Theorems on list-coloring

N.Vedavathi [1] , Dharmaiah Gurram[1]
1.Asst.Professor in Mathematics,K L University,Guntur (dist)A.P,India-522502

Abstract

Graph coloring is a well-known and well-studied area of graph theory with many applications. In this paper, we will consider two generalizations of graph coloring. In particular, list-coloring and sum-list-coloring.

Key Words: Four Color Theorem, Thomassen's 5-list-coloring .

Introduction

We begin by defining a graph and the different types of colorings explored in this paper.

Definition 1.1. A graph G is an ordered pair (V, E) where elements of V are called vertices and elements of E are two element subsets of V called edges. If x, y $\in$ V and {x, y} $\in$ E, then it is said that x and y are adjacent, denoted x ~ y.

For simplicity of notation, we will use x y to denote an edge {x, y} $\in$ E. In this dissertation we will only be looking at connected simple graphs, those which contain no loops or multiple edges.

Definition 1.2. For a graph G = (V, E), an assignment c : V → N is a coloring of G. Furthermore, this coloring is proper if c(u) ≠ c(v) for all u v $\in$ E. If c uses only the colors {1, 2, . . . ,k}, then c is a k-coloring. When such a proper k-coloring exists, G is said to be k-colorable.

Throughout this dissertation, we will look at a generalization of coloring called list-coloring.

Definition 1.3. For a graph G = (V, E), let L : V → 2^N be an assignment of lists of colors to N the vertices of G. A coloring c : V→ N is an L-coloring or list-coloring of G if c(v) $\in$ L(v) for all v $\in$ V . Furthermore, this coloring is proper if c(u) ≠ c(v) for al l u v $\in$ E. When such an L-coloring exists, G is said to be L-colorable.

It will be assumed for the remainder of this thesis that all colorings are proper, unless otherwise noted.

Definition 1.4. Let G = (V, E) be a graph on n vertices and f : V → N be a size function that assigns to each vertex of G a list size. Let an f - assignment L:V → 2^N be an assignment N of lists of colors to the vertices of G such that |L(v)| = f(v) for all v $\in$ V . The graph G is said to be f- choosable if G is L-colorable for every f-assignment L. A choosable size function is called a choice function.

For example, let G be the 3-cycle (u, v, w). Let f ≡ 2 be a size function for G, and then G is not f - choosable. This is because if L is an f-assignment where the lists assigned to each vertex are identical, then the graph is not L-colorable. However, let g be a size function for G such that g(u) = g(v) = 2, g(w) = 3. Then G is g - choosable. The 3-cycle can always be colored from lists of these sizes as follows: First choose a color from L(u) to assign to u. Next there is at least one color in L(v) to assign to v, so assign such a color to v. Finally, there is at least one color in L(w) that may be assigned to w, so assign such a color to w. This will always yield a proper L-coloring of the 3-cycle.

With respect to the colorings defined above, there are some graph parameters that are utilized.

Definition 1.5. The minimum value of k for which a graph G is k-colorable is the chromatic number χ(G).

Definition 1.6. A graph G is said to be k-list-colorable if f ≡ k is a choice function for G. For a choice function f define max(f) := $\max_{v\in V(G)}$ f (v). The list chromatic number χl(G), or l v V (G) choice number ch(G), is the minimum of max(f) over all choice functions for G.

Definition 1.7. For a choice function f define size(f):= $\sum_{v\in V(G)}$ f (v). The sum choice number χ_{SC}(G) is the minimum of size (f) over all choice functions for G.

One of the problems we will be looking at in this dissertation involves assigning some lists of size 1 to certain vertices of a graph.

Definition 1.8. For a graph G = (V, E) and a subset P ⊂ V of vertices, let f : V→N be a size function for G. If f(v) = 1 for all v ∈ P , f (v) = k for all v ∈ V - P, and f is a choice function for G, then it is said that a pre coloring of P is extendable to a k-list-coloring of G.

For a graph G = (V, E) on n vertices ordered v_1, v_2 , . . . ,v_n, we may write a size function f : V → N for G as (G; f (v_1), f (v_2), . . ., f (v_n)) or the vector (f (v_1), f (v_2), . . ., f (v_n)) when G is clear.

We say that (G; f (v_1), f (v_2), . . ., f (v_n)) is good if f is a choice function for G. If G is not f - choosable, then we say that (G; f (v_1), f (v_2), . . ., f (v_n)) is bad.

Given two size functions f and f for G, if f(v_i) ≤ f (v_i) for all i = 1, 2, . . ., n, then we say that f ≤ f . This inequality is strict if f (v_i) < f (v_i) for some i. If G is f - choosable and f ≤ f , then G is also f - choosable. Similarly, if G is not f -choosable and f ≤ f , then G is not f -choosable. If (G; f (v_1), f (v_2), . . .,

$f(v_n))$ is good and $f \le f$, then $(G; f(v_1), f(v_2), \ldots, f(v_n))$ is also good. Also, $(G; f(v_1), f(v_2), \ldots, f(v_n))$ is bad and $f \le f$, then $(G; f(v_1), f(v_2), \ldots, f(v_n))$ is also bad.

A fundamental problem in coloring the vertices of a graph is determining the optimal choice function for a given graph. In this case, we consider optimality in the sense of min $\|f\|$ and min $\|f\|_\infty$. In other words, minimizing the L^∞ and L^1 norms, $\|f\|_\infty = \max_{i=1,\ldots,n} |f(v_i)|$ and $\|f\|_1 = \sum_{i=1}^{n} |f(v_i)|$ $|f(v_i)|$, respectively.

The graph parameter χl (G) corresponds to min $\|f\|_\infty$ over all choice functions f for G. The newer graph parameter χSC (G) corresponds to min $\|f\|_1$ over all choice functions f for G. When a choice function f is such that f (v) = 1 for some vertices v in G, then this corresponds to the coloring extension problem mentioned earlier. In this thesis, we investigate coloring extension problems on planar graphs and sum-list-coloring.

1.1 COLORING

Of the various ways to color the vertices of a graph, the most well-studied is the traditional notion of graph coloring. Some of the first problems in graph coloring date back to the late 1800s and the Four Color Theorem.

Theorem 1.9 (Four Color Theorem). Any planar graph is 4-colorable.

The graph K_4 is an example of a planar graph for which $\chi(K_4) = 4$. This shows that for an arbitrary planar graph, three colors are not enough. There are many results on planar graphs that are 3-colorable if they do not contain cycles of certain lengths.. The Four Color. Theorem was originally posed in 1852 by Francis Guthrie, and ultimately proved by Appel and Haken [9]. The proof of the Four Color Theorem has a long and storied past and the proof itself is very involved. It involves showing that a minimal counterexample to the theorem does not exist. This is done, in part, by providing an unavoidable set of configurations along with a set of reducible configurations. The proof also relies heavily on computers.

1.2 LIST-COLORING

List-coloring was first introduced by Vizing and independently by Erdos, Rubin, and Taylor . In graph coloring, one seeks to minimize the number of colors used. Similarly, in list-coloring, one seeks to minimize the list size.

Erd os et al. came up with the notion of list-coloring in an attempt to solve a problem of Jeffrey Dinitz posed at the Tenth Southeastern Conference on Combinatorics, Graph Theory, and Computing at Boca Raton in April 1979 . The problem was stated as follows:

Question 1.10. Given an m × m array of m-sets, is it always possible to choose one element from each set, keeping the chosen elements distinct in every row, and distinct in every column?

This problem can be stated in terms of list-coloring as follows:

Question 1.11. Let G = (V, E) be a graph on m^2 vertices. Let $V = \{v_i: 1 \le i \le m, 1 \le j \le m\}$ and let E be defined so that $v_{i,j} \sim v_{i,j}$ if i = i or j = j . To each vertex, assign an arbitrary set of m colors. Can G always be colored from the assigned lists?

So this question is asking whether or not $(K_m \times K_m) = m$, where $K_m \times K_m$ is the Cartesian product .

While it is di cult to compute χl (G) for an arbitrary graph G, there is an upper bound on χl (G) based on the maximum degree Δ(G).

Lemma 1.12. χl (G) = Δ(G) + 1.

Proof. Let $\Delta := \Delta(G)$ and assign arbitrary lists L of size $\Delta + 1$ to each vertex of G. Let $v_1, \ldots, v_n$ be an arbitrary ordering of the vertices of G. Use this ordering to L-color the vertices of G. This will provide a proper L-coloring of G because each v_i is adjacent to at most Δ vertices and at least one element of $L(v_i)$ will be available to assign to v_i.

One of the most celebrated results in list-coloring is the following theorem of Thomassen which shows that there exist graphs with arbitrarily large maximum degree that are 5-list-colorable.

Theorem 1.13 (Thomassen's 5-list-coloring theorem). Let G = (V, E) be a plane graph, let C be the cycle that corresponds to the boundary of a face of G, and let u, v $\in$ V (C) such that u ~ v. Let L : $V \rightarrow 2^N$ be an assignment of lists of colors to vertices of G such that N $|L(u)| = |L(v)| = 1$ and $L(u) \neq L(v)$; $|L(w)| = 3$ for all w $\in$ V (C) - {u, v}; and $|L(w)| = 5$ for all w $\in$ V - V (C). Then G is L-colorable.

If a planar graph does not contain cycles of certain lengths, then it is 4-list-colorable. There are also similar results for determining planar graphs that are 3-list-colorable. For extensive literature on list-colorings of planar graphs we refer more.

Here, we brie y discuss Thomassen's 5-list-coloring theorem and some related results. One thing that Thomassen's 5-list-coloring theorem tells us is that planar graphs are 5-list-colorable.

For this reason, the result can be thought of as the list-coloring version of the famous Four Color Theorem.

While the proof of the Four Color Theorem is quite long and relies heavily on the use of computers, the proof of Thomassen's 5-list-coloring theorem is a short induction argument .

Additionally, for planar graphs, lists of size 4 are not enough. There exist multiple examples of planar graphs that are not 4-list-colorable. One of the first examples was constructed by Voigt and had 238 vertices. This was improved in the years that followed. Gutner and Voigt and Wirth both came up with constructions of examples with 75 vertices, and Mirzakhani presented an example with only 63 vertices. Each of these constructions uses multiple copies of a smaller graph as a building block to create a counterexample.

1.3 THE RELATIONSHIP BETWEEN COLORING AND LIST-COLORING

Graph coloring is a special case of list-coloring where the lists assigned to each vertex are identical. For this reason, $\chi(G) \le \chi l(G)$ for all graphs G. In other words, if G is k-list- colorable, then G is k-colorable. The converse, however, is not true. There are graphs that are k-colorable, but not k-list-colorable. For example, the graph $K_{3,3}$ is bipartite and hence 2-colorable, but it is not 2-list-colorable. It is known that $K_{3,3}$ is 3-list-colorable .

It is known that bipartite graphs are 2-colorable because all the vertices in each partite set can be assigned the same color. However, there exist bipartite graphs whose list chromatic number is arbitrarily large. For example, if m = $\binom{2k-1}{k}$, then $K_{m,m}$ let $K_{m,m}$ = (A $\cup$ B, E) and assume $K_{m,m}$ is k-list-colorable. In both partite sets A and B assign to each vertex one of the m distinct possible k-subsets of {1, 2, .. . , 2k - 1} as the list of available colors for that vertex. Any coloring of the vertices in A from the lists of colors assigned to them must use k distinct colors. Otherwise, there would be a vertex in A with no color assigned to it. This is because there does not exist a subset of k - 1 colors of which at least one of these colors appears in every k-set assigned to the vertices of A. Thus, there is vertex in B which cannot be colored because its list is identical to the set of k colors assigned to all of the vertices of A. This is a contradiction which implies that $K_{m,m}$ is not k-list-colorable and $\chi l(K_{m,m}) > k$.

1.4 SUM-LIST-COLORING

Sum-list-coloring was introduced by Isaak in 2002. It is a fairly new topic in graph theory, so there is much to be discovered. In particular, it is a survey of all sum-list-coloring results up to 2007. In sum-list-coloring, the list sizes are allowed to vary and one seeks to minimize the sum of list sizes over all vertices.

For any graph G, the sum choice number is bounded above by $\chi SC(G) = |V(G)| + |E(G)|$, as provided by a greedy coloring. See Lemma 1.14 for a proof of this result. When equality holds in the previous inequality, G is said to be sc-greedy. The sum |V (G)| + |E(G)| is called the greedy b ound and denoted by GB(G), or GB when G is implied.

Lemma 1.14. For any graph G, $\chi SC(G) = |V(G)| + |E(G)|$.

Proof. Let $v_1, \ldots, v_n$ be an ordering of the vertices. Let $f(v_1) = 1 + |\{v_1 : j < i \text{ and } v_1, v_n \in E(G)\}|$. A greedy coloring using this ordering and arbitrary lists of the prescribed sizes provides a proper coloring for any such list assignment.

Observe that list-coloring and the list chromatic number, or choice number, $\chi l(G)$ are related to sum-list-coloring and the sum choice number: $\chi SC(G)/n \le \chi l(G)$. Moreover, for some graphs G, it is the case that $\chi SC(G)/n$ is significantly smaller than (G). In particular, Furedi and Kantor [30] proved the following:

Theorem 1.15 . There exist constants c_1, c_2 such that for all m $\ge$ 4 and n $\ge 50m^2$ log

$$2n + c_1 m\sqrt{n\log m} \le \chi SC(K_{m,n}) \le 2n + c_2 m\sqrt{n\log m}$$

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This implies there exists a choice function f for such a $K_{m,n}$ whose average list size does not necessarily increase with the average degree. Note that as n approaches infinity, the average degree approaches 2m. Furthermore,

$$\lim_{m\to\infty, n>>m^2\log m} \frac{|E(K_{m,n})|}{m+n} = \infty,$$

$$\lim_{m\to\infty, n>>m^2\log m} \frac{\chi SC(K_{m,n})}{m+n} = 2$$

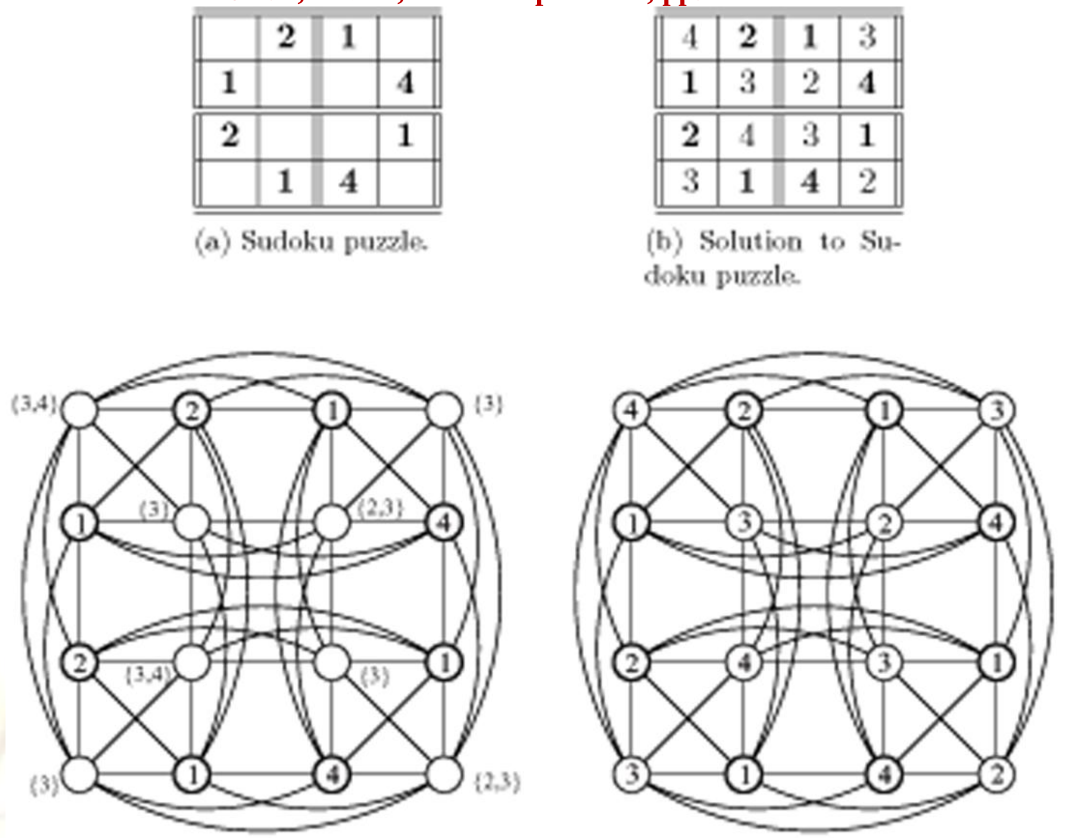
where the first limit looks at the average degree and the second looks at the average list size. See [30] for more on this result. Alon [4] showed that is bounded below by a function of the average degree:

Theorem 1.16 . For some constant c and a graph G with average degree d,

$$\chi l(G) \ge c\frac{\log d}{\log\log d}$$

It can thus be observed that when the list sizes are allowed to vary, this result no longer holds.

To show that $\chi SC(G) = m$, one must provide a choice function f of size m for G and show that for each size function g of size m - 1, there is a g-assignment that does not have a proper coloring. Chapter 6 will provide examples of certain graphs that are sc-greedy and determine information about the sum choice number of other graphs.



(a) Sudoku puzzle. (b) Solution to Sudoku puzzle.

(c) Graph for Sudoku puzzle with precoloring. (d) Coloring extension of graph for Sudoku puzzle.

Figure 1.1: Sudoku puzzle and corresponding graph.

Conclusions

we discussed about Four Color Theorem, Thomassen's 5-list-coloring .

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