Proof of the Energy-Time uncertainty principle in Powerful Statement

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ABSTRACT

In this paper we have tried to perform a comprehensive summery of Energy-Time uncertainty principle. At first we review the history of uncertainty principle then the most well-known arguments between Einstein and Bohr. Our main aim is to provide an acceptable relationship for Energy-Time uncertainty principle and proof to it.

I. Introduction

The momentum-position uncertainty principle $\Delta P\Delta x \geq \hbar$ has an energy-time analog, $\Delta E\Delta t \gg \hbar$ .Though this must be a different kind of relationship to the momentum-position one, because t is not a dynamical variable, so this can't have anything to do with non- commutation.

There are many quantum mechanics books and papers about energy-time uncertainty principle. After reading many of them we asked ourselves: just what is the energy time uncertainty principle, anyhow? After looking in many quantum mechanics papers and books, We was astonished to find (this is 2013!) that only one of them [7] managed to give an acceptable statement of it, where by "acceptable" we mean an inequality in which all variables have a mathematical definition.

Section II gives a history of uncertainty principles and section III gives a careful review on energy-time uncertainty principle and finally we present a detailed proof for Thirring's view.

II. History of uncertainty principle

This section gives a history of uncertainty principles, including an amusing debunking of common myths about the Bohr-Einstein box debate. For another history quite complementary to ours, see [1].

The uncertainty principle expresses the physical content of quantum theory in a qualitative way [2]. The uncertainty principle was first proposed by Heisenberg in 1927. It basically states that is not possible to specify exactly and simultaneously the values of both members of a pair of physical variables which describe the behavior of an atomic system.

In a sense the principle can also be seen as a type of constraint. The members of a pair are canonically conjugate to each other in a Hamiltonian way. The most well- known example is the coordinate x of a particle (position in one dimension) and its corresponding momentum component P_x :

$$\Delta x \Delta P_x \geq \frac{\hbar}{2} \qquad (1)$$

Another example is the angular momentum component J_z of a particle and the angular position $\emptyset$ in the perpendicular (xy) plane:

$$\Delta \emptyset \Delta J_z \geq \hbar \qquad (2)$$

In classical mechanics these extreme situations complement each other and both variables can be determined simultaneously. Both variables are needed to fully describe the system under consideration. In quantum theory, (1) states that one cannot precisely determine a component of momentum of a particle without loosing all information of the corresponding position component at a specific time. If the in- between extremes case is considered, the product of the uncertainty in position and the uncertainty in the corresponding momentum must numerically be equal to, at least, $\hbar/2$ To understand the physical meaning of the uncertainty principle, Bohr in 1928 stated the complementary principle. This principle shows the fundamental limits on the classical concept that a system's behavior can be described independently of the observation procedure. The complementary principle states that "atomic phenomena cannot be described with the completeness demanded by classical dynamics" [2]. Basically the principle states that experimental apparatus cannot be used to determine a measurement more precisely than the limit given by the uncertainty principle. In a sense when a measurement is done to determine the value of one of a pair of canonically conjugate variables, the second variable experiences a shift in value. This shift cannot be calculated exactly without interfering with the measurement of the

Among the most amusing historical developments in the early history of quantum mechanics were the oft-recounted debates between A.Einstein and N.Bohr. The legends associated with these debates are positively permeated with the same aura as the famous story of George Washington and the Cherry Tree. Einstein at the 1930 Solvay conference [3], [4] presented to Bohr the following attack on quantum mechanics via the energy time uncertainty principle. Keep in mind that in 1930, it was still 15 years before the first precise statement of this principle had been made [5], so Bohr and Einstein were arguing about an issue neither understood. (They both thought the uncertainty principle was $\Delta E \Delta t \geq \frac{\hbar}{2}$ and with the interpretations of the Δ's the same as in the $\Delta p \Delta x$ principle. This interpretation has never been justified, and also note that the coefficient $\frac{1}{2}$ on the right hand side is also unjustified, at least in [1].) Einstein considered a box being weighed by a spring scale and containing photons. A cuckoo-clock mechanism mounted on the box opens a door on the box's side for a time duration t, specified extremely precisely, and with a precision apparently independent of any quantity having to do with photon energies. The weight of the box before and after the measurement seems to be measurable extremely precisely, and hence, since $E = mc^2$, the energy of the escaping photons is deducible extremely precisely. Hence we get c (where ΔE and Δt denote uncertainties in the values of t and of the energy E of the photons that escaped through the door) arbitrarily small, refuting quantum mechanics. Of course, of the three formulations of the energy-time principle given here, and Finkel's [6], none of them pertain to uncertainty in the time lapse at all, so this entire idea of Einstein's was based on a false premise. But neither Einstein nor Bohr were aware of that, and so both thought this was a tremendously dangerous attack on the foundations of quantum mechanics. So then (the usual tale proceeds) Bohr after great effort refuted Einstein's attack by invoking general relativity (!), as follows the uncertainty Δz in the vertical position z of the spring scale obeys $\Delta z \Delta p_z \geq \frac{\hbar}{2}$. Einstein presumably wants $\Delta p_z \ll g m_{escaped\ photons} t$, where g is the acceleration of gravity. But now, the clock mounted on the box, according to general relativity, differs by Δt from the time on a clock attached rigidly to the earth, where $\Delta t \sim \frac{g t \Delta z}{c^2}$. Hence our knowledge of the time lapse will suffer from an uncertainty of order $\frac{h}{m_{escaped\ photons} c^2}$, saving the uncertainty principle $\Delta E \Delta t > h$ and quantum mechanics. The Fairy tale concludes: Einstein was so stunned at this use of his own theory of general relativity against him, that he conceded defeat. Actually, Bohr's counterattack, although startling, had two fatal flaws. First, we do not see what Δp_z has to do with $g m_{escaped\ photons} t$. We could agree only to perform weightings after exponentially damping the box's motion by placing it in a bath of viscous oil; any uncertainties in the weighing would then seem to be of a fixed magnitude exponentially independent of anything else. (The box would be in a minimum uncertainty state after the damping, roughly.) second, neither gravitational time dilation nor Δz need have anything to do with the case. This is because Einstein could instead have measured the mass of the box with an electric field (after charging the box with a known amount of charge). The capacitor plates generating the electric field could have been superconductively shorted during the door-cycling but charged to 1 volt for the weightings- say by connection to an arbitrary enormous external capacitor bank providing a 1 volt reference. Thus, both steps in Bohr's argumentation depended on an extremely specific scenario and don't impact on even slightly altered scenarios much less the whole of physics.

III. The energy Time uncertainty principle

In trying to change time, as the classical external parameter, into an observable, one cannot deduce the time-energy uncertainty relation:

$$\Delta t \Delta E \geq \frac{\hbar}{2}; \tag{3}$$

Where t = time , E = energy

From kinematical point of view, as time does not belong to the algebra of observables [8]. In spite of this, (1) is generally regarded as being true. The relation (3), unlike other canonical pairs, is not the consequence of fundamental quantum incomplementarity of two canonical variables. The time-energy uncertainty relation is very different to the standard quantum uncertainty relation, such as the position momentum one. The precise meaning of the time-energy relation is still not exactly known. The problem lies in the fact that one cannot give the precise meaning to the quantity Δt . This is because time is not a standard quantum mechanical observable associated with an Hermitian operator. If such an operator canonically conjugate to the Hamiltonian did exist, then, t could be defined conventionally and the uncertainty principle could be applied to the physical quantity corresponding to the time operator.

Of course there have been a few attempts in the literature to formulate and prove energy time uncertainty relations. But we start to theorem of Mandelstam and Tamm[5].

We know that for any observable O that not depend explicitly on time

$$< [O, H] >= i\hbar \frac{d < O >}{dt}$$

Now suppose we choose A=O and B=H(O and H are operator), where H is the Hamiltonian. From $\Delta A \Delta B \geq \frac{|\langle [A,B] \rangle|}{2}$ we get

$$(\Delta O)(\Delta H) \geq \frac{1}{2} < i[O, H] >= \frac{\hbar}{2} \frac{d<O>}{dt} \tag{4}$$

Now we define $\Delta E \equiv \Delta H$ and define

$$\Delta t \equiv \frac{\Delta O}{\frac{d<O>}{dt}} \tag{5}$$

Then by the combine above equations we obtain (3). This is the energy time uncertainty principle where Δt is the amount of time it takes for the expectation value of the observable O to change by one standard deviation as follows

$$\Delta O \equiv \frac{d<O>}{dt}\Delta t \qquad (6)$$

If ΔE is small then the rate of change of all observables must be very gradual or, put the other way around, if any observable is changing rapidly then the uncertainty in the energy must be large.

Thirring's [7] actually gives an inequality in which every term in the formula has a mathematical definition. Here we present a detailed proof.

Suppose Ψ is a wavefunction evolving with time t, $e^{\frac{-iHt}{\hbar}}$ to where H is a Hamiltonian operator. Suppose the probability that an energy measurements on Ψ would yield a value in an energy interval of width ΔE, is $\geq 1-\delta$. Then

$$\int_{-\infty}^{\infty}\left|< \Psi \mid e^{\frac{-iHt}{\hbar}} \Psi >\right|^2 dt \geq \frac{(1-\delta)^2}{h\Delta E}$$

Proof

Suppose Ψ is a wave function so that, with time it changes into $e^{\frac{-iHt}{\hbar}}\Psi$, which we obtain by applying the time-evolution operator. The probability measure on Ψ after time of t is as $\left|\left\langle \Psi \middle| e^{\frac{-iHt}{\hbar}} \Psi\right\rangle\right|^2$.

Then life time of Ψ is equal to

$$\tau(\Psi) = \int_{-\infty}^{\infty}\left|\left\langle \Psi \middle| e^{\frac{-iHt}{\hbar}} \Psi\right\rangle\right|^2 dt \qquad (7)$$

By use the Fourier transform and Parseval's equality we have

$$\tau(\Psi) = \int_{-\infty}^{\infty}\langle\Psi|\Psi(t)\rangle\langle\Psi|\Psi(t)\rangle dt = \int_{-\infty}^{\infty}\langle\Psi|\Psi(\omega)\rangle\langle\Psi|\Psi(\omega)\rangle d\omega$$

In above relation the second integral shows that frequency space and $\Psi(t) = e^{\frac{-Ht}{\hbar}}\Psi$. So we calculate the Fourier transform function of $\Psi(t)$ then we can write

$$\Psi(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(t)e^{i\omega t}dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\frac{i}{\hbar}(H-\omega)t}dt = \frac{1}{\sqrt{2\pi}}[2\pi\hbar\delta(H-\omega)\Psi]$$

Thus we get

$$\tau(\Psi) = 2\pi\hbar^2\int_{-\infty}^{\infty}|\langle\Psi|\delta(H-\omega)\Psi\rangle|^2 d\omega \qquad (8)$$

Now we are concerned with the extent to which the state at a later time t is similar to the state at t=0, we therefore construct the inner product between the two states at different times

$$C(t) = \langle\Psi|\Psi(t)\rangle \quad , \quad \Psi = \sum_E C_E|\varphi> \quad , \quad \Psi(t) = \sum_E C_E^{\cdot}|\acute{\varphi}>$$

So we get the correlation amplitude $C(t)$, as following

$$C(t) = \sum_E |C_E|^2 e^{\frac{-iEt}{\hbar}}$$

The correlation amplitude that starts with unity at t=0 to decrease in magnitude with time. The probability measure energy on Ψ at ΔE is equal to

$$\sum_{E\in\Delta E}|C_E|^2 \geq 1-\delta$$

According to Cauchy-Schwarz inequality (for integrals) we get

$$(1-\delta)^2 \leq 2\pi\hbar^2\left(\int_{E_1}^{E_2}\langle\Psi|\delta(H-\omega)\Psi\rangle\, d\omega\right)^2$$

Then

$$(1-\delta)^2 \leq \int_{E_1}^{E_2}d\acute{\omega}\int_{E_1}^{E_2}2\pi\hbar^2|\langle\Psi|\delta(H-\omega)\rangle|^2 d\omega = \frac{\Delta E}{\hbar}\tau(\Psi)$$

Since $\int_{E_1}^{E_2}d\acute{\omega} = \frac{1}{\hbar}\int_{E_1}^{E_2}dE = \frac{\Delta E}{\hbar}$ So we get $\Delta E\,\tau(\Psi) \geq \hbar(1-\delta)^2$ then by the (8) we obtain

$$\int_{-\infty}^{\infty}\left|\left\langle\Psi\middle|e^{\frac{-Ht}{\hbar}}\Psi\right\rangle\right|^2 dt \geq \frac{\hbar(1-\delta)^2}{\Delta E}$$

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