Isomorphism Theorems for Fuzzy Submodules of G-Modules

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ABSTRACT
In this paper we give isomorphism theorems for fuzzy submodules of G-modules.

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I. INTRODUCTION

The theory of group representation (G-module theory) was developed by Frobenious G[1962]. Soon after the concept of fuzzy sets was introduced by Zadeh [1] in 1965.Fuzzy subgroup and its important properties were defined and established by Rosenfeld [2] in 1971.After that in the year 2004 Shery Fernandez [3] introduced fuzzy parallels of the notions of G-modules.

In this paper we give isomorphism theorems for fuzzy submodules of G-modules.The idea came for proving these results from Mordeson and Malik [6] which was originally proved for L-submodules of R-modules M.
Throughout the paper we use the notation $\approx$ for homomorphism, $\cong$ for isomorphism.

II. PRELIMINARIES

To prove isomorphism theorems for fuzzy submodules of G-modules,we use the following definitions.

Definition 2.1 [5]: Let G be a finite group. A vector space M over a field K is called a **G-module** if for every g $\in$ G and m $\in$ M,there exist a product (called the action of G on M) m.g $\in$ M satisfying the following axioms :

$$(i)\, m.1_G = m \forall m \in M$$

$$(ii)\, m.(g.h) = (m.g).h$$

$$(iii)(k_1m_1 + k_2m_2).g = k_1(m_1.g) + k_2(m_2.g)$$

$$\forall k_1, k_2 \in K, m, m_1, m_2 \in M, g, h \in G$$

Where 1_G being identity element in G.

Definition-2.2[5]: Let M and M' be G-modules. A mapping $\phi : M \to M'$ is a **G-module homomorphism** if

$$(i)\phi(k_1m_1 + k_2m_2) = k_1\phi(m_1) + k_2\phi(m_2)$$

$$(ii)\phi(g.m) = g.\phi(m)$$

$$\forall k_1, k_2 \in K, m, m_1, m_2 \in M, g \in G$$

Further in above definition if ϕ is one-one and onto, then ϕ is an isomorphism.

Definition-2.3[5]: Let $\phi : M \to M'$ be a G-module homomorphism. The set of all $m \in M$ such that $\phi(m) = 0$,the zero element of M' is called the **kernel of** ϕ and is denoted by $\ker \phi$

In other words

$$\ker \phi = \{m \in M : \phi(m) = 0\}$$

Definition-2.4[4]: Let M be a G-module. A vector subspace N of M is a **G-submodule** if N is also a G-module under the same action of G.

Definition-2.5[5]: If M is a G-module and N is a G-submodule of M, then M/N is a G-module which is called **Quotient G-modules**.

Let $g \in G$ and $x + N \in \frac{M}{N}$. Define the action of G on M/N by

$$g(x + N) = gx + N \in \frac{M}{N}$$

This satisfies all the condition of G-module and therefore M/N is a G-module.

Definition-2.6[5]: Let G be a finite group and M be a G-module over K, Which is a subfield of C. Then a **fuzzy G-module** on M is a fuzzy subset μ of M such that

$$(i)\mu(ax + by) \geq \mu(x) \wedge \mu(y) \forall a, b \in K, x, y \in M.$$

$$(ii)\mu(gm) \geq \mu(m) \forall g \in G, m \in M.$$

Definition-2.7[6]: Let $\mu \in G^M$ (Where G^M denotes the fuzzy power set of G-module M).Then μ is called a **fuzzy submodule** of G-module M, if

$$(i)\mu(0) = 1_G$$

$$(ii)\mu(gm) \geq \mu(m) \forall g \in G, m \in M$$

$$(iii)\mu(m_1 + m_2) \geq \mu(m_1) \wedge \mu(m_2) \forall m_1, m_2 \in M$$

Definition-2.8[6]: Let $\nu \in G(M)$ (Where G (M) denotes the set of all fuzy submodules of G-module M) and let N be a submodule of M. Define

$\xi \in G^{\frac{M}{N}}$ (Where $G^{\frac{M}{N}}$ is the fuzzy power set of G-module M/N) as follows:

$$\xi([x]) = \vee\{\nu(u) : u \in [x]\} \;\; \forall x \in M$$

Where M/N denotes the quotient module of M w.r.t. N and [x] represents the coset x + N.

Then $\xi \in$ G (M/N)

Definition-2.9[6]: If $\mu \in G(M)$ then $\mu^* = \{x \in X : \mu(x) \succ 0\}$ is called **support of** μ.

III. MAIN RESULTS

We give the following results.

Theorem-3.1 (First isomorphism theorem)

Let $\nu \in G(M)$ and $\xi \in G(N)$ such that $\nu \approx \xi$. Then there exists $\mu \in G(M)$ such that $\mu \subseteq \nu$ and $\frac{\nu}{\mu} \cong \xi$.

Proof Since $\nu \approx \xi$ there exists an epimorphism f of M onto N such that $f(\nu) = \xi$.

Define $\mu \in G^M$ as follows

$$\mu(x) = \begin{cases} \nu(x), x \in \ker f \\ 0, x \notin \ker f \end{cases}$$

Then $\mu \in G(M)$ and $\mu \subseteq \nu$.

If $x \in \ker f$ then $yxy^{-1} \in \ker f, \forall y \in M$.

$$\mu(yxy^{-1}) = \nu(yxy^{-1}) \geq \nu(x) \wedge \nu(y) = \mu(x) \wedge \nu(y)$$

If $x \in G \setminus \ker f$ then $\mu(x) = 0$ and so $\mu(yxy^{-1}) \geq \mu(x) \wedge \nu(y)$.

Hence μ is normal G-submodule of ν. Also

$\nu \overset{f}{\approx} \xi \Rightarrow f(\nu) = \xi$.

Which further implies $(f(\nu))^* = \xi^*$

It follows that $f(\nu^*) = \xi^*$.

Let $g = f_{|\nu^*}$ then g is a homomorphism of ν^* onto ξ^* and kerg = μ^* by definition of μ. Then there exists an isomorphism h of ν^* / μ^* onto ξ^* such that

$$h(x\mu^*) = g(x) = f(x) \forall x \in \nu^*$$

For such an h, we have

$$h(\nu/\mu)(z) = \vee\{(\nu/\mu)(x\mu^*) : x \in \nu^*, h(x\mu^*) = z\}$$

$$= \vee\{\vee\{\nu(y) : y \in x\mu^*\} : x \in \nu^*, g(x) = z\}$$

$$= \vee\{\nu(y) : y \in \nu^*, g(y) = z\}$$

$$= \vee\{\nu(y) : y \in G, f(y) = z\}$$

$$= \xi(z), \forall z \in \xi^*$$

$$\therefore \frac{\nu}{\mu} \overset{h}{\cong} \xi.$$

Theorem-3.2 (Second isomorphism theorem)

Let μ, ν be fuzzy submodules of G-module M with $\mu(0) = \nu(0)$ then

$$\frac{\nu}{(\mu \cap \nu)} \cong \frac{(\mu + \nu)}{\mu}$$

Proof We have μ^* is normal G-submodule of M. By the second isomorphism theorem for module

$$\frac{\nu^*}{(\mu^* \cap \nu^*)} \cong \frac{\mu^* + \nu^*}{\mu^*}$$

One can verify that

$$(\mu \cap \nu)^* = (\mu^* \cap \nu^*)$$

$$(\mu + \nu)^* = \mu^* + \nu^*$$

Consequently we have

$$\frac{\nu^*}{(\mu \cap \nu)^*} \overset{f}{\cong} \frac{(\mu + \nu)^*}{\mu^*}$$

Where f is given by

$$f(x(\mu \cap \nu)^*) = x\mu^*, \forall x \in \nu^*$$

Thus

$$f\left(\frac{\nu}{(\mu \cap \nu)}\right)(y\mu^*) = \left(\frac{\nu}{(\mu \cap \nu)}\right)(y(\mu \cap \nu)^*)$$

(Since f is one-one)

$$= \vee\{\nu(z) : z \in y(\mu \cap \nu)^*\}$$

$$= \vee\{(\mu + \nu)(z) : z \in y(\mu^* \cap \nu^*)$$

$$\leq \vee\{(\mu + \nu)(z) : z \in y\mu^*\}$$

$$= \left(\frac{(\mu + \nu)}{\mu}\right)(y\mu^*), \forall y \in \nu^*$$

Hence

$$f\left(\frac{\nu}{(\mu \cap \nu)}\right) \subseteq \frac{(\mu + \nu)}{\mu}$$

$$\therefore \frac{\nu}{(\mu \cap \nu)} \cong \frac{(\mu + \nu)}{\mu}$$

Theorem-3.3 (Third isomorphism Theorem)

Let μ, ν, ξ be fuzzy submodules of G-module M with $\mu(0) = \nu(0)$ and $\mu \subseteq \nu \subseteq \xi$ then

$$\frac{(\xi/\mu)}{(\nu/\mu)} \cong \frac{\xi}{\nu}.$$

Proof Since $\mu \subseteq \nu \subseteq \xi$ then μ^* is submodule of ν^* and both μ^* and ν^* are submodule of ξ^*. By the third isomorphism theorem for modules

$$\frac{(\xi^*/\mu^*)}{(\nu^*/\mu^*)} \overset{f}{\cong} \frac{\xi^*}{\nu^*}$$

Where f is defined by

$$f(x\mu^*(\nu^*/\mu^*)) = x\nu^*, \forall x \in \xi^*$$

Thus

$$f\left(\frac{(\xi/\mu)}{(\nu/\mu)}\right)(x\nu^*) = \left(\frac{(\xi/\mu)}{(\nu/\mu)}\right)(x\mu^*(\nu^*/\mu^*))$$

$$= \vee\{(\xi/\mu)(y\mu^*) : y \in \xi^*, y\mu^* \in x\mu^*(\nu^*/\mu^*)\}$$

$$= \vee\{\vee\{\xi(z) : z \in y\mu^*\} : y \in \xi^*, y\mu^* \in x\mu^*(\nu^*/\mu^*)\}$$

$$= \vee\{\xi(z) : z \in \xi^*, f(z) \in x\nu^*\}$$

$$= \left(\frac{\xi}{\nu}\right)(x\nu^*)$$

Hence

$$\frac{(\xi/\mu)}{(\nu/\mu)} \overset{f}{\cong} \frac{\xi}{\nu}.$$

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