Convergence Of Cosine Sums In Metric L

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Abstract

The aim of this paper is to study the L^1-convergence of modified cosine sums [4] . The results obtained generalize the results of [4] and deduce a well known result [6] as a corollary.

1. Introduction.

Consider the cosine series

(1.1) $$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx .$$

Let $S_n(x)$ denote the partial sum of (1.1) and

$f(x) = \lim_{n\to\infty} S_n(x)$.

The problem of L^1 – convergence, via Fourier coefficients, consists of finding the properties of Fourier coefficients such that the necessary and sufficient condition for

$\| S_n(x) - f(x) \| = o(1), \quad n\to\infty$, is given in the form $a_n \log n = o(1), \ n\to\infty$, where

$\| . \|$ denotes the L^1-norm.

Convex sequence. A sequence $\{a_k\}$ is said to be convex if

$\Delta^2 a_k \geq 0$ for every k where $\Delta^2 a_k = \Delta a_k - \Delta a_{k+1}$ and $\Delta a_k = a_k - a_{k+1}$.

Quasi- Convex sequence A sequence $\{a_k\}$ is said to be quasi-convex if

$$\sum_{k=1}^{\infty} k \ |\Delta^2 a_k| < \infty.$$

The class of all such sequences is an extension of the class of convex null sequences. The class of quasi-convex sequences is a subclass of BV class $(\sum_{k=1}^{\infty} |\Delta a_k| < \infty)$, the class of all null sequences of bounded variation.

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The class S[5.cf .1]. A null sequence $\{a_k\}$ belongs to the class S if there exists a sequence $\{A_k\}$ such that

(1.2) $$A_k \downarrow 0 , \ k\to\infty ,$$

(1.3) $$\sum_{k=0}^{\infty} A_k < \infty ,$$

(1.4) $$|\Delta a_k| \leq A_k \text{ for all } k .$$

The class S is the extension of the class of quasi-convex sequences. Since a quasi-convex null sequence satisfies conditions of the class S, if we choose $A_n = \sum_{m=n}^{\infty} |\Delta^2 a_m|$.

Concerning the convergence of (1.1) in L-metric, the following results are known.

Theorem A [1]. If $\{a_k\}$ is a null convex sequence, then the cosine series (1.1) is the Fourier series of its sum f, and

$$\| S_n(x) - f(x) \| = o(1), \ n\to\infty$$

if and only if

$$a_n \log n = o(1), \ n\to\infty .$$

Theorem B [1]. If $a_k = o(1), \ k \to \infty$, and the series $\sum_{k=1}^{\infty} k \ |\Delta^2 a_k| < \infty$. then the cosine series (1.1) is the Fourier series of its sum f, and

$$\| S_n(x) - f(x) \| = o(1), \quad n\to\infty,$$

if and only if

$$a_n \log n = o(1), \quad n\to\infty$$

Teljakovskii generalized Theorem B by establishing the following Theorem :

Theorem C[6]. Let $\{a_k\}$ belong to the class S. Then the cosine series (1.1) is the Fourier series of its sum f and

$$\| S_n(x) - f(x) \| = o(1), \quad n\to\infty$$

if and only if

$$a_n \log n = o(1), n\to\infty .$$

Teljakovskii, thus showed that the class S is also a class of L^1-convergence which in turn led to numerous, more general results.

Kumari and Ram [4] introduced a new modified cosine sum
$f_n(x) =$

$$\frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx ,$$

and proved

Theorem D. Let (1.1) belong to the class S.
If $\lim_{n\to\infty} | a_{n+1} | \log n = 0$, then

$$\| f(x) - f_n(x) \| = o(1), \ n \to \infty .$$

2. Lemmas

The following lemmas are required for the proofs of our results.

Lemma 1.[2]. If $| c_k | \le 1$, then

$$\int_0^{\pi} \left| \sum_{k=0}^{n} c_k D_k(x) \right| dx \le C\,(n+1) ,$$

where C is a positive constant.

Lemma 2[3]. Let $D_n(x)$, $\bar{D}_n(x)$ and $K_n(x)$ denote Dirichlet, conjugate Dirichlet and Fejer kernels respectively, then

$$K_n(x) = D_n(x) - (1/(n+1)) \ \bar{D}'_n(x)$$

3. Results .We prove the following theorem :

Theorem . Let $\{a_k\}$ belong to the class S. Then

$$\| f(x) - f_n(x) \| = o(1), \ n \to \infty .$$

Corllary . If $\{a_k\}$ belongs to the class S, then

$$\| S_n(x) - f(x) \| = 0, \ n \to \infty$$

if and only if

$$a_n \log n = o(1), \ n \to \infty .$$

The theorem generalizes Theorem D and corollary is Theorem C of Teljakovskii.

Proof of Theorem . We have

(3.1) $f_n(x) =$

$$\frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \times$$

$$\left[\Delta\left(\frac{a_k}{k}\right) + \Delta\left(\frac{a_{k+1}}{k+1}\right) + \ldots + \Delta\left(\frac{a_n}{n}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k \cos kx \left[\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1} \sum_{k=1}^{n} k \cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1} \ \bar{D}'_n(x)$$

$$= S_n(x) - \frac{a_{n+1}}{n+1} \ \bar{D}'_n(x).$$

Using Abel transformation and lemma 2,

$$f_n(x) = \sum_{k=0}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) - \frac{a_{n+1}}{n+1} \ \bar{D}'_n(x)$$

$$= \sum_{k=0}^{n-1} \Delta a_k D_k(x) + a_n D_n(x) - a_{n+1} D_n(x) + a_{n+1} K_n(x)$$

$$= \sum_{k=0}^{n-1} \Delta a_k D_k(x) + a_{n+1} K_n(x)$$

So,

$f(x) - f_n(x)$

$$= \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) - a_{n+1} K_n(x)$$

Abel transformation with lemma1 yield,

$$\int_0^{\pi}\left|f(x)-f_n(x)\right|dx$$

$$\leq \int_0^{\pi}\left|\sum_{k=n+1}^{\infty}\Delta a_k D_k(x)\right|dx + \int_0^{\pi}\left|a_{n+1}K_n(x)\right|dx$$

$$\leq \int_0^{\pi}\left|\sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x)\right|dx$$

$$+\left|a_{n+1}\right| \int_{-\pi}^{\pi} K_n(x)$$

$$= \int_0^{\pi}\left|\sum_{k=n+1}^{\infty}\Delta A_k \sum_{j=0}^{k}\frac{\Delta a_j}{A_j}D_j(x)\right|dx+\pi\left|a_{n+1}\right|,$$

$$\leq C\sum_{k=n+1}^{\infty}(k+1)\Delta A_k \;+\pi\left|a_{n+1}\right|$$

since $\int_{-\pi}^{\pi} K_n(x)=\pi$, $\{a_k\}$ is null sequence and under the assumed hypothesis $\sum_{k=n+1}^{\infty}(k+1)\Delta A_k$ converges, the right hand side tends to zero as $n\to\infty$ and this gives

$$\lim_{n\to\infty}\int_0^{\pi}\left|f(x)-f_n(x)\right|dx=0.$$

This completes the proof of our theorem.

Proof of Corollary . We have

$$\int_0^{\pi} | f(x) - S_n(x) | \, dx =$$

$$\int_0^{\pi} | f(x) - f_n(x) + f_n(x) - S_n(x) | \, dx$$

$$\leq \int_0^{\pi} | f(x) - f_n(x) | dx$$

$$+ \int_0^{\pi} | f_n(x) - S_n(x) | dx$$

$$\leq \int_0^{\pi} | S_n(x) - f_n(x) | dx + \int_0^{\pi} |a_{n+1} D_n(x)| dx$$

$$+ \int_{-\pi}^{\pi} |a_{n+1}K_n(x)| dx$$

and

$$\int_0^{\pi} | a_{n+1} D_n(x) | dx$$

$$\leq \int_0^{\pi} | f_n(x)-S_n(x) | dx + \int_{-\pi}^{\pi} | a_{n+1}K_n(x) | dx$$

$$\leq \int_0^{\pi} | f(x) - S_n(x) | dx + \int_{-\pi}^{\pi} | a_{n+1}K_n(x) | dx .$$

Since $\int_{-\pi}^{\pi} | D_n(x) | dx$ behave like $a_{n+1}\log n$ for large values of n and $\lim_{n\to\infty}\int_0^{\pi}\left|f(x)-f_n(x)\right|dx=0$ by our theorem, the corollary follows.

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