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Design of Full Order Observer Using Generalized Matrix Inverse for Linear Time Invariant Systems

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Abstract- In this paper a full order observer has been designed using generalized matrix inverse. The design method resolves the state vector into two unique components, of which one is known and the other is unknown. This method does not assume any structure of the observer and imposes no restriction on the output distribution matrix. Condition of existence of such observer is presented with proof. An illustrative numerical example of two loop missile autopilot is also included with simulation results.
Keywords- Das & Ghoshal Observer (DGO), Full Order Observer, Generalized Matrix Inverse, Linear Time Invariant (LTI) Systems, Missile Autopilot.

I. INTRODUCTION

The problem of state estimation of a linear time invariant system (LTI) from the knowledge of its input and output has been discussed by several authors. In 1964 D.G. Luenberger first introduced the concept and it has been shown in [1] that the state vector of a linear system can be reconstructed from a pre supposed dynamics if the difference between the assumed structure and actual system forms a homogeneous ordinary differential equation. The concept is further developed in [2], where the special topics of identity observer, reduced order observer, linear functional observer, stability properties and dual observers are also discussed. In [3] it has been shown that state vector of an nth order system with m independent outputs can be reconstructed with an observer of order (n-m). S.D.G. Cumming [4] also presented a simple design of stable state observers with reduced dynamics. Such observers with reduced order dynamics result in great reduction in observer complexity. In [5] O' Reilly proposed a construction method of full order observer which also presupposed the observer structure. In 1981 G. Das and T.K. Ghoshal presented a novel approach to reduced order observer design using generalized matrix inverse [6]. Here the observer structure is not pre assumed and leads to a step by step procedure for observer design. In [6] it has been shown that the DGO is computationally simple as the use of generalized matrix inverse avoids the matrix operations like co-ordinate transformation and matrix partition to design the observer parameter matrices. In [7] it is shown that computation of generalized matrix inverse is not harder than matrix multiplication. In [8] a detailed comparative study has been carried out between reduced order Luenberger observer [1,2] and reduced order Das & Ghoshal observer (DGO) [6]. A method for designing Luenberger full order observer using generalized matrix inverse for both time variant and time invariant linear systems is proposed in [9]. Stefen Hui and Stanislaw H. Zak [10] used projection operator approach (Projection Method Observer or PMO) to estimate the states for the systems with both known and unknown inputs. Here the construction procedure considers the decomposition of the state vector into known and unknown components. Unlike DGO, where the unknown component is the orthogonal projection of the known component, PMO considers the components which are skew symmetric in nature. A detailed and exhaustive comparative study, between the unknown input reduced order DGO and the PMO, has been discussed in [11] and it has been shown that the unknown input DGO is computationally simple compared to PMO in observing the states of a system in presence of unknown inputs.. In [12] reduced order Das & Ghoshal observer had been extended to full-order observer using the principle of generalized matrix inverse. The work presented in [12] has been extended in this article to obtain simpler and computationally less complex design of full order observer using the concept of generalized matrix inverse. Similar to [6] & [12], this construction procedure also does not pre assume the observer structure. The observer dynamics in the proposed design is simpler than that in [12] and imposes no restriction on the output distribution matrix.

The following notations will be used here:-
R represents the field of real numbers; m x n denotes the dimension of a matrix with m rows and n columns. A^g denotes the Moore-Penrose generalized inverse of matrix A. A^T is the transpose of A and I represents the identity matrix of appropriate dimension. $R(X)$ represents rank of any matrix X.

II. MATHEMATICAL PRELIMINARIES

If $A \in R^{m\times n}$ is a matrix and a matrix $A^g \in R^{n\times m}$ exists that satisfies the four conditions below,

$$AA^g = (AA^g)^T \quad (1)$$
$$A^gA = (A^gA)^T \quad (2)$$
$$AA^gA = A \quad (3)$$
$$A^gAA^g = A^g \quad (4)$$

Then the matrix A^g is called the Moore-Penrose generalized matrix inverse of A and is unique for each A . If a system of linear equation is given by,

$$Ax = y \quad (5)$$

Where $A \in R^{m\times n}$ is a known matrix, $y \in R^{m\times 1}$ is a known vector and $x \in R^{n\times 1}$ is an unknown vector.
Then (5) is consistent if and only if,

$$AA^gy = y \quad (6)$$

If (5) is consistent then general solution of (5) is given by,

$$x = A^gy + (I - A^gA)v \quad (7)$$

([14] Graybill 1969 p.104). Where $v \in R^{n\times 1}$denotes an arbitrary vector having elements as arbitrary functions of time.

III. CONSTRUCTION PROCEDURE

Consider an LTI system described by,

$$\dot{x} = Ax + Bu,\ x_0 = x(0) \quad (8)$$
$$y = Cx \quad (9)$$

Where x is an (n x 1) unknown state vector to be estimated, x_0 is the initial condition of $x. u$ is (b x 1) input vector, y is (m x 1) output vector. A, B, C are known matrices of appropriate dimensions. We assume that the $\{A, C\}$ pair is completely observable which implies that the simultaneous solution for (8) and (9) for x is unique, when x_0, u and y are given.

Lemma

For (n x n) matrix A and (m x n) matrix C, the pair $\{A, CA\}$ is observable if the pair $\{A, C\}$ is observable and A is non singular matrix.

Proof.

If the pair $\{A, C\}$ is observable then the observability matrix Q is of full column rank.

$$\text{Where, } Q = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix}$$

Now the observability matrix for the pair $\{A, CA\}$ is given by,

$$Q' = \begin{bmatrix} CA \\ CA^2 \\ \cdot \\ \cdot \\ CA^n \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix} A = QA$$

Where, R(Q) = n and A is non-singular.
Hence, R(Q') = n. In other words, Q' is of full column rank if A is non-singular.
This completes the proof. of the lemma.

Now the general solution of (9) can be written as,

$$x = C^gy + (I - C^gC)v \quad (10)$$

Where C^g is the (n x m) generalized matrix inverse of C and v is an (n x 1) vector whose elements are arbitrary functions of time. Let's consider

$$(I - C^gC)v = h \quad (11)$$

By putting (11) in (10) we get,

$$x = C^gy + h \quad (12)$$

Now, putting (12) in (8) we get,

$$\dot{h} = Ah + AC^gy + Bu - C^g\dot{y} \quad (13)$$

Again from (8), (9) and (12) we get,

$$\dot{y} = C\dot{x}$$
$$\text{Or, } \dot{y} = C(Ax + Bu)$$
$$\text{Or, } \dot{y} = CAh + CAC^gy + CBu \quad (14)$$

Equation (13) is the dynamic equation and (14) is the output equation. h is the state vector, $AC^gy + Bu - C^g\dot{y}$ is the input vector and $\dot{y} - CAC^gy - CBu$ is the output vector. Thus using Luenberger's equation ([2], 3.4) Observer dynamic equation can be directly written as,

$$\dot{\tilde{h}} = (A - KCA)\tilde{h} + (B - KCB)u + (AC^g - KCAC^g)y + (K - C^g)\dot{y} \quad (15)$$

To estimate state variables using (15),$\{A, CA\}$ pair must be observable. In the lemma, it has been already shown that if $\{A, C\}$ pair is observable, then $\{A, CA\}$ pair is also observable when A matrix is non singular. Therefore the proposed observer will be effective only if, A matrix is of full rank. Then poles can be arbitrarily placed with the help of Ackerman's formula.

Here K is the gain matrix of (n x m) dimension. It can be determined using Ackerman's pole placement algorithm. In any case, if$\{A, CA\}$ pair is not observable, at least the system should be detectable to make the proposed observer work.

For arbitrary initial conditions of $\tilde{h}$, it will tend to h as time approaches to infinity. That means h can be observed from (15).

In order to eliminate the first derivative of y from (15), we replace $\tilde{h}$ with $\{\tilde{q} + (K - C^g)y\}$. Now the observer dynamic equation becomes,

$$\dot{\tilde{q}} = (A - KCA)\tilde{q} + (B - KCB)u + (AK - KCAK)y \quad (16)$$

This equation is valid for a class of linear time invariant systems with observable $\{A, CA\}$ pair.
Observed state vector $\tilde{x}$ can be expressed as,

$$\tilde{x} = \tilde{q} + Ky \quad (17)$$

$\tilde{x}$ would tend to $x$as $\tilde{h}$ tends to h when time approaches infinity. It is worth mentioning that, the observer is of order n. This completes the construction procedure of a full order observer.

IV. NUMERICAL EXAMPLE

Here we have taken the state space model of flight path rate demand autopilot in pitch plane as an example. It has been obtained from literature [13]. The A, B, C and D matrices of the state space model are written below,

$$A=\begin{bmatrix} -\frac{1}{T_a} & \frac{1+\sigma^2 w_b^2}{T_a} & -\frac{K_b\sigma^2 w_b^2}{T_a} & -K_b\sigma^2 w_b^2 \\ -\frac{1+T_a^2 w_b^2}{T_a(1+\sigma^2 w_b^2)} & \frac{1}{T_a} & \frac{(T_a^2-\sigma^2)K_b w_b^2}{T_a(1+\sigma^2 w_b^2)} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -w_a^2 & -2\zeta_a w_a \end{bmatrix}$$

$$B=\begin{bmatrix} 0 \\ 0 \\ 0 \\ K_q w_a^2 \end{bmatrix} \qquad C=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad D=0$$

State variables of the above state space model are as follows,

$x_1=\dot{\gamma}$ (Flight path rate)

$x_2=q$ (Body rate in pitch)

$x_3=\eta$ (Elevator deflection)

$x_4=\dot{\eta}$ (Rate of elevator deflection)

The following numerical data have been taken for the class of missile considered here.

$$T_a=2.85\sec, w_b=5.6rad/\sec, \sigma^2=.00142\sec^2$$

$$\zeta_a=0.6, K_b=-0.1437, w_a=180rad/\sec, K_q=-1.4$$

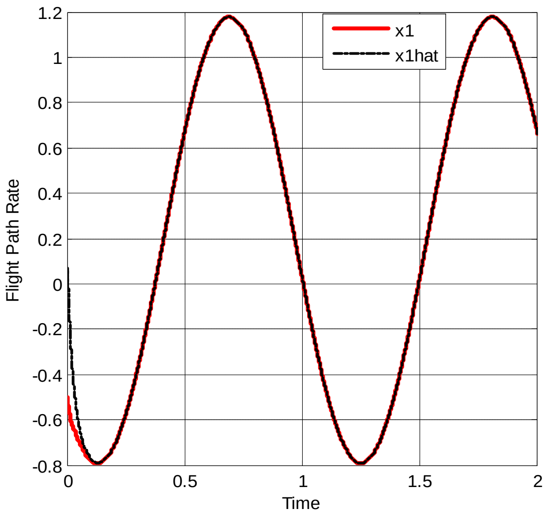

Figure 4.1: Flight Path Rate

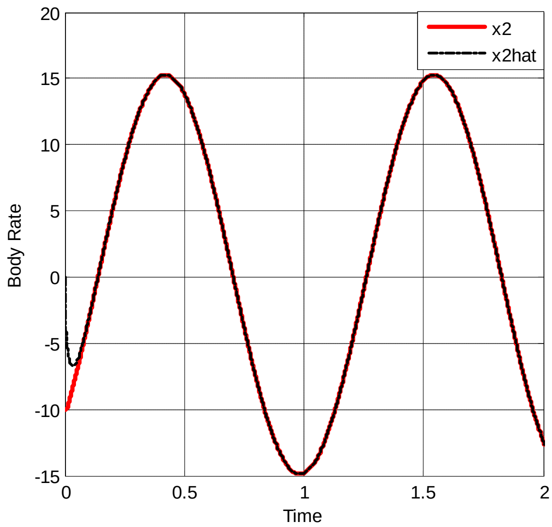

Figure 4.2: Body Rate in Pitch

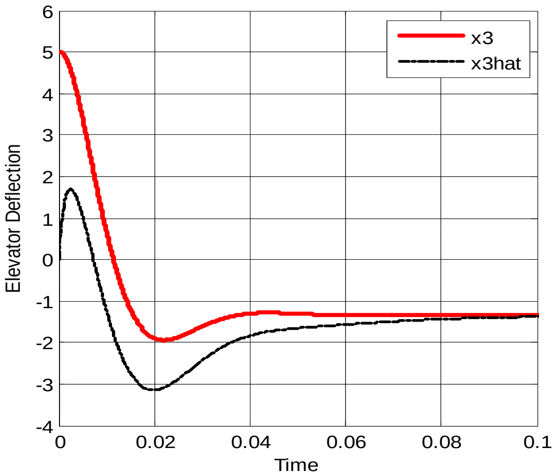

Figure 4.3: Elevator Deflection

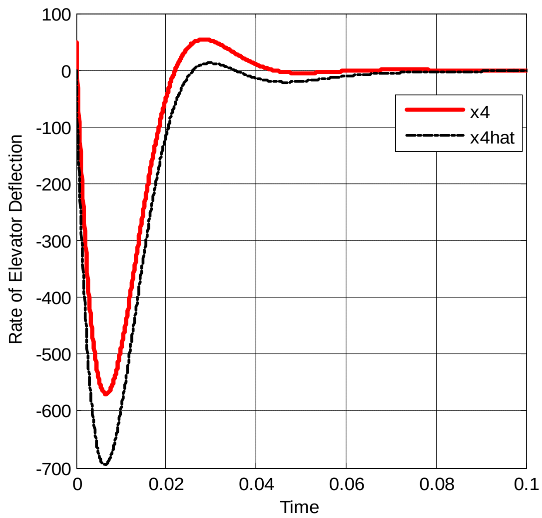

Figure 4.4: Rate of Elevator Deflection

V. DISCUSSIONS

In this paper the full order observer is constructed by applying the theory of generalized matrix inverse directly. The two key equations for observer construction, (13) and (14) are obtained directly from (9) and (8) using generalized matrix inverse. Original states and the corresponding estimated states of the above missile autopilot system are plotted using MATLAB (Figure 4.1 to Figure 4.4). The red lines indicate the original system state (X_i) while the black chain dotted lines indicate the estimated states (X_i hat where $i = 1,2,3,4$). System's initial condition is taken as $[-.5 \quad -10 \quad 5 \quad 50]^T$ Observer poles are placed at -47, -52, -600 and -700. Then observer gain is determined by using Ackerman's method. The observer gain is denoted by the symbol K. Here,

$$K = \begin{bmatrix} -13 & 510 & -445 & 79934 \\ -3 & 23 & -20 & -95 \end{bmatrix}^T$$

The simulation results show that, the observer dynamics starts from zero initial condition and merges to the system dynamics within a very short time period.

VI. CONCLUSION

An easy and step by step procedure to construct full order observer has been proposed in this article. The observer is simpler in structure and computationally less complex .This method also does not presuppose the observer structure and does not require any co-ordinate transformation, even if the output distribution matrix is not in standard form or not of full rank. Numerical example of two loop missile autopilot is also given to show the effectiveness of the procedure.

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