Improvement In Acceptable Uncertainty Bounds By Dynamic State Feedback Controller For Second Order Systems

T. Sasi Kanth
Department of Electrical Engineering, JNT University, GIET, Rajahmundry.
K.Anand
Asst prof, Department of Electrical Engineering, JNT University, GIET, Rajahmundry

Abstract-- **The tolerable uncertainty bound of a system with constant uncertainty may be improved by means of dynamic state feedback controller compared to that achievable by using static state feedback one. However, for all the systems with constant uncertainty in input matrix, the uncertainty bound may not be improved by dynamic state feedback controller. So, a categorization of systems is needed for which a dynamic state feedback controller can improve the tolerable uncertainty bound. In this paper, a categorization of second order systems with input matrix uncertainty is made in order to investigate whether possibilities of improvement in robustness margin exist or not by using dynamic state feedback controller. The categorization is further used to provide some new examples for which possibility of i mproving tolerable uncertainty exists.**

Keywords: Uncertain Systems; Stability; Dynamic Feedback Controller

I. INTRODUCTION

Robust control design for uncertain system has been extensively studied for past few years. In many literatures, a lot of attention has been given to robust stabilization. A popular approach to deal with uncertain systems is the so called "quadratic stability" [11]. A relationship between the quadratic stability and the control has been established [8, 11] for continuous time domain case, and attempt has been made for discrete time system. A robust H_∞ performance problem has been addressed for a class of system with norm bounded parametric uncertainty in some of the system matrices [9, 10]. A dynamic state feedback controller is used for performance improvement [1] and may mimic a PI-type controller which is a subset of the PID state-feedback one [6]. It is well known that, for systems with time-invariant uncertainties, the robust margin in terms of the tolerable uncertainty bound may be improved by using a dynamic state feedback controller compared to the static feedback one [1, 5]. So far, some example cases have been studied for the improvement of tolerable uncertainty bounds using the dynamic state-feedback controllers [1]. However, it is not clear from existing literature whether there exist some more examples of such systems or a class of systems for which the improvement in tolerable uncertainty bound holds. In this paper, we attempt to classify a class of second order systems for which such improvement is possible by using suitable dynamic state feedback controller compared to that achievable by using static feedback controller.

For the purpose, one firstly requires to obtain an approximate tolerable uncertainty bound achievable by static state feedback controllers. This has been computed by using a method of contradiction [1]. In this paper, we attempt to characterize the class of second order systems for which this method of contradiction applies. Subsequently, we have developed an algorithm to test and verify whether there is possibility that dynamic feedback controllers may improve the tolerable uncertainty bound or not. Using this proposed algorithm, we have generated some new examples and verified that there exists a dynamic feedback controller, which indeed improves the uncertainty bound for these cases.

II. STABILIZATION OF SYSTEMS WITH INPUT MATRIX UNCERTIANTY

A class of systems with time-invariant uncertain input matrix may be described as

$$\dot{x}(t) = Ax(t) + (B + \Delta B(r))u(t) \tag{1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, r is the uncertain parameter in the control input matrix.

The static state feedback controller of the form

$$u(t) = Kx(t) \tag{2}$$

which can stabilize the system with an appropriate value of controller gain. Or alternatively by using dynamic state feedback controller of the form

$$\begin{aligned}\dot{z}(t) &= A_c z(t) + B_c x(t)\\ u(t) &= C_c z(t) + D_c x(t)\end{aligned} \tag{3}$$

where $z(t) \in R^{n_c}$ is the state of the controller, A_c, B_c, C_c, D_c are constant matrices in appropriate dimensions.

If a system is controllable, all the poles of the closed loop system can be placed anywhere in the left half s-plane, obviously there is nothing more to attain by using dynamic state feedback controller [2]. If the uncertainty parameter is time varying then the stability bound of the system cannot be enhanced by dynamic state feedback controller [4]. If the uncertainty parameter is time invariant then the stability bound of the system can be enhanced by dynamic state feedback controller rather than static state feedback controller, which has been shown in the paper [1].

A. A Characterization

To categorize those systems whose stability bound can be enhanced by using a dynamic state feedback controller, we consider a general second order uncertain system with constant uncertainty of the form where $x(t) \in R^2$ is the state vector and $u(t) \in R$ is the control input , $\bar{b}_1 = b_1 + r$ and r is the parametric uncertainty in the control input matrix that may take arbitrary value within a range, i.e.,

$$\dot{x}(t) = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} x(t) + \begin{bmatrix} \bar{b}_1 \\ b_2 \end{bmatrix} u(t) \qquad (4)$$

$|r| \leq \bar{r}$,where r

For this system, consider a static state feedback controller of the form

$$u(t) = \begin{bmatrix} k_1 & k_2 \end{bmatrix} x(t) \qquad (5)$$

The closed loop equation is then given by

$$\dot{x}(t) = \begin{bmatrix} a_1 + \bar{b}_1 k_1 & a_2 + \bar{b}_1 k_2 \\ a_3 + b_2 k_1 & a_4 + b_2 k_2 \end{bmatrix} x(t) \qquad (6)$$

Equation (6) is in the form of $\dot{x}(t) = \bar{A}x(t)$. The characteristic equation can be represented by $|sI - \bar{A}| = 0$, which yields

$$\begin{aligned} &s^2 - s\left\{\bar{b}_1 k_1 + b_2 k_2 + (a_1 + a_4)\right\} + \\ &\left\{(a_4 \bar{b}_1 - a_2 b_2)k_1 + (a_1 b_2 - a_3 \bar{b}_1)k_2 + (a_1 a_4 - a_2 a_3)\right\} = 0 \end{aligned} \qquad (7)$$

To guarantee the stability of the system, each of the coefficients should be positive which can be interpreted as:

$$P\begin{bmatrix} k_1 & k_2 & 1 \end{bmatrix}^T < 0, \quad (8)$$

$$Q\begin{bmatrix} k_1 & k_2 & 1 \end{bmatrix}^T > 0, \quad (9)$$

where $P = \begin{bmatrix} \bar{b}_1 & b_2 & (a_1 + a_4) \end{bmatrix}$ and $Q = \begin{bmatrix} (a_4 \bar{b}_1 - a_2 b_2) & (a_1 b_2 - a_3 \bar{b}_1) & (a_1 a_4 - a_2 a_3) \end{bmatrix}$.

Note that, both the inequalities will contradict each other when P becomes equal to Q . Assuming that the nominal system (for $r = 0$) is stabilizable, since P and Q are continuous on the uncertain parameter r , there exists a $\bar{r}$ satisfying $|r| \leq \bar{r}$ such that P at $\bar{r}$ becomes equal to Q at $-\bar{r}$. It is clear from the above that, (8) and (9) will contradict each other and solution of k_1 and k_2 will not exist, i.e., the system is not stabilizable anymore. It can be easily derived that such a contradictory inequality appears when

$$b_1 + \bar{r} = (b_1 - \bar{r})a_4 - a_2 b_2 \qquad (10)$$

$$b_2 = a_1 b_2 - a_3(b_1 - \bar{r}) \qquad (11)$$

$$a_1 + a_4 = a_1 a_4 - a_2 a_3 \qquad (12)$$

Note that, the last condition can be interpreted as $\text{trace}(A) = \det(A)$. From the above conditions, we develop an algorithm using which one can obtain analytical tolerable ranges of r for a given system. It is presented next.

B. When This Proof of Contradiction Applies?

The following algorithm presents a method using which one can determine the tolerable uncertainty bound using static feedback controllers.

Algorithm 1:

1: $\text{Trace}(A) = \text{Det}(A)$

2: $a_4 = +1$, (This is not that restrictive as it appears since one can always interchange the state definitions, i.e., can interchange a_1 and a_4)

3: $a_2 \neq \dfrac{a_4 + 1}{a_4 - 1 - a_3(a_4 + 1)}$

4: Finally, if $a_1 = \dfrac{a_4 + a_2 a_3}{a_4 - 1}$ and $b_2 = \dfrac{2a_3 b_1}{(a_1 - 1)(a_4 + 1) - a_2 a_3}$.

Using the above equations, one may check whether the contradiction applies to a given system (4) and subsequently can determine the analytical range of the tolerable uncertainty by using static state feedback controllers. As a converse, using the same, one may test a given system (4) for which dynamic controller may enhance the tolerable uncertainty bound.

Remark 1: If the system satisfies the above condition then one may obtain the tolerable uncertainty bounds for any static state feedback controller by following the method of contradiction as

$$|r| < \frac{b_1(a_4 - 1) - a_2 b_2}{1 + a_4}$$

Next, we present a lemma that will be used to determine the tolerable uncertainty bounds for a given controller case, which will be used to determine the tolerable uncertainty bound for a given dynamic feedback controller.

Lemma 1 [3]: For two appropriate dimensional matrices M_0 and M_1 with M_0 is Hurwitz, any matrix belonging to the set of the matrices

$$M = \{M_r = M_0 + rM_1, r \in [r_{\min}, r_{\max}]\} \qquad (13)$$

remains stable for any $r \in (r_{\min} \quad r_{\max})$, where

$$r_{\min} = \frac{1}{\lambda_{\min}(-(M_0 \oplus M_0)^{-1}(M_1 \oplus M_1))} \qquad (14)$$

$$r_{\max} = \frac{1}{\lambda_{\max}(-(M_0 \oplus M_0)^{-1}(M_1 \oplus M_1))} \qquad (15)$$

$\lambda_{\max}(X)$ denotes the maximum positive real eigenvalue of X (if X has no positive eigenvalue then $\lambda_{\max}(X) = 0^+$) and $\lambda_{\min}(X)$ denotes the minimum real eigenvalue of X (if X has no negative eigenvalue then $\lambda_{\min}(X) = 0^-$).

III. NUMERICAL EXAMPLES

In this section, we demonstrate two new examples for which the method of contradiction applies and there exists dynamic state feedback controller that improves the tolerable uncertainty bound.

Example 1: Considering a second order system with constant uncertainty with input control matrix

$$\dot{x}(t) = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} -1+r \\ -2 \end{bmatrix} u(t) \quad (16)$$

Appling a static state feedback controller of the form $u = k_1 x_1(t) + k_2 x_2(t)$, the closed loop equation can be

$$\dot{x}(t) = \begin{bmatrix} 1+(-1+r)k_1 & -1+(-1+r)k_2 \\ 1-2k_1 & 2-2k_2 \end{bmatrix} x(t) \quad (17)$$

The characteristic equation is

$$s^2 + s\{k_1(1-r)+2k_2-3\} - \{k_1(4-2r)+k_2(1+r)-3\} = 0 \quad (18)$$

For stability of the system,

$$\{k_1(1-r)+2k_2-3\} > 0 \quad (19)$$

$$\{k_1(4-2r)+k_2(1+r)-3\} < 0 \quad (20)$$

At $r=-1$, from Eqn. (19)

$$\{2k_1+2k_2-3\} > 0 \quad (21)$$

From Eqn. (20)

$$\{6k_1-3\} < 0 \quad (22)$$

At $r=1$, from Eqn. (19)

$$\{2k_1-3\} > 0 \quad (23)$$

From Eqn. (20)

$$\{2k_1+2k_2-3\} < 0 \quad (24)$$

Equations (21) and (24) are contradicting each other so the system can be stabilized by a static feedback controller when $|r|<1$.

Considering a dynamic feedback controller,

$$\begin{aligned} \dot{z}(t) &= -122.5501x_1(t) + 12.06766x_2(t) - 2.3592z(t) \\ u(t) &= -40.8416x_1(t) + 51.1517x_2(t) + 0.8173z(t) \end{aligned} \quad (25)$$

The parameters of the dynamic controller have been searched by fminsearch program of MATLAB®.

The closed loop equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 41.8416 & -52.1517 & -0.8173 \\ 82.6832 & -100.3033 & -1.6346 \\ -122.5501 & 12.0676 & -2.3592 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} + r \begin{bmatrix} -40.8416 & 51.1517 & 0.8173 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} \quad (26)$$

Now, the range of the tolerable uncertainty $r \in (r_{min} \quad r_{max})$ may be determined using Lemma 1. For the purpose, we express the above closed loop system as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{z} \end{bmatrix} = M \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix}$$

where M is a set of perturbed matrices

$$M = M_0 + rM_1$$

M_0 and M_1 are $n \times n$ matrices with M_0 strictly stable .The maximal range of r for M to be strictly stable using Lemma 1 is obtained as

$$r_{min} = -1.1709 \text{ and } r_{max} = 1.2655$$

Here the dynamic controller enhances the stability bound.

Example 2: We have considered another example to justify more appropriately that by referring the above algorithm, one can develop new examples,

$$\dot{x}(t) = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 1+r \\ 2 \end{bmatrix} u(t) \quad (27)$$

The above system obeys the method of contradiction for $|r|<1$. Hence, the system cannot be stabilized by a static state feedback controller for $|r| \geq 1$. However, considering a dynamic state feedback controller

$$\begin{aligned} \dot{z}(t) &= -0.2957x_1(t) + 11.6418x_2(t) - 5.1051z(t) \\ u(t) &= 1.0538x_1(t) - 3.8586x_2(t) - 0.1906z(t) \end{aligned} \quad (28)$$

the tolerable uncertainty bound using Lemma 1 is obtained as:

$$r_{min} = -1.1910 \text{ and } r_{max} = 1.1952$$

Clearly this controller improves the tolerable uncertainty bound compared to that tolerable by using any static feedback one.

Note that, the tolerable uncertainty bounds obtained using dynamic feedback controller are not the optimal one since one may design a different controller which may yield larger bounds than the present one.

IV. DISCUSSION

The above approach with the developed simple Algorithm is an easy way to check the category of systems with constant uncertainty for robust stability improvement. With this, one can check whether any dynamic controller of the form (3) can improve the uncertainty bound or not. Considering two numerical examples (16) and (27), the uncertainty bound is improved by dynamic controller (25) and (28), respectively, which gives the validation for the algorithm 1. This developed algorithm can be used for improved robust stabilization in many applications, such as any second order approximated model of DC Motor, Inverted pendulum etc.

V. CONCLUSION

Categorization of second order uncertain systems with time-invariant uncertainty in the input matrix in order to improve the tolerable uncertainty bound using suitable dynamic feedback controller has been presented in this paper. Two new examples have been demonstrated for which it has been seen that dynamic feedback controller indeed enhances the tolerable uncertainty bound compared to the static one for several systems. The developed algorithm for category of system with constant uncertainty may be motivation for developing an algorithm for systems with constant uncertainty in all the terms of the system matrices and also in control of input matrices. There is no proper procedure for designing the dynamic controller for this robust improvement. So, one may work to develop a proper procedure for designing the controller.

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