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Differential Transform Method for the Heat Equation

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ABSTRACT
Differential transform method (DTM) is the multi-step form of numerical method to solve differential equations. The method was first introduced by Zhou (1986) and is used for giving an exact value for nth derivative to solve analytical functions having known or unknown boundary conditions. This method offers analytical solution in polynomial terms. The method is rather different than the traditional form of Taylor series method offering computations with required derivatives in data function (Tabatabaei et al., 2011). The problems and approximate solutions are elaborated here to give a better insight of the DTM. The method is clearly useful in solving linear as well as non-linear value problems for the analysis of electric circuits. Thus, it possesses the potential to reduce the size of computational work.

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I. DIFFERENTIAL TRANSFORM METHOD

For the method of reaction equation containing non-local boundary condition, it is quite crucial for several methods in treating the equations (Abbasbandy et al, 2007). For resolving problems, the numerical solutions are proposed of immense help (Selvi& Ramesh, 2017). The numerical treatment thus helps in offering the reaction diffusion equation class with non-local boundary condition through the use of DTM.

In order to solve the heat equation, the variable is of the form

$$u_t = f(x,y,z)u_{xx} + g(x,y,z)u_{yy} + h(x,y,z)u_{zz}, \quad \text{(Tabatabaei et al., 2011)}$$

$$0 < x < a, 0 < y < b, 0 < z < c, t > 0$$

Here, Neumann boundary conditions are as below

$$u_x(0,y,z,t) = f_1(y,z,t), \qquad u_x(a,y,z,t) = f_2(y,z,t),$$
$$u_y(x,0,z,t) = g_1(x,z,t), \qquad u_y(x,b,z,t) = g_2(x,z,t),$$
$$u_z(x,y,0,t) = h_1(x,y,t), \qquad u_z(x,y,c,t) = h_2(x,y,t),$$

Furthermore, an initial condition is as below

$$u(x,y,z,0) = \phi(x,y,z)$$

On taking into account the below problem of heat equation

$$u_t = u_{xx} + u^m (1)$$

$$u(x,0) = f(x) \qquad (2)$$

Here, the function u(x, t) is analytical as well as differential in continuous manner as compared to the space 'x' and time 't' for the interest. Here

$$u_k(x) = \frac{1}{k!}\frac{\partial^k}{\partial t^k}u(x,t)]$$

For t-dimensional spectrum, the $a_k(x)$ transforms towards the below

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x)t^k$$

The above two equations (1 and 2) for a(x,t) could be further elaborated as below

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{\partial^k}{\partial t^k}u(x,t)]t^k$$

The use of DTM for obtaining solution for the two equations is through placing differential transform for both these sides, below is the equation

$$(k+1)u_{k+1}(x) = [\frac{\partial^k}{\partial t^k}u(x)\sum_{k=0}^{\infty} f_k(x)]$$

$u(x) = f(x)$
Here,
$f_k(x)$ is

$$f_0(x) = u_0^m(x)$$
$$f_1(x) = mu_0^{m-1}(x)u_1(x)$$
$$f_2(x) = \frac{1}{2}m(m-1)u_0^{m-2}(x)u_1^2(x) + mu_0^{m-1}(x)u_2(x)$$
$$f_3(x) = \frac{1}{6}m(m-1)(m-2)u_0^{m-3}(x)u_1^3(x) + m(m-1)u_0^{m-2}(x)u_2(x)u_1(x) + mu_0^{m-1}(x)u_3(x)$$

Here, we get

$$u(x,t) = u_0(x) + u_1(x)t + u_2(x)t^2 + u_3(x)t^3 + \cdots + u_k(x)t^k$$

Thus, the below equation is obtained from the above

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$$

This method is helpful in obtaining reliable outcomes for various science scenarios. The method however consists of few shortcomings too. The use of this method allows in obtaining series of solutions, which may not reveal an actual behaviour associated with the problem (Selvi and Ramesh, 2017). It only offers an approximation for the solution contained in small region. However, other methods could offer successful results for linear as well as non-linear value problems. The technique is thus quite efficient for analytical solution associated with differential and integral equations.

The studies have clearly shown various kinds of applications of differential equation for solving the Eigen value problems as well as partial differential equations. Their usage in solving one dimensional problem was through constructing Eigen value with normalised function for differential equations of 2nd, and 4th order (Tabatabaei et al, 2011). Another one is based on the 2 dimensional differential transformations for partial differential equation of 1st and 2nd order having constant coefficients (Ali and Raslan, 2009). In such scenarios, the differential equation is derived followed by obtaining results to compare against other forms of analytical methods.

II. APPLICATION

In the case of one-dimensional model of heat equation

$$u_t = \frac{1}{2}x^2 u_{xx} \qquad (3)$$

$$0 < t < 1, \quad t > 0$$

Here, boundary conditions are as below

$$u(0,t) = 0, \quad (1,t) = e^t \qquad (4)$$

With initial conditions as

$$u(x,0) = x^2 \qquad (5)$$

The use of DTM in equation 3 could help in obtaining below

$$U(k,h+1) = \frac{1}{2(h+1)}\sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r-2,h-s)(k-r+2)(k-r+1)U(k-r+2,s), \qquad (6)$$

By making use of DTM for their boundary conditions in equation 4, below is the result

$$U(0,h) = 0, \quad U(1,h) = \frac{1}{h!}$$

On the basis of equation 5, it can be written

$$U(k,0) = \delta(k-2) = \begin{cases} 1 & k = 2 \\ 0 & k \neq 2 \end{cases}$$

The use of k, h into the equation 6 could help in finding equation as below

$$U(k,h+1) = 0, \quad k = 0,1,2,3,\ldots \quad, h = 0,1,2,3,\ldots \quad, U(2,h) = \frac{1}{h!}$$

The use of inverse transformation could thus help in obtaining the solution as

$$u(x,t) = x^2 + x^2 t + \frac{1}{2!}x^2t^2 + \frac{1}{3!}x^2t^3 + \frac{1}{4!}x^2t^4 + \ldots \qquad (7)$$
$$u(x,t) = x^2 e^t$$

In another case of non-linear heat equation,

$$u_t = u_{xx} - 2u^3$$

$$u(x,0) = \frac{1+2x}{x^2+x+1}$$

Here

$$F_0(x) = \left(\frac{1+2x}{x^2+x+1}\right)^3 \quad \text{and} \quad u_1(x) = \frac{-6(1+2x)}{(x^2+x+1)^2}$$

$$F_1(x) = \frac{-18(1+2x)^3}{(x^2+x+1)^2} \quad \text{and} \quad u_2(x) = \frac{36(1+2x)}{(x^2+x+1)^3}$$

$$F_2(x) = \frac{216(1+2x)^3}{(x^2+x+1)^5} \quad \text{and} \quad u_3(x) = \frac{-216(1+2x)}{(x^2+x+1)^4}$$

On using equation 7, below is the result

$$u(x,t) = \frac{1+2x}{x^2+x+1} - \frac{6(1+2x)}{(x^2+x+1)^2}t + \frac{36(1+2x)}{(x^2+x+1)^3}t^2 - \frac{216(1+2x)}{(x^2+x+1)^4}t^3 + \cdots$$

The DTM is formed as per the Taylor series expansion (Tabatabaei et al, 2011). This differential transform method helps in producing the solutions of varying forms. The differential equations have gained significant attention in the recent period. With the increase in complexities of solving numerous applications in engineering and physics, it was found as an ideal solution. The fractional differential equation offers solution of approximation, which is free of exact solution (Bhadauria et al, 2011). Thus, instead of basing on the exact solution, numerical technique adopts an approximation method. This is thus found of immense help in reaching the solution. Differential equation for heat equation thus forms the wide array of solutions to promote the results being obtained of reliable standard. The DTM is rather easier as compared to traditional form of Taylor series method that demands the use of symbolic computations for required derivatives in data function (Bhadauria et al, 2011). The Taylor series method is rather an expensive form of method with higher order. Therefore, DTM is applicable in diverse scenarios of heat wave for one, two as well as three-dimensional heat model.

III. CONCLUSION

The study here clearly elaborates the use of differential transform method for solving heat equations. The use of methods could help in resolving such equations. It is thus learnt from the study that DTM is an ideal method for usage in producing reliable findings. The technique helps in reducing computational complexities for the case of recurrence equations. The DTM is thus an ideal method and the solution can be obtained of close value. The findings can be further improved through working on more scenarios to obtain the solution. This could help in optimally utilising this easy form of DTM. The method is thus useful to solve the partial differential equations having variable coefficients.

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