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Degree Of Approximation Of Functions By Modified Partial Sum Of Their Conjugate Fourier Series By Generalized Matrix Mean

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Abstract

The paper studies the degree of approximation of conjugate of a 2π -periodic Lebesgue integrable function f by using modified partial sum of its conjugate Fourier series by generalized matrix mean in generalized Holder metric.

Keyword. Banach Space, generalized Holder metric and regular generalized matrix.

1. Definition and Notation

The following definitions will be used throughout the paper (see Zygmund [8] p.16,42, [5] p.2,49 and [3]).

(i) The space $L_p[-\pi,\pi]$ includes the space of all 2π -periodic Lebesgue integrable continuous functions defined in $[-\pi,\pi]$ with *p*-norm given by

$$\|f\|_p = \begin{cases} \sup_t |f(t)| & ; p=\infty \\ \left[\int_0^{2\pi} |f(t)|^p dt\right]^{\frac{1}{p}} & ; p \ge 1 \\ \int_0^{2\pi} |f(t)|^p dt & ; 0<p<1. \end{cases}$$

(ii) $w(\delta) = w(\delta, f) = \sup_{0\le h\le\delta} \|f(x+h) - f(x)\|_c$

when $p=\infty$, is called the modulus of continuity

$$w_p(\delta) = w_p(\delta, f) = \sup_{0\le h\le\delta} \|f(x+h) - f(x)\|_p$$

is called the integral modulus of continuity.

$$w_p^{(2)}(\delta) = w_p^{(2)}(\delta, f) = \sup_{0\le h\le\delta} \|f(x+h) + f(x-h) - 2f(x)\|_p$$

is called the integral modulus of smoothness.

(iii) The Lipschitz condition is given by

$$\sup_{0\le h\le\delta} \frac{\|f(x+h)-f(x)\|_c}{|\delta|^\beta} \le K \text{ (+ve constant) when } p=\infty$$

or $\sup_{0\le h\le\delta} \frac{\|f(x+h)-f(x)\|_p}{|\delta|^\beta} \le K$ (+ve constant) when $0<p<\infty$.

(iv) The Holder metric space H_α is defined by

$$H_\alpha = \left\{ f \in C_{2\pi} : |f(x) - f(y)| \le K|x-y|^\alpha ; K>0, 0\le\alpha\le1 \right\}$$

with Holder metric induced by the norm

$$\|f\|_\alpha = \|f\|_c + \sup_{x\ne y} \Delta^\alpha f(x,y) = \sup_{t\in[-\pi,\pi]} |f(t)| + \sup_{x\ne y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

where $\Delta^\alpha f(x,y) = \frac{|f(x)-f(y)|}{|x-y|^\alpha}$ and 0 for $\alpha = 0$.

(v) A normed linear space which is complete in the metric defined by its norm is called Banach Space.

(vi) The generalized Holder metric space $H(\alpha, p)$ is defined by

$$H(\alpha,p) = \left\{ f \in L_p : \|f(x+h) - f(x)\|_p \le K|h|^\alpha \right\}$$

where K > 0(constant), 0< $\alpha \le 1$ and $0<p\le\infty$. Also the metric given by

$$\|f\|_{(\alpha,p)} = \|f\|_p + \sup_h \Delta^\alpha f(x+h,x) = \|f\|_p + \sup_h \frac{\|f(x+h)-f(x)\|_p}{|h|^\alpha}$$

and $\|f\|_{(o,p)} = \|f\|_p$ for $\alpha = 0$, is called generalized Holder metric.

(vii) $H(\alpha,p)$ is a complete normed linear space and hence a Banach space for $0<p<1$.

Also $H(\alpha,\infty) = H_\alpha$.

(viii)A generalized matrix $M = \left(m_{nk}(i)\right) \in \tau$ is said to be regular if

$$\|M\| = \sup_{n,i} \sum_{k=0}^{\infty} \left|m_{nk}(i)\right| < \infty \text{ and } \sum_{k=0}^{\infty} m_{nk}(i) \to 1 \text{ as } n \to \infty \text{ uniformly in } i.$$

(ix) Let $f \in L_p[-\pi,\pi]; p \geq 1,$ be a 2π -periodic Lebesgue integrable function.

Then the conjugate Fourier series of f at $t = x,$ is given by

$$f(x) = \sum_{k=1}^{\infty} \left\{a_k \sin(kx) - b_k \cos(kx)\right\}; \tag{1}$$

where $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)\,dt$

and $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)\,dt.$

The following notations shall be used throughout the paper.

$\tilde{D}_n^*(t) = \dfrac{1-\cos(nt)}{2\tan\left(t/2\right)}$ is called Dirichlet's modified conjugate kernel.

$$\psi_x(t) = \frac{1}{2}\left[f(x+t) - f(x-t)\right]$$

$$\phi_n(i) = \sum_{k=0}^{\infty} \left|m_{n,k}(i) - m_{n,k+1}(i)\right|$$

$$\tilde{K}_n^*(i;t) = \sum_{k=0}^{\infty} m_{nk}(i) \frac{\left\{1-\cos(kt)\right\}}{2\tan\left(t/2\right)} \tag{2}$$

$$\tilde{H}_n^*(i;t) = \sum_{k=0}^{\infty} m_{nk}(i) \frac{\cos(kt)}{2\tan\left(t/2\right)} \tag{3}$$

$\tilde{f}(x) = -\dfrac{2}{\pi}\displaystyle\int_0^{\pi} \dfrac{\psi_x(t)}{2\tan\left(t/2\right)}dt$ is called conjugate function of $f(x)$.

$$\tilde{f}(x,\varepsilon) = \tilde{f}\left(x, \pi/\xi_n\right) = -\frac{2}{\pi}\int_{\pi/\xi_n}^{\pi} \frac{\psi_x(t)}{2\tan\left(t/2\right)}dt \; ; \tag{4}$$

where $\varepsilon = \dfrac{\pi}{\xi_n}(\to 0)$ is very small positive number.

$$\tilde{S}_n^*(x) = -\frac{2}{\pi}\int_0^{\pi} \psi_x(t)\tilde{D}_n^*(t)\,dt \tag{5}$$

is called modified n^{th} partial sum of conjugate Fourier series of $f(x)$ given by (1).

$$M\left(\tilde{S}_n^*(x)\right) = \sum_{k=0}^{\infty} m_{nk}(i)\tilde{S}_k^*(x) \tag{6}$$

uniformly in i, provided the series exists for each n, which is called the matrix transformation of $\tilde{S}_n^*(x)$.

$$\tilde{l}_n^*(i;x) = M\left(\tilde{S}_n^*(x)\right) - \tilde{f}\left(x, \frac{\pi}{\xi_n}\right) \tag{7}$$

2. Introduction

Chandra [1] and Sahney have determined the degree of approximation of a function belonging to *Lip* α by $(C,1),(C,\delta)$ and (N,p_n) means. In 1981, Quereshi discussed the degree of approximation of conjugate of a function belonging to *Lip* α and *Lip* (α,p) by (N,p_n) means of conjugate series of a Fourier series. In 2000, Shyam Lal [4] determined the degree of approximation of conjugate of function belonging to weighted class $W(L^p,\xi(t))$ by matrix means of conjugate series of Fourier series. Also in 2001, G.Das, R.N.Das and B.K.Ray[3] studied the degree of approximation in same direction using infinite matrix mean in generalized Holder metric.

The objective of the present paper is to study more comprehensively the result of G.Das,R.N.Das & B.K.Ray[3] by generalized matrix mean.

3. Result

In this paper we have studied the degree of approximation of conjugate function of $f(x)$ by modified partial sum of its conjugate Fourier series by generalized matrix mean in generalized Holder metric i.e.

$$\left\|\tilde{t}_n^*(i;x)\right\|_{(\beta,p)}=\left\|\sum_{k=0}^{\infty}m_{nk}(i)\tilde{S}_k^*(x)-\tilde{f}\left(x,\frac{\pi}{\xi_n}\right)\right\|_{(\beta,p)}$$

uniformly in *i*.

The following lemma will be required for establishing the theorem.

Lemma.

Let $0<p\le\infty$.

Then (a) $\|\psi_x(t)\|_p\le w_p(\delta,f)$

and (b) $\|\psi_{x+y}(t)-\psi_x(t)\|_p\le K\|f(x+t+y)-f(x+t)\|_p$;where K > 0 (constant).

Proof.

(a) For $p\ge1$ and by Minkowski's inequality, we have

$$\int_0^{2\pi}|f(x+t)-f(x-t)|^p dx\le\left\{\int_0^{2\pi}|f(x+t)-f(x)|^p dx\right\}^{\frac{1}{p}}+\left\{\int_0^{2\pi}|f(x)-f(x-t)|^p dx\right\}^{\frac{1}{p}}$$

and for $0<p<1$, we have by modified Minkowski's inequality

$$\int_0^{2\pi}|f(x+t)-f(x-t)|^p dx\le\int_0^{2\pi}|f(x+t)-f(x)|^p dx+\int_0^{2\pi}|f(x)-f(x-t)|^p dx$$

$$\Rightarrow\sup_{0\le t\le\delta}\|f(x+t)-f(x-t)\|_p\le\sup_{0\le t\le\delta}\|f(x+t)-f(x)\|_p+\sup_{0\le t\le\delta}\|f(x)-f(x-t)\|_p$$

$$\le2\sup_{0\le t\le\delta}\|f(x+t)-f(x)\|_p$$

$$\Rightarrow\sup_{0\le t\le\delta}\left\|\frac{f(x+t)-f(x-t)}{2}\right\|_p\le\sup_{0\le t\le\delta}\|f(x+t)-f(x)\|_p$$

$$\Rightarrow\|\psi_x(t)\|_p\le w_p(\delta,f)$$

(b) Now

$$\psi_{x+y}(t)-\psi_x(t)=\frac{1}{2}\left[f(x+y+t)-f(x+y-t)-f(x+t)+f(x-t)\right]$$

$$=\frac{1}{2}\left[f(x+t+y)-f(x+t)\right]+\frac{1}{2}\left[f(x-t)-f(x-t+y)\right]$$

By Minkowski's inequality for $p\ge1$ and $0<p<1$ separately, we get

$$\|\psi_{x+y}(t)-\psi_x(t)\|_p\le\frac{1}{2}\|f(x+t+y)-f(x+t)\|_p+\frac{1}{2}\|f(x-t)-f(x-t+y)\|_p$$

$$\le K\left\| f(x+t+y) - f(x+t)\right\|_p \text{ ;where K > 0 is a constant}$$

which establishes the lemma.

Theorem.

If for $p \ge 1, 0 \le \beta < \alpha \le 1$ and for positive increasing sequences (ξ_n) and (η_n) such that $\xi_n \le \eta_n$ and $M = (m_{nk}(i)) \in \tau$ (space of regular matrices) such that $\sum_{k=\eta_n+1}^{\infty} k^2 |m_{nk}(i)| = O(\eta_n)$ and $f \in H(\alpha, p)$, then

$$\left\| \tilde{l}_n^*(i;x) \right\|_{(\beta,p)} = O(1)\left[\xi_n^{-\alpha} + \xi_n^{\beta-\alpha}\left\{ \frac{1}{\eta_n} + \log\left(\frac{\eta_n}{\xi_n}\right)\right\}^{\frac{\beta}{\alpha}} + \phi_n(i)\begin{cases} \xi_n^{1-\alpha}\left(1+\xi_n^{\beta}\right) & ;0<\alpha<1 \\ \log \xi_n + \xi_n^{\beta}\left(\log \xi_n\right)^{1-\beta} & ;\alpha = 1 \end{cases} \right]$$

uniformly in *i*.

Proof.

The equation (7) can be written as

$$\tilde{l}_n^*(i;x) = M\left(\tilde{S}_n^*(x)\right) - \tilde{f}\left(x, \frac{\pi}{\xi_n}\right)$$

$$= \sum_{k=0}^{\infty} m_{nk}(i)\tilde{S}_k^*(x) - \tilde{f}\left(x, \frac{\pi}{\xi_n}\right)$$

$$= \sum_{k=0}^{\infty} m_{nk}(i)\left\{ -\frac{2}{\pi}\int_0^{\pi} \psi_x(t)\frac{1-\cos(kt)}{2\tan\left(\frac{t}{2}\right)}dt\right\} + \sum_{k=0}^{\infty} m_{nk}(i)\left\{ \frac{2}{\pi}\int_{\frac{\pi}{\xi_n}}^{\pi} \frac{\psi_x(t)}{2\tan\left(\frac{t}{2}\right)}dt\right\}$$

$$\left(\sum_{k=0}^{\infty} m_{nk}(i) \to 1 \text{ as } n \to \infty\right)$$

$$= -\frac{2}{\pi}\int_0^{\pi} \psi_x(t)\left[\sum_{k=0}^{\infty} m_{nk}(i)\left\{\frac{1-\cos(kt)}{2\tan\left(\frac{t}{2}\right)}\right\}\right]dt + \frac{2}{\pi}\int_{\frac{\pi}{\xi_n}}^{\pi} \psi_x(t)\left[\sum_{k=0}^{\infty} m_{nk}(i)\frac{1}{2\tan\left(\frac{t}{2}\right)}\right]dt$$

(provided the change of order of summation is permitted)

$$= -\frac{2}{\pi}\int_0^{\frac{\pi}{\xi_n}} \psi_x(t)\left[\sum_{k=0}^{\infty} m_{nk}(i)\left\{\frac{1-\cos(kt)}{2\tan\left(\frac{t}{2}\right)}\right\}\right]dt + \frac{2}{\pi}\int_{\frac{\pi}{\xi_n}}^{\pi} \psi_x(t)\left[\sum_{k=0}^{\infty} m_{nk}(i)\left\{\frac{\cos(kt)}{2\tan\left(\frac{t}{2}\right)}\right\}\right]dt$$

$$= -\frac{2}{\pi}\int_0^{\frac{\pi}{\xi_n}} \psi_x(t)\tilde{K}_n^*(i;t)dt + \frac{2}{\pi}\int_{\frac{\pi}{\xi_n}}^{\pi} \psi_x(t)\tilde{H}_n^*(i;t)dt \qquad (8)$$

Now $\left|\tilde{K}_n^*(i;t)\right| \le \sum_{k=0}^{\infty} |m_{nk}(i)| \left| \frac{1-\cos(kt)}{2\tan\left(\frac{t}{2}\right)} \right|$

$$\le \sum_{k=0}^{\infty} |m_{nk}(i)| \frac{2}{2\tan\left(\frac{t}{2}\right)} \qquad \left(\because |1-\cos(kt)| \le 2\right)$$

$$\le \frac{2}{t}\sum_{k=0}^{\infty} |m_{nk}(i)| \qquad \left(\because 2\tan\left(\frac{t}{2}\right) \ge t \ \ for\ 0 \le t \le \pi\right)$$

$$\le \frac{2}{t}\|M\|$$

$$= O\left(t^{-1}\right)$$

$$\Rightarrow \left|\tilde{K}_n^*(i;t)\right| = O\left(t^{-1}\right) \tag{9}$$

Also for $p \ge 1, 0 < \alpha \le 1$ and $f \in H(\alpha, p)$,

$$\left\| f(x+t) - f(x) \right\|_p = O\left(|t|^\alpha\right) \tag{10}$$

By lemma(a), $\left\|\psi_x(t)\right\|_p \le \underset{0\le t\le\delta}{Sup}\left\| f(x+t) - f(x)\right\|_p = O\left(|t|^\alpha\right)$ (by (10))

$$\Rightarrow \left\|\psi_x(t)\right\|_p = O\left(|t|^\alpha\right) \tag{11}$$

Again $\left\|\psi_{x+y}(t) - \psi_x(t)\right\|_p$

$$\le \left\|\psi_{x+y}(t)\right\|_p + \left\|\psi_x(t)\right\|_p \qquad \text{(by Minkowski's inequality)}$$

$$= O\left(|t|^\alpha\right) + O\left(|t|^\alpha\right) \qquad \text{(by (11))}$$

$$\Rightarrow \left\|\psi_{x+y}(t) - \psi_x(t)\right\|_p = O\left(|t|^\alpha\right) \tag{12}$$

Consider $\left\|\tilde{l}_n^*(i;x+y) - \tilde{l}_n^*(i;x)\right\|_p$

$$= \left\| -\frac{2}{\pi}\int_0^{\frac{\pi}{\xi_n}} \left\{\psi_{x+y}(t) - \psi_x(t)\right\}\tilde{K}_n^*(i;t)\,dt + \frac{2}{\pi}\int_{\frac{\pi}{\xi_n}}^{\pi} \left\{\psi_{x+y}(t) - \psi_x(t)\right\}\tilde{H}_n^*(i;t)\,dt \right\|_p$$

$$\le \frac{2}{\pi}\int_0^{\frac{\pi}{\xi_n}} \left\|\psi_{x+y}(t) - \psi_x(t)\right\|_p \left|\tilde{K}_n^*(i;t)\right| dt + \frac{2}{\pi}\int_{\frac{\pi}{\xi_n}}^{\pi} \left\|\psi_{x+y}(t) - \psi_x(t)\right\|_p \left|\tilde{H}_n^*(i;t)\right| dt$$

$$= I_1 + I_2 \ \ \text{(say)} \tag{13}$$

where $I_1 = \frac{2}{\pi}\int_0^{\frac{\pi}{\xi_n}} \left\|\psi_{x+y}(t) - \psi_x(t)\right\|_p \left|\tilde{K}_n^*(i;t)\right| dt$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{\xi_n}} O\left(|t|^\alpha\right) O\left(t^{-1}\right) dt \qquad \text{(by (12) and (9))}$$

$$= O(1)\int_0^{\frac{\pi}{\xi_n}} t^{\alpha-1}\,dt$$

$$= O(1)\left[\frac{t^{\alpha}}{\alpha}\right]_0^{\frac{\pi}{\xi_n}}$$

$$= \frac{1}{\xi_n^{\alpha}}O(1) \qquad (14)$$

Also by Abel's transformation for $\frac{\pi}{\xi_n} \le t \le \pi$,

$$\sum_{k=0}^{\infty} m_{nk}(i)\cos(kt) = \lim_{n\to\infty}\left[\sum_{k=0}^{n-1} S_k\left\{m_{n,k}(i) - m_{n,k+1}(i)\right\} + S_n m_{n,n}(i)\right];$$

$$\left(where \quad S_n = \sum_{k=0}^{n} cos(kt) = \frac{\sin\left(\frac{n+1}{2}\right)t\cos\left(\frac{nt}{2}\right)}{\sin\left(\frac{t}{2}\right)}\right) \text{(see[2],[8])}$$

$$= \sum_{k=0}^{\infty}\left\{m_{n,k}(i) - m_{n,k+1}(i)\right\}\frac{\sin\left(\frac{k+1}{2}\right)t\cos\left(\frac{kt}{2}\right)}{\sin\left(\frac{t}{2}\right)} \text{ as } \lim_{n\to\infty} m_{n,n}(i) = 0$$

$$\Rightarrow \left|\sum_{k=0}^{\infty} m_{nk}(i)\cos(kt)\right| \le \frac{1}{\sin\left(\frac{t}{2}\right)}\sum_{k=0}^{\infty}\left|m_{n,k}(i) - m_{n,k+1}(i)\right| \quad \left(\because \left|\frac{\sin\left(\frac{k+1}{2}\right)t\cos\left(\frac{kt}{2}\right)}{\sin\left(\frac{t}{2}\right)}\right| \le \frac{1}{\sin\left(\frac{t}{2}\right)}\right)$$

$$= O\left(t^{-1}\right)\sum_{k=0}^{\infty}\left|m_{n,k}(i) - m_{n,k+1}(i)\right|$$

$$= O\left(t^{-1}\right)\phi_n(i)$$

$$\Rightarrow \left|\sum_{k=0}^{\infty} m_{nk}(i)\cos(kt)\right| = O\left(t^{-1}\right)\phi_n(i) \text{ for } \frac{\pi}{\xi_n} \le t \le \pi \qquad (15)$$

and $I_2 = \frac{2}{\pi}\int_{\frac{\pi}{\xi_n}}^{\pi}\left\|\psi_{x+y}(t) - \psi_x(t)\right\|_p\left|\tilde{H}_n^*(i;t)\right|dt$

$$= \frac{2}{\pi}\int_{\frac{\pi}{\xi_n}}^{\pi} O\left(|t|^{\alpha}\right)\left|\sum_{k=0}^{\infty} m_{nk}(i)\frac{\cos(kt)}{2\tan\left(\frac{t}{2}\right)}\right|dt \qquad \text{(by (12))}$$

$$= \frac{2}{\pi}\int_{\frac{\pi}{\xi_n}}^{\pi} O\left(|t|^{\alpha}\right)O\left(t^{-1}\right)\left|\sum_{k=0}^{\infty} m_{nk}(i)\cos(kt)\right|dt \quad \left(\because 2\tan\left(\frac{t}{2}\right) \ge t \; for\, 0 \le t \le \pi\right)$$

$$= \frac{2}{\pi}\int_{\frac{\pi}{\xi_n}}^{\pi} O\left(|t|^{\alpha}\right)O\left(t^{-1}\right)O\left(t^{-1}\right)\phi_n(i)dt \qquad \text{(by (15))}$$

$$= O(1)\phi_n(i)\int_{\frac{\pi}{\xi_n}}^{\pi} t^{\alpha-2}dt$$

$$= O(1)\phi_n(i)\begin{bmatrix} \dfrac{t^{\alpha-1}}{\alpha-1} & ;0<\alpha<1 \\ \log t & ;\alpha=1 \end{bmatrix}_{\frac{\pi}{\xi_n}}^{\pi}$$

$$= O(1)\phi_n(i)\begin{bmatrix} \xi_n^{1-\alpha} & ;0<\alpha<1 \\ \log\xi_n & ;\alpha=1 \end{bmatrix} \qquad (16)$$

Further order estimates for I_1 and I_2 can be obtained as follows

$$I_1 = \frac{2}{\pi}\int_0^{\frac{\pi}{\eta_n}}\left\|\psi_{x+y}(t)-\psi_x(t)\right\|_p\left|\tilde{K}_n^*(i;t)\right|dt + \frac{2}{\pi}\int_{\frac{\pi}{\eta_n}}^{\frac{\pi}{\xi_n}}\left\|\psi_{x+y}(t)-\psi_x(t)\right\|_p\left|\tilde{K}_n^*(i;t)\right|dt$$

$$= I_{11}+I_{12} \quad \text{(say)} \qquad (17)$$

Also $\left|\tilde{K}_n^*(i;t)\right| = \left|\sum_{k=0}^{\infty} m_{nk}(i)\left\{\dfrac{1-\cos(kt)}{2\tan\left(\frac{t}{2}\right)}\right\}\right|$

$$\le \sum_{k=0}^{\eta_n}\left|m_{nk}(i)\right|\left|\frac{\sin^2\left(\frac{kt}{2}\right)}{\tan\left(\frac{t}{2}\right)}\right| + \sum_{k=\eta_n+1}^{\infty}\left|m_{nk}(i)\right|\left|\frac{\sin^2\left(\frac{kt}{2}\right)}{\tan\left(\frac{t}{2}\right)}\right|$$

$$\le \sum_{k=0}^{\eta_n}\left|m_{nk}(i)\right|\left|\frac{k^2t^2}{4\tan\left(\frac{t}{2}\right)}\right| + \sum_{k=\eta_n+1}^{\infty}\left|m_{nk}(i)\right|\left|\frac{k^2t^2}{4\tan\left(\frac{t}{2}\right)}\right| \qquad \left(\because \left|\sin\left(\frac{kt}{2}\right)\right| \le \left|\frac{kt}{2}\right|\right)$$

$$\le \sum_{k=0}^{\eta_n}\left|m_{nk}(i)\right|\eta_n^2\left(\frac{t}{2}\right) + \sum_{k=\eta_n+1}^{\infty}\left|m_{nk}(i)\right|k^2\left(\frac{t}{2}\right) \qquad \left(\because 2\tan\left(\frac{t}{2}\right) \ge t\right)$$

$$= O(t)\left[\sum_{k=0}^{\eta_n}\left|m_{nk}(i)\right|\eta_n^2 + \sum_{k=\eta_n+1}^{\infty}k^2\left|m_{nk}(i)\right|\right]$$

$$= O(t)\left[O(\eta_n)+O(\eta_n)\right] \left(\because \sum_{k=0}^{\infty}\left|m_{nk}(i)\right| < \infty \text{ and by necessary condition of theorem}\right)$$

$$\Rightarrow \left|\tilde{K}_n^*(i;t)\right| = O(t)O(\eta_n) \qquad (18)$$

Again by lemma (b), $\left\|\psi_{x+y}(t)-\psi_x(t)\right\|_p = O\left(|y|^\alpha\right)$ (19)

Now $I_{11} = \dfrac{2}{\pi}\displaystyle\int_0^{\frac{\pi}{\eta_n}}\left\|\psi_{x+y}(t)-\psi_x(t)\right\|_p\left|\tilde{K}_n^*(i;t)\right|dt$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{\eta_n}} O\left(|y|^\alpha\right)O(\eta_n)O(t)\,dt \qquad \text{(by (18),(19))}$$

$$= O\left(|y|^\alpha\right)O(\eta_n)\int_0^{\frac{\pi}{\eta_n}} t\,dt$$

$$= O\left(|y|^\alpha\right)O(\eta_n)\left[\frac{t^2}{2}\right]_0^{\frac{\pi}{\eta_n}}$$

$$= O\left(|y|^{\alpha}\right) O(\eta_n) O\left(\frac{1}{\eta_n^2}\right)$$

$$= \frac{1}{\eta_n} O\left(|y|^{\alpha}\right) \tag{20}$$

and $I_{12} = \frac{2}{\pi}\int_{\pi/\eta_n}^{\pi/\xi_n} \left\|\psi_{x+y}(t) - \psi_x(t)\right\|_p \left|\tilde{K}_n^*(i;t)\right| dt$

$$= \frac{2}{\pi}\int_{\pi/\eta_n}^{\pi/\xi_n} O\left(|y|^{\alpha}\right) \left|\sum_{k=0}^{\infty} m_{nk}(i) \left\{\frac{1-\cos(kt)}{2\tan\left(\frac{t}{2}\right)}\right\}\right| dt \qquad \text{(by (19))}$$

$$= O\left(|y|^{\alpha}\right)\int_{\pi/\eta_n}^{\pi/\xi_n} O\left(t^{-1}\right)\left|\sum_{k=0}^{\infty} m_{nk}(i)\left\{2\sin^2\left(\frac{kt}{2}\right)\right\}\right| dt \qquad \left(\because 2\tan\left(\frac{t}{2}\right) \ge t\right)$$

$$= O\left(|y|^{\alpha}\right)\int_{\pi/\eta_n}^{\pi/\xi_n} O\left(t^{-1}\right)\left\{\sum_{k=0}^{\infty} 2\left|m_{nk}(i)\right|\right\} dt \qquad \left(\because \left|\sin^2\left(\frac{kt}{2}\right)\right| \le 1\right)$$

$$= O\left(|y|^{\alpha}\right)\int_{\pi/\eta_n}^{\pi/\xi_n} \|M\| O\left(t^{-1}\right) dt$$

$$= O\left(|y|^{\alpha}\right)\int_{\pi/\eta_n}^{\pi/\xi_n} \frac{1}{t} dt$$

$$= O\left(|y|^{\alpha}\right)[\log t]_{\pi/\eta_n}^{\pi/\xi_n}$$

$$= O\left(|y|^{\alpha}\right)\log\left(\frac{\eta_n}{\xi_n}\right) \tag{21}$$

Hence $I_1 = I_{11} + I_{12} = O\left(|y|^{\alpha}\right)\left[\frac{1}{\eta_n} + \log\left(\frac{\eta_n}{\xi_n}\right)\right]$ (22)

Combining (14) and (22), we get for $0 \le \beta < \alpha \le 1$

$$I_1 = I_1^{\frac{\beta}{\alpha}} I_1^{\left(1-\frac{\beta}{\alpha}\right)}$$

$$= \left[O\left(|y|^{\alpha}\right)\left\{\frac{1}{\eta_n} + \log\left(\frac{\eta_n}{\xi_n}\right)\right\}\right]^{\frac{\beta}{\alpha}} \left[O(1)\frac{1}{\xi_n^{\alpha}}\right]^{1-\frac{\beta}{\alpha}}$$

$$= O\left(|y|^{\beta}\right)\xi_n^{\beta-\alpha}\left\{\frac{1}{\eta_n} + \log\left(\frac{\eta_n}{\xi_n}\right)\right\}^{\frac{\beta}{\alpha}} \tag{23}$$

Also $I_2 = \frac{2}{\pi}\int_{\pi/\xi_n}^{\pi} \left\|\psi_{x+y}(t) - \psi_x(t)\right\|_p \left|\tilde{K}_n^*(i,t)\right| dt$

$$= \frac{2}{\pi}\int_{\frac{\pi}{\xi_n}}^{\pi} O\left(|y|^{\alpha}\right)O\left(t^{-1}\right)\left|\sum_{k=0}^{\infty} m_{nk}(i)\cos(kt)\right|dt \quad \text{(by (19) and } 2\tan\frac{t}{2}\geq t\text{)}$$

$$= O\left(|y|^{\alpha}\right)\int_{\frac{\pi}{\xi_n}}^{\pi} O\left(t^{-1}\right)O\left(t^{-1}\right)\phi_n(i)\,dt \qquad \text{(by (15))}$$

$$= O\left(|y|^{\alpha}\right)\int_{\frac{\pi}{\xi_n}}^{\pi} \phi_n(i)t^{-2}dt$$

$$= O\left(|y|^{\alpha}\right)\phi_n(i)\int_{\frac{\pi}{\xi_n}}^{\pi} t^{-2}dt$$

$$= O\left(|y|^{\alpha}\right)\phi_n(i)\left[\frac{t^{-1}}{-1}\right]_{\frac{\pi}{\xi_n}}^{\pi}$$

$$= \xi_n\phi_n(i)O\left(|y|^{\alpha}\right) \tag{24}$$

Combining (16), (24), we have for $0\leq\beta<\alpha\leq 1$

$$I_2 = I_2^{\frac{\beta}{\alpha}} I_2^{\left(1-\frac{\beta}{\alpha}\right)}$$

$$= \left\{\xi_n\phi_n(i)O\left(|y|^{\alpha}\right)\right\}^{\frac{\beta}{\alpha}}\begin{Bmatrix} O(1)\phi_n(i)\xi_n^{1-\alpha} & ;0<\alpha<1 \\ O(1)\phi_n(i)\log\xi_n & ;\alpha=1\end{Bmatrix}^{1-\frac{\beta}{\alpha}}$$

$$= O\left(|y|^{\beta}\right)\phi_n(i)\begin{cases} \xi_n^{1+\beta-\alpha} & ;0<\alpha<1 \\ \xi_n^{\beta}\left(\log\xi_n\right)^{1-\beta} & ;\alpha=1\end{cases} \tag{25}$$

Hence $\left\|\tilde{l}_n^*(i;x+y)-\tilde{l}_n^*(i;x)\right\|_p \leq I_1+I_2$

$$= O\left(|y|^{\beta}\right)\xi_n^{\beta-\alpha}\left\{\frac{1}{\eta_n}+\log\left(\frac{\eta_n}{\xi_n}\right)\right\}^{\frac{\beta}{\alpha}} + O\left(|y|^{\beta}\right)\phi_n(i)\begin{Bmatrix} \xi_n^{1+\beta-\alpha} & ;0<\alpha<1 \\ \xi_n^{\beta}\left(\log\xi_n\right)^{1-\beta} & ;\alpha=1\end{Bmatrix}$$

$$\Rightarrow \sup_{n,y\neq 0}\frac{\left\|\tilde{l}_n^*(i;x+y)-\tilde{l}_n^*(i;x)\right\|_p}{|y|^{\beta}} = O(1)\left[\begin{matrix} \xi_n^{\beta-\alpha}\left\{\frac{1}{\eta_n}+\log\left(\frac{\eta_n}{\xi_n}\right)\right\}^{\frac{\beta}{\alpha}} \\ +\phi_n(i)\begin{Bmatrix} \xi_n^{1+\beta-\alpha} & ;0<\alpha<1 \\ \xi_n^{\beta}\left(\log\xi_n\right)^{1-\beta} & ;\alpha=1\end{Bmatrix}\end{matrix}\right] \tag{26}$$

Further $\left\|\tilde{l}_n^*(i;x)\right\|_p$

$$\leq \frac{2}{\pi}\int_0^{\frac{\pi}{\xi_n}}\left\|\psi_x(t)\right\|_p\left|\tilde{K}_n^*(i;t)\right|dt + \frac{2}{\pi}\int_{\frac{\pi}{\xi_n}}^{\pi}\left\|\psi_x(t)\right\|_p\left|\tilde{H}_n^*(i;t)\right|dt$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{\xi_n}} O\left(|t|^{\alpha}\right)O\left(t^{-1}\right)dt + \frac{2}{\pi}\int_{\frac{\pi}{\xi_n}}^{\pi} O\left(|t|^{\alpha}\right)\left|\sum_{k=0}^{\infty} m_{nk}(i)\frac{\cos(kt)}{2\tan\left(\frac{t}{2}\right)}\right|dt$$

(using (11) in both integrals & (9) in 1st integral)

$$= O(1)\int_0^{\frac{\pi}{\xi_n}} t^{\alpha-1}dt + O(1)\int_{\frac{\pi}{\xi_n}}^{\pi} t^{\alpha}\, O\left(t^{-1}\right)\left|\sum_{k=0}^{\infty} m_{nk}(i)\cos(kt)\right|dt \qquad \left(\because 2\tan\left(\frac{t}{2}\right)\geq t\right)$$

$$= O(1)\left[\frac{t^{\alpha}}{\alpha}\right]_0^{\frac{\pi}{\xi_n}} + O(1)\int_{\frac{\pi}{\xi_n}}^{\pi} t^{\alpha-1} O\left(t^{-1}\right)\phi_n(i)\,dt \qquad \text{(by (15))}$$

$$= O(1)\frac{1}{\xi_n^{\alpha}} + O(1)\phi_n(i)\int_{\frac{\pi}{\xi_n}}^{\pi} t^{\alpha-2}\,dt$$

$$= O(1)\xi_n^{-\alpha} + O(1)\phi_n(i)\left[\begin{matrix}\dfrac{t^{\alpha-1}}{\alpha-1} & ;0<\alpha<1\\ \log t & ;\alpha=1\end{matrix}\right]_{\frac{\pi}{\xi_n}}^{\pi}$$

$$= \xi_n^{-\alpha}O(1) + O(1)\phi_n(i)\left[\begin{matrix}\xi_n^{1-\alpha} & ;0<\alpha<1\\ \log\xi_n & ;\alpha=1\end{matrix}\right.$$

$$\Rightarrow \left\|\tilde{l}_n^*(i;x)\right\|_p = O(1)\left[\xi_n^{-\alpha} + \phi_n(i)\left\{\begin{matrix}\xi_n^{1-\alpha} & ;0<\alpha<1\\ \log\xi_n & ;\alpha=1\end{matrix}\right\}\right] \qquad (27)$$

Adding (26) and (27), we have

$$\left\|\tilde{l}_n^*(i;x)\right\|_{(\beta,p)} = \left\|\tilde{l}_n^*(i;x)\right\|_p + \sup_{n,y\neq 0}\frac{\left\|\tilde{l}_n^*(i;x+y)-\tilde{l}_n^*(i;x)\right\|_p}{|y|^{\beta}}$$

$$= O(1)\left[\xi_n^{-\alpha} + \xi_n^{\beta-\alpha}\left\{\frac{1}{\eta_n}+\log\left(\frac{\eta_n}{\xi_n}\right)\right\}^{\frac{\beta}{\alpha}} + \phi_n(i)\left\{\begin{matrix}\xi_n^{1-\alpha}\left(1+\xi_n^{\beta}\right) & ;0<\alpha<1\\ \log\xi_n + \xi_n^{\beta}\left(\log\xi_n\right)^{1-\beta} & ;\alpha=1\end{matrix}\right\}\right] \qquad (28)$$

Hence the result follows.
This completes the proof of theorem.

4. Corollaries

Using the above theorem the following two corollaries can be established.

Corollary1

If for $p\geq 1, 0\leq\beta<\alpha\leq 1, f\in H(\alpha,p)$ and $M=\left(m_{nk}(i)\right)$ is a lower triangular matrix of non-negative real numbers with monotonic increasing in *k* such that $\sum_{k=0}^{n} m_{nk}(i)\to 1$ as $n\to\infty$ uniformly in *i*, then

$$\left\|\tilde{l}_n^*(i;x)\right\|_{(\beta,p)} = O(1)\left[n^{-\alpha}\left(1+n^{\beta-\frac{\beta}{\alpha}}\right) + m_{n,n}(i)\left\{\begin{matrix}n^{1-\alpha}\left(1+n^{\beta}\right) & ;0<\alpha<1\\ \log n + n^{\beta}\left(\log n\right)^{1-\beta} & ;\alpha=1\end{matrix}\right\}\right]$$

Proof.

Let $p\geq 1, 0\leq\beta<\alpha\leq 1, f\in H(\alpha,p)$.

Let $M=\left(m_{nk}(i)\right)$ be a lower triangular matrix of non-negative real numbers with monotonic increasing in *k* such that $\sum_{k=0}^{n} m_{nk}(i)\to 1$ as $n\to\infty$ uniformly in *i* i.e.

$$m_{nk}(i)\geq 0,\ m_{n,k}(i)\leq m_{n,k+1}(i)\ \forall k=0,1,2\ldots,n-1 \text{ and } \sum_{k=0}^{n} m_{nk}(i)\to 1 \text{ as } n\to\infty \text{ uniformly in } i.$$

Choose $\xi_n = \eta_n = n$.

Then $\sum_{k=\eta_n+1}^{\infty} k^2 |m_{nk}(i)| = \sum_{k=n+1}^{\infty} k^2 |m_{nk}(i)|$

$$= (n+1)^2 \times 0 + (n+2)^2 \times 0 + ...$$

$$= O(n) \tag{29}$$

Also $\phi_n(i) = \sum_{k=0}^{\infty} |m_{n,k}(i) - m_{n,k+1}(i)| = \sum_{k=0}^{n} |m_{n,k+1}(i) - m_{n,k}(i)|$

$$= \sum_{k=0}^{n-1} |m_{n,k+1}(i) - m_{n,k}(i)| + |m_{n,n+1}(i) - m_{n,n}(i)|$$

$$= \sum_{k=0}^{n-1} \{m_{n,k+1}(i) - m_{n,k}(i)\} + m_{n,n}(i) \qquad (\because m_{n,n+1}(i) = 0)$$

$$= \{m_{n,1}(i) - m_{n,0}(i)\} + \{m_{n,2}(i) - m_{n,1}(i)\} + ... + \{m_{n,n}(i) - m_{n,n-1}(i)\} + m_{n,n}(i)$$

$$= -m_{n,0}(i) + m_{n,n}(i) + m_{n,n}(i)$$

$$= 2m_{n,n}(i) - m_{n,0}(i)$$

$$\le 2m_{n,n}(i)$$

$$\Rightarrow \phi_n(i) = O(m_{n,n}(i)) \tag{30}$$

Clearly all conditions of theorem are hold.

Hence $\left\| l_n^*(i;x) \right\|_{(\beta,p)} = O(1)\left[n^{-\alpha} + n^{\beta-\alpha}\left(n^{-\frac{\beta}{\alpha}} \right) + O(m_{n,n}(i)) \begin{cases} n^{1-\alpha}(1+n^{\beta}) & ;0<\alpha<1 \\ \log n + n^{\beta}(\log n)^{1-\beta} & ;\alpha=1 \end{cases} \right]$

$$= O(1)\left[n^{-\alpha}\left(1 + n^{\beta-\frac{\beta}{\alpha}} \right) + m_{n,n}(i) \begin{cases} n^{1-\alpha}(1+n^{\beta}) & ;0<\alpha<1 \\ \log n + n^{\beta}(\log n)^{1-\beta} & ;\alpha=1 \end{cases} \right] \tag{31}$$

This establishes corollary 1.

Corollary 2

If for $p \ge 1, 0 \le \beta < \alpha \le 1, f \in H(\alpha,p)$ and $M = (m_{nk}(i))$ is a lower triangular matrix of non-negative real numbers with monotonic decreasing in *k* such that $\sum_{k=0}^{n} m_{nk}(i) \to 1$ as $n \to \infty$ uniformly in *i*, then

$$\left\| l_n^*(i;x) \right\|_{(\beta,p)} = O(1)\left[n^{-\alpha}\left(1 + n^{\beta-\frac{\beta}{\alpha}} \right) + m_{n,0}(i) \begin{cases} n^{1-\alpha}(1+n^{\beta}) & ;0<\alpha<1 \\ \log n + n^{\beta}(\log n)^{1-\beta} & ;\alpha=1 \end{cases} \right]$$

Proof.

Let $p \ge 1, 0 \le \beta < \alpha \le 1, f \in H(\alpha,p)$.

Let $M = (m_{nk}(i))$ be a lower triangular matrix of non-negative real numbers with monotonic decreasing in *k* such that $\sum_{k=0}^{n} m_{nk}(i) \to 1$ as $n \to \infty$ uniformly in *i* i.e.

$$m_{nk}(i) \ge 0,\ m_{n,k}(i) \ge m_{n,k+1}(i)\ \forall k = 0,1,2...,n-1 \text{ and } \sum_{k=0}^{n} m_{nk}(i) \to 1 \text{ as } n \to \infty \text{ uniformly in } i.$$

Choose $\xi_n = \eta_n = n$.

Then $\sum_{k=\eta_n+1}^{\infty} k^2 |m_{nk}(i)| = \sum_{k=n+1}^{\infty} k^2 |m_{nk}(i)|$

$$= (n+1)^2 \times 0 + (n+2)^2 \times 0 + ...$$

$$= O(n) \tag{32}$$

Also $\phi_n(i) = \sum_{k=0}^{\infty} \left| m_{n,k}(i) - m_{n,k+1}(i) \right|$

$$= \sum_{k=0}^{n} \left| m_{n,k}(i) - m_{n,k+1}(i) \right|$$

$$= \sum_{k=0}^{n} \left\{ m_{n,k}(i) - m_{n,k+1}(i) \right\}$$

$$= \left\{ m_{n,0}(i) - m_{n,1}(i) \right\} + \left\{ m_{n,1}(i) - m_{n,2}(i) \right\} + \ldots + \left\{ m_{n,n}(i) - m_{n,n+1}(i) \right\} \qquad \left(\because m_{n,n+1}(i) = 0 \right)$$

$$= m_{n,0}(i)$$

$$\Rightarrow \phi_n(i) = m_{n,0}(i) \tag{33}$$

Clearly all conditions of theorem are hold.

Hence $\left\| l_n^*(i;x) \right\|_{(\beta,p)} = O(1)\left[n^{-\alpha} + n^{\beta-\alpha}\left(n^{-\frac{\beta}{\alpha}} \right) + m_{n,0}(i) \begin{Bmatrix} n^{1-\alpha}\left(1+n^{\beta}\right) & ;0<\alpha<1 \\ \log n + n^{\beta}(\log n)^{1-\beta} & ;\alpha = 1 \end{Bmatrix} \right]$

$$= O(1)\left[n^{-\alpha}\left(1 + n^{\beta - \frac{\beta}{\alpha}} \right) + m_{n,0}(i) \begin{Bmatrix} n^{1-\alpha}\left(1+n^{\beta}\right) & ;0<\alpha<1 \\ \log n + n^{\beta}(\log n)^{1-\beta} & ;\alpha = 1 \end{Bmatrix} \right] \tag{34}$$

This establishes corollary 2.

Remark

The above result improves the result of G. Das, R. N. Das and B. K. Ray [3] taking modified partial sum $\tilde{S}_n^*(x)$ of conjugate Fourier series of $f(x)$ in place of $\tilde{S}_n(x)$ (see [8]).

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