Collocation Approximation Method For Special Linear Integro-Differentialequations Using Chebyshev Basis Function

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Abstract

In this paper, we consider the solution of first and second order Linear integro-differential by the use of trial solution formulated as Chebyshev form of Fourier cosine series. The behaviour of solution for different degrees (N) of the trial solution is carefully studied and illustrative examples are included to demonstrate the validity and applicability of the techniques.

Key words: Chebyshev-collocation, Integro-differentials, Trial solution

1 INTRODUCTION

The profound use of first-kind Chebyshev polynomials especially in approximation theory has been on a vast increase since its discovery. In recent years, it enjoys major application in formulation of basis function as well as perturbation tools.

This in essence is due to the pivotal role of minimax approximation of function by polynomials which these polynomials efficiently play in the field of approximation (see ref. [2] and [11]).

On a general note, orthogonal functions and polynomials have received considerable attention especially in the solution of differential and integral problems, where the main characteristics of the technique is to reduce these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem.

The aim of this work however, is to apply the first kind of this orthogonal polynomial in the solution of integro-differential equation of the form:

$$P\, y''(x) + Q\, y'(x) + R\, y(x) + \lambda\int_a^b K(x, t)\, y(t)dt = f(x) \quad (1)$$

together with the boundary conditions:

$y(a) + y'(a) = A$
$y(b) + y'(b) = B$

For a given interval, (1) has a unique solution $y(x) : [a, b] \to R$ which is continuously differentiable(see [2], [5]).

To solve the problem of this sort, [5] applied collocation method with collocation points chosen as prescribed in [3], while [6] applied Galerkin weighted residual method.

In order to enhance produced results, a number of authors have equally carried out a perturbed version of the above method [9, 11], while others applied methods like (see [1], [8], and [10]). Quite a number of these methods go with complex derivations [2].

On the other hand, many researchers have successfully constructed trial solution with Chebyshev polynomial for the solution of differential equations, for instance Fox and Parker [7] applied it for the solution of second order IVP while others applied it to higher order differential equations. In all of these, it is noticed that solution with the use of high degree approximating polynomials.

In a bid to further enhance the solution of equation (1) with less complexity, we employ in this work a derived trial solution formulated with Chebyshev basis function. The application of this polynomial is therefore to utilize its error minimizing capability in enhancing the solution of the considered problem in the interval [a, b] within which the used Chebyshev polynomial are shifted from the natural interval [-1, 1].

2 DERIVATION OF TRIAL SOLUTION USING CHEBYSHEV POLYNOMIALS

The word "trial solution" here refers to the specific form which the solution (1) is to take. From Fourier series, it follows that:

$$PN(x) = \frac{1}{2}a_0 + \sum_{K=1}^{\infty} (a_k \cos kx + b_k \sin kx), \quad (2)$$

With the Fourier coefficients, defined by;

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx \;; \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx \quad (3)$$

is the least square approximation to f(x), with unit weight function in
$-\pi \le x \le \pi$.

From the work of Fox and Parker [7], it is established that the cosine series;

$$f(x) = \frac{1}{2}a_0 + \sum_{K=1}^{\infty} a_k \cos k\, dx,\ a_k = \frac{2}{\pi}\int_{-1}^{1}(1-x^2)^{-1/2} f(x) T_k(x)\, dx$$

has a very high rate of convergence and in Chebyshev form; it is written as:

$$y_N(x, a) = \frac{1}{2}a_0 + \sum_{K=1}^{\infty} a_r T_r(x) \qquad (5)$$

where x represents all the independent variables in the problems coefficients are the Degree Of Freedom (DOF).

$T_r(x)$ are Chebyshev polynomials in variable x, defined as a set of orthogonal polynomials of degree r given by:

$$T_r(x) = \cos\{r\cos^{-1}\frac{(2x-b-a)}{b-a}\}; \qquad (6)$$

and satisfies the recurrence relation;

$$T_{r+1}(x) = 2\frac{(2x-b-a)}{b-a}T_r(x) - T_{r-1}(x); \qquad (7)$$

which is valid within the interval $a \leq x \leq b$.

3 NUMERICAL TECHNIQUES

This technique involves the determination of approximants a_r by first sub-stituting (5) into (1) so as to yield a residual equation of the form:

$$P\bar{y}''_N(x) + Q\bar{y}'_N(x) + R\bar{y}_N(x) + \lambda\int_a^b K(x,t)\bar{y}(t)dt = f(x) \qquad (8)$$

It should be noted that equation (5) is substituted in both x and t that is for y(x) and y(t) so that we have residual equation of the form (8). And the attached initial/boundary conditions are equally imposed on the trial solution (5) to yield a system of equations that is equal to the boundary conditions in number.

The residual equation (8) is thereafter collocated at the zeros of a Chebyshev polynomial $T_r(x)$ which is chosen to have r in such a way that its addition to the number of given initial/boundary conditions is equal to the number if unknown approximates a_r in (5). This is to ensure a number of collocation equations that is equal to the number of unknown a_r.

Collocation at zeros of Chebyshev polynomial as suggested by Gauss-Lobatto [12]. Collocation methods require that at the zero of the relevant polynomial the residual equation is satisfied, thus yielding a number of collocation equations. These equations in conjunction with the resulting equations from imposed conditions are solved to determine the unknown approximants a_r, which are thereafter substituted into the (5) to yield the desired approximate solution.

4 NUMERICAL EXAMPLES AND RESULTS

We apply this method on some special problems. A varying degree of trial solution is used (N =2, 4 and 8) to allow for an in dept examination of the results produced. The entire solution steps are automated with the use of symbolic algebraic program MATLAB.

The entire process is automated by the use of a symbolic algebraic program MATLAB 7.9

Problem 1:

Solve the integro-differential equation

$$y'(x) = -\int_0^1 e^{st} y(t)dt + y(x) = \frac{1 - e^1 + x}{1 + x} \qquad (9)$$

Subject to initial condition y(0) = 1

The analytical solution is given as $y(x) = e^x$

Problem 2:

Determine the approximate solution the IVP;

$$y'(x) = \int_0^1 (4x + 2t)\, y(t)dt + y(x) - \cos(2x) - 2\sin(2x)\frac{1}{2}\sin(4x) \qquad (10)$$

with initial condition y(0) = 1,

The analytical solution is given as y(x) = cos (2x)

Problem 3:

Solve the integro-differential equation

$$y''(x) + xy'(x) - xy(x) = e^x - 2\sin x + \int_{-1}^{1}\sin(x)e^{-t}dt$$

with initial condition y(0) = 1, y′ (0) = 1

the exact solution is $y(x) = e^x$

Remark: All the examples we solved, the exact solutions are known in closed form, hence, we have defined our error as:

$$\text{Error} = |\, y(x) - y_N(x)\,|;\ a \leq x \leq b$$

TABLES OF ERRORS

Tables 1: Table of error for problem 1

X	N=2	N=4	N=8
0	3.1974e^{-16}	8.9038e^{-06}	1.0121e^{-05}
0.1	5.4094e^{-04}	8.6603e^{-06}	8.9536e^{-06}
0.2	4.4562e^{-04}	3.7704e^{-05}	1.0636e^{-05}
0.3	2.7736e^{-04}	9.7740e^{-05}	1.5452e^{-05}
0.4	1.6202e^{-03}	1.9104e^{-04}	2.3657e^{-05}
0.5	3.5758e^{-03}	3.2046e^{-04}	3.5492e^{-05}
0.6	6.1379e^{-03}	4.8940e^{-04}	5.1178e^{-05}
0.7	9.3006e^{-03}	7.0180e^{-04}	7.0924e^{-05}
0.8	1.3059e^{-02}	9.6218e^{-04}	9.4922e^{-05}
0.9	1.7407e^{-02}	1.2756e^{-03}	1.2336e^{-04}
1.0	2.2343e^{-02}	1.6476e^{-03}	1.5641e^{-04}

Tables 2: Table of error for problem 2

X	N=2	N=4	N=8
0	1.0779e^{-02}	2.0063e^{-04}	1.4642e^{-05}
0.1	7.3867e^{-03}	6.5487e^{-04}	2.1026e^{-05}
0.2	8.0231e^{-03}	6.2818e^{-04}	3.0939 e^{-05}
0.3	1.0779e^{-02}	5.0912e^{-05}	6.8622 e^{-05}
0.4	1.3293e^{-02}	7.0673e^{-04}	1.5941 e^{-06}
0.5	1.3655e^{-02}	1.3261e^{-03}	2.1612 e^{-06}
0.6	1.1137e^{-02}	1.6132e^{-03}	2.2515 e^{-06}
0.7	6.4669e^{-03}	1.5262e^{-03}	2.8817 e^{-06}
0.8	1.5556e^{-03}	1.1480e^{-03}	2.2002 e^{-06}
0.9	1.2367e^{-03}	6.1306e^{-04}	4.2080 e^{-06}
1.0	4.4409e^{-02}	1.7188e^{-05}	9.5997 e^{-07}

Tables 3: Table of error for problem 3

X	N=2	N=4	N=8
0	2.2204e^{-16}	8.8818e^{-16}	0
0.1	6.5101e^{-03}	2.4023e^{-04}	4.1727e^{-06}
0.2	2.5321e^{-02}	1.9743 e^{-03}	1.2527e^{-05}
0.3	5.5270e^{-02}	6.6394 e^{-03}	6.8321e^{-05}
0.4	9.5072e^{-02}	1.5553 e^{-02}	2.9425e^{-04}
0.5	1.4330e^{-01}	2.9898 e^{-02}	3.1836e^{-04}
0.6	1.9840e^{-01}	5.0714 e^{-02}	3.9837e^{-04}
0.7	2.5862e^{-01}	7.8873 e^{-02}	5.3651e^{-04}
0.8	3.2205e^{-01}	1.1507 e^{-01}	8.6212e^{-04}
0.9	3.8656e^{-01}	1.5980 e^{-01}	1.6454e^{-04}
1.0	4.4980e^{-01}	2.1334 e^{-01}	3.1826e^{-03}

5 CONCLUSION

Table 1, 2, and 3 show the numerical solutions obtained in term of errors for the integro-differential equations solved through Chebyshev Polynomial basis function. We observed from the examples solved that for varying degrees of N that as N increases the results obtained converge closely to the exact solution, this is clearly portrayed in the tables of results given above. In practical situation, accuracy of a numerical method is determined by the consistency of successive approximations and the rate of decrease of the coefficients in the various series. These are clearly noticed in this method. We thus conclude that the numerical method is feasible and effective for the class of problems considered.

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