On Integrability and L^P-convergence of modified trigonometric sums

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Abstract

We study modified cosine and sine sums and obtain necessary and sufficient conditions for the integrability and L^p-convergence of these modified sums. We deduce the result of Ul'yanov [4, cf.1] as corollary from our results. We also obtain stronger result than that of Rees and Stanojević [3, Theorem 1].

1. Introduction

Let us consider the series

(1.1) $f(x) = \dfrac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$,

(1.2) $\bar{f}(x) = \sum_{k=1}^{\infty} a_k \sin kx$.

We know that the functions f(x) and $\bar{f}(x)$ defined by the series (1.1) and (1.2) for $a_k \downarrow 0$ can be non-summable. Ul'yanov [4, cf. 1] obtained following result concerning the integrability of $| f |^p$ and $| \bar{f} |^p$, for any p provided $0 < p < 1$ and $< a_k >$ a null sequence of bounded variation.

Theorem A. If the sequence $< a_k >$ satisfies the condition $a_k \to 0$ and $\Sigma |\Delta a_k| < +\infty$, then for any p, $0 < p < 1$, we have

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \left| f(x) - S_n(x) \right|^p dx = 0,$$

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \left| \bar{f}(x) - \bar{S}_n(x) \right|^p dx = 0 ,$$

where $S_n(x)$ and $\bar{S}_n(x)$ are the partial sums of the series (1.1) and (1.2).

Concerning the series (1.2),the following theorem is well known:

Theorem B [1]. Let $\bar{f}(x) = \sum_{n=1}^{\infty} a_n \sin nx$ where $\Delta a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$, then $\bar{f} \varepsilon L^1 [0, \pi]$.

However, there is no known analogue of theorem B for the cosine series.

Hooda [2] *et al.* introduced new modified cosine sums as

$$(1.3)\quad f_n(x) = \left(\frac{1}{2}\right)\left[a_1 + \sum_{k=0}^{n} \Delta^2 a_k\right] + \sum_{k=1}^{n}\left[a_{k+1} + \sum_{j=k}^{n} \Delta^2 a_j\right] \cos kx,$$

and here sine sums as

$$(1.4)\quad \bar{f}_n(x) = \sum_{k=1}^{n}\left[a_{k+1} + \sum_{j=k}^{n} \Delta^2 a_j\right] \sin kx ,$$

and studied these sums for convergence. We give the necessary and sufficient conditions for the integrability and L^p–convergence of these modified sums. We deduce Theorem A as corollary of our theorems , obtain stronger result than that of Rees of Stanojević [3, Theorem1] and an analogue of theorem B for the cosine series. However, we shall consider only (1.3), as the results for (1.4) can be obtained by slight modifications.

2. Main Results

We prove the following results :

Theorem 1. If the sequence $\{a_n\}$ satisfies the conditions $a_n \to 0$ and $\Sigma |\Delta a_n| < \infty$,then for any p, 0<p<1,

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \left| f(x) - f_n(x) \right|^p dx = 0 ,$$

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \left| \bar{f}(x) - \bar{f}_n(x) \right|^p dx = 0 .$$

2000 AMS Mathematics Subject Classification : 42A20, 42A32.

Theorem 2. Let $a_n \to 0$, then

(i) $f(x) = \lim_{n\to\infty} f_n(x)$ exists, and

(ii) $f \in L^1[0,\pi]$.

This result is stronger than Theorem 1 of Rees and Stanojević [3] and provides an analogue of Theorem B for the cosine series for these types of sums.

3. Proofs of Main Results

Proof of Theorem 1. We have

(3.1) $$f_n(x)=\left(\frac{1}{2}\right)\left[a_1+\sum_{k=0}^{n}\Delta^2 a_k\right] + \sum_{k=1}^{n}\left[a_{k+1}+\sum_{j=k}^{n}\Delta^2 a_j\right]\cos kx$$

$$= (1/2)[a_0 - a_{n+1} + a_{n+2}] + \sum_{k=1}^{n} [a_k - a_{n+1} + a_{n+2}] \cos kx$$

$$= (a_0/2) - (1/2)\Delta a_{n+1} + \sum_{k=1}^{n} a_k \cos kx - \Delta a_{n+1}\sum_{k=1}^{n} \cos kx$$

$$= (a_0/2) + \sum_{k=1}^{n} a_k \cos kx - \Delta a_{n+1}\left(\sum_{k=1}^{n}\cos kx + \left(\frac{1}{2}\right)\right)$$

$$= S_n(x) - \Delta a_{n+1} D_n(x),$$

where $D_n(x) = (1/2) + \cos x + \ldots + \cos nx$ represents Dirichlet's kernel.

Using Abel's transformation, we get

(3.2) $$f_n(x) = \sum_{k=1}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) - \Delta a_{n+1} D_n(x)$$

$$= \sum_{k=1}^{n} \Delta a_k D_k(x) + a_{n+2} D_n(x).$$

Now,

$$f(x) - f_n(x) = \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+2} D_n(x) \text{ for } x \neq 0.$$

This means

$$|f(x) - f_n(x)|^p \le \left(\frac{2}{|x|}\right)^p \left[\sum_{k=n+1}^{\infty}|\Delta a_k| + |a_{n+2}|\right]^p$$

and therefore,

$$\int_{-\pi}^{\pi} |f(x)- f_n(x)|^p\,dx \le 2^p\left[|a_{n+2}| + \sum_{k=n+1}^{\infty}|\Delta a_k|\right]^p \int_{-\pi}^{\pi}\frac{dx}{x^p}$$

$$\to 0 \text{ as } n\to\infty.$$

A similar argument is also valid for $\bar{f}(x) - \bar{f}_n(x)$.

Corollary 1. From relation (3.1), we have

$$\int_{-\pi}^{\pi} |f(x) - S_n(x)|^p\,dx = \int_{-\pi}^{\pi} | f(x) - f_n(x) + f_n(x) - S_n(x)|^p\,dx$$

$$\le \int_{-\pi}^{\pi} | f(x) - f_n(x)|^p\,dx + \int_{-\pi}^{\pi} | f_n(x) - S_n(x)|^p\,dx$$

$$= \int_{-\pi}^{\pi} | f(x) - f_n(x)|^p\,dx + \int_{-\pi}^{\pi} |\Delta a_{n+1} D_n(x)|^p\,dx$$

Now,

$$\int_{-\pi}^{\pi} |\Delta a_{n+1} D_n(x)|^p\,dx$$

$$\leq \int_{-\pi}^{\pi} \left(\frac{2}{|x|}\right)^p |\Delta a_{n+1}|^p \, dx$$

$$= 2^p |\Delta a_{n+1}|^p \int_{-\pi}^{\pi} (dx/x^p)$$

$$\to 0 \text{ as } n\to\infty .$$

Also $\lim_{n\to\infty} \int_{-\pi}^{\pi} |f(x) - f_n(x)|^p \, dx = 0$ by our Theorem 1. Hence the corollary follows.

Corollary2. $\lim_{n\to\infty} \int_{-\pi}^{\pi} | \bar{f}(x) - \bar{S}_n(x) |^p \, dx = 0$ follows by the similar arguments as in corollary 1.

Corollaries (1) and (2) yield Theorem A of Ul'yanov.

Proof. of Theorem 2. (i) We have

$$f_n(x) = \sum_{k=0}^{n} \Delta a_k D_k(x) + a_{n+2} D_n(x)$$

Using Abel's transformation,

$$(3.3)\ f_n(x) = \sum_{k=0}^{n-1} (k+1)\Delta^2 a_k F_k(x)$$

$$+ (n+1) \Delta a_n F_n(x) + a_{n+2} D_n(x)$$

where $F_n(x)$ denotes Féjer kernel.

Since $F_n(x) = O(1/ nx^2)$ for $x \neq 0$, and $a_n \to 0$, the last two terms of (3.3) tend to zero. Also

$$0 \leq \sum_{k=0}^{n-1} \Delta^2 a_k (k+1) F_k(x)$$

$$\leq (C/ x^2) (a_0 - \Delta a_n),$$

so $\lim_{n\to\infty} \sum_{k=0}^{n} (k+1) \Delta^2 a_k F_k(x)$ always exists for $x \neq 0$ and $a_n \to 0$ which proves the first part.

(ii) Now , $f(x) = \sum_{k=0}^{\infty} \Delta^2 a_k (k+1) F_k(x), \ x \neq 0$

Integrating term by term, we get

$$\int_{0}^{\pi} f(x)\, dx = (\pi/2) \sum_{k=0}^{n} (k+1) \Delta^2 a_k$$

$$= (\pi/2) a_0 < \infty .$$

References

[1] N.K. Bari, *A Treatise on trigonometricseries* (London : Pregamon Press) Vol. II (1964).

[2] N.Hooda ,B.Ram and S.S.Bhatia, On L^1-Convergence of a modified cosine sum, *Soochow J. Math.***28(3)**(2002),305-310.

[3] C.S. Rees and Č.V. Stanojević, Necessary and sufficient condition for Integrability of certain cosine sums, *J. Math. Anal. Appl.* **43**(1973), 579-586.

[4] P.L. Ul'yanov, Application of A-integration to a class of trigonometric series (in Russian) *MC* **35** (1954), 469-490.