phi Z}Y + \nabla_Z^{\perp}\phi Y - \nabla_Y^{\perp}\phi Z - \phi[Z,Y].$

Also for nearly hyperbolic Sasakian manifold, we have

(3.15) $(\bar{\nabla}_Z\phi)Y + (\bar{\nabla}_Y\phi)Z = 2g(Z,Y)\xi - 2\eta(Y)Z - 2\eta(Z)Y - 2\eta(Z)\eta(Y)\xi.$

Adding (3.14) and (3.15), we obtain

$$2(\bar{\nabla}_Z\phi)Y = 2g(Z,Y)\xi - 2\eta(Y)Z - 2\eta(Z)Y - 2\eta(Z)\eta(Y)\xi - A_{\phi Y}Z + A_{\phi Z}Y + \nabla_Z^{\perp}\phi Y - \nabla_{\phi Y}^{\perp}Z - \phi[Y,Z].$$

Subtracting (3.14) from (3.15), we get

$$2(\bar{\nabla}_Y\phi)Z = 2g(Y,Z)\xi - 2\eta(Y)Z - 2\eta(Z)Y - 2\eta(Z)\eta(Y)\xi - A_{\phi Z}Y + A_{\phi Y}Z - \nabla_Z^{\perp}\phi Y + \nabla_Y^{\perp}\phi Z - \phi[Z,Y]$$

Hence the lemma is proved. □

Corollary 3.5. If M be a $\xi-$ vertical CR-submanifold of a nearly hyperbolic Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection. Then

$$2(\bar{\nabla}_Z\phi)Y = 2g(Y,Z)\xi + A_{\phi Z}Y - A_{\phi Y}Z + \nabla_Z^{\perp}\phi Y - \nabla_Y^{\perp}\phi Z - \phi[Z,Y],$$

and $$2(\bar{\nabla}_Y\phi)Z = 2g(Y,Z)\xi + A_{\phi Z}Y - A_{\phi Y}Z - \nabla_Z^{\perp}\phi Y + \nabla_Y^{\perp}\phi Z + \phi[Z,Y]$$

for any $Y,Z \in D^{\perp}$.

Lemma 3.6. Let M be a CR-submanifold of a nearly hyperbolic Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection, then

(3.16) $2(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y] + 2g(X,Y)\xi - 2\eta(Y)X - 2\eta(X)Y - 2\eta(X)\eta(Y)\xi,$

(3.17) $2(\bar{\nabla}_Y\phi)X = A_{\phi Y}X - \nabla_X^{\perp}\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \phi[X,Y] + 2g(X,Y)\xi - 2\eta(Y)X - 2\eta(X)Y - 2\eta(X)\eta(Y)\xi$

for any $X \in D$ and $Y \in D^{\perp}$.

Proof. Let $X \in D$ and $Y \in D^{\perp}$, then from Gauss and Weingarten formula, we have

(3.18) $\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X).$

Also, we have

(3.19) $\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = (\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X + \phi[X,Y].$

From (3.18) and (3.19), we find

(3.20) $(\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y].$

Also from nearly hyperbolic Sasakian manifold, we have

(3.21) $(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = 2g(X,Y)\xi - 2\eta(Y)X - 2\eta(X)Y - 2\eta(X)\eta(Y)\xi.$

Adding (3.20) and (3.21), we get

$$2(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X.Y] + 2g(X,Y)\xi - 2\eta(Y)X - 2\eta(X)Y - 2\eta(X)\eta(Y)\xi.$$

Subtracting (3.20) from (3.21), we have

$$(2\bar{\nabla}_Y\phi)X = A_{\phi Y}X - \nabla_X^{\perp}\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \phi[X.Y] + 2g(X,Y)\xi - 2\eta(X)Y - 2\eta(Y)X - 2\eta(X)\eta(Y)\xi.$$

Hence the lemma is proved. □

Corollary 3.7. If M be a $\xi-$horizontal CR-submanifold of a nearly hyperbolic Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection. Then

$$2(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X.Y] + 2g(X,Y)\xi - 2\eta(X)Y$$

and

$$2(\bar{\nabla}_Y\phi)X = A_{\phi Y}X - \nabla_X^{\perp}\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \phi[X.Y] + 2g(X,Y)\xi - 2\eta(X)Y$$

for any $X \in D$ and $Y \in D^{\perp}$.

Corollary 3.8. If M be a $\xi-$ vertical CR-submanifold of a nearly hyperbolic Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection. Then

$$(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y] + 2g(X,Y)\xi - 2\eta(Y)X$$

and

$$(\bar{\nabla}_Y\phi)X = A_{\phi Y}X - \nabla_X^{\perp}\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \phi[X,Y] + 2g(X,Y)\xi - 2\eta(Y)X$$

for any $X \in D$ and $Y \in D^{\perp}$.

IV. Parallel Distribution

Definition 4.1. The horizontal (resp., vertical) distribution $D(\text{resp.}, D^{\perp})$ is said to be parallel [19] with respect to the connection on M *if* $\nabla_X Y \in D$ (resp., $\nabla_Z W \in D^{\perp}$) for any vector field $X,Y \in D$ $(resp., W,Z \in D^{\perp})$.

Theorem 4.2. Let M be a CR-submanifold of a nearly hyperbolic Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection. If the horizontal distribution D is parallel, then

(4.1) $h(X,\phi Y) = h(Y,\phi X)$

for all $X,Y \in D$.

Proof. Using parallelism of horizontal distribution D, we have

(4.2) $\nabla_X(\phi Y) \in D$ and $\nabla_Y(\phi X) \in D$ for any $X, Y \in D$.
From (3.2), we have

(4.3) $2g(X,Y)\xi + 2Bh(X,Y) = 0.$

Using (2.20) in (4.3), we get

(4.4) $\phi h(X,Y) = Bh(X,Y) + Ch(X,Y).$

Using (4.3) and (4.4), we obtain

(4.5) $\phi h(X,Y) = -g(X,Y)\xi + Ch(X,Y).$

Now, from (3.3), we have

(4.6) $2Ch(X,Y) = h(X,\phi Y) + h(Y,\phi X).$

Using (4.5) and (4.6), we have

(4.7) $2\phi h(X,Y) + 2g(X,Y)\xi = h(X,\phi Y) + h(Y,\phi X).$

Replacing X by ϕX in (4.7), we get

(4.8) $2\phi h(\phi X,Y) + 2g(\phi X,Y)\xi = h(\phi X,\phi Y) + h(Y,X).$

Replacing Y by ϕY in (4.7), we get

(4.9) $2\phi h(X,\phi Y) + 2g(X,\phi Y)\xi = h(X,Y) + h(\phi Y,\phi X).$

From (4.8) and (4.9), we obtain

$$h(\phi X,Y) = h(X,\phi Y) \quad \text{for all } X,Y \in D.$$

Hence the lemma is proved. □

Theorem 4.3. Let M be a $\xi-$vertical CR-submanifold of a nearly hyperbolic Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection, if the distribution $D^\perp$ is parallel with respect to the connection on M, then

(4.10) $\left(A_{\phi Z}Y + A_{\phi Y}Z\right) \in D^\perp$ for any $Y,Z \in D^\perp$.

Proof: Let $Y, Z \in D^\perp$. Using Gauss and Weingarten formula (2.16) and (2.17), we get

(4.11) $(\bar{\nabla}_Y\phi)Z = -A_{\phi Z}Y + \nabla_Y^\perp\phi Z - \phi(\nabla_Y Z) - \phi h(Y,Z).$

Interchanging Y and Z, we have

(4.12) $(\bar{\nabla}_Z\phi)Y = -A_{\phi Y}Z + \nabla_Z^\perp\phi Y - \phi(\nabla_Z Y) - \phi h(Z,Y).$

Adding above two equations, we get

(4.13) $(\bar{\nabla}_Y\phi)Z + (\bar{\nabla}_Z\phi)Y = -A_{\phi Z}Y - A_{\phi Y}Z + \nabla_Y^\perp\phi Z + \nabla_Z^\perp\phi Y - \phi(\nabla_Y Z) - \phi(\nabla_Z Y) - 2\phi h(Z,Y).$

Using (2.13), we have

(4.14) $2g(Y,Z)\xi - 2\eta(Z)Y - 2\eta(Y)Z - 2\eta(Y)\eta(Z)\xi = -A_{\phi Z}Y - A_{\phi Y}Z + \nabla_Y^\perp\phi Z + \nabla_Z^\perp\phi Y - \phi(\nabla_Y Z) - \phi(\nabla_Z Y) - 2\phi h(Z,Y).$

Taking inner product with $X \in D$ in (4.14), we find

$$g\left(A_{\phi Z}Y,X\right) + g\left(A_{\phi Y}Z,X\right) + g(\phi\nabla_Y Z,X) + g(\phi\nabla_Z Y,X) = 0.$$

If $D^\perp$ is parallel then $\nabla_Y Z \in D^\perp$ and $\nabla_Z Y \in D^\perp$, so that

$$g\left(A_{\phi Z}Y + A_{\phi Y}Z.X\right) = 0.$$

Consequently, we have

$$\left(A_{\phi Z}Y + A_{\phi Y}Z\right) \in D^\perp \text{ for any } Y,Z \in D^\perp.$$

Hence the lemma is proved. □

Definition 4.4. A CR-submanifold is said to be mixed-totally geodesic if $h(X,Z) = 0$ for all $X \in D$ and $Z \in D^\perp$.

Definition 4.5. A Normal vector field $N \neq 0$ is called $D-$parallel normal section if $\nabla_X^\perp N = 0$ for all $X \in D$.

Theorem 4.6. Let M be a mixed totally geodesic $\xi-$vertical CR-submanifold of a nearly hyperbolic Sasakian manifold $\bar{M}$ with quarter symmetric non-metric connection. Then the normal section $N \in \phi D^\perp$ is $D-$parallel if and only if $\nabla_X\phi N \in D$ for all $X \in D$.

Proof. Let $N \in \phi D^\perp$ and $X \in D, Y \in D^\perp$. Then from (3.2), we have

(4.16) $2Bh(X,Y) = Q\nabla_Y(\phi X) - QA_{\phi Y}X.$

For mixed geodesic, we have $h(X,Y) = 0$. Thus, equation (4.16) takes the form

(4.17) $Q\nabla_Y(\phi X) = QA_{\phi Y}X.$

Using (4.17) in (3.3), we get

(4.18) $\phi Q\nabla_X(\phi N) = \nabla_X^\perp N.$

Thus, if the normal section $N \neq 0$ is D-parallel, then using (4.18) and definition 4.5, we get

(4.19) $\phi\nabla_X(\phi N) = 0,$

which implies that $\nabla_X(\phi N) = 0$ for all $X \in D$.
The converse part easily follows from (4.18).
Hence the theorem is proved. □

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