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An Extended Generalized Estimator of Ratio (Product) Of Parameters in Double Sampling

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ABSTRACT
An extended class of generalized double sampling estimator using information on an auxiliary variable is proposed for the estimation of ratio (product) of population parameters. Bias and mean square error are found, and the properties of the generalized estimator are studied. Classes of estimators depending on optimum and estimated optimum values in the sense of minimum mean square error are also investigated.
KEY-WORDS: Bias and mean square error, Efficiency, Optimum and estimated optimum estimators.

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I. INTRODUCTION

Let a first phase large simple random sample of size n' be drawn from a population of size N and only the auxiliary variable x_2 be observed on this first phase sample, and further, let both the study variables (y, x_1) and the auxiliary variable x_2 be observed on the second phase simple random sample of size n from the first phase sample of size n'. For population values $\{ Y_i \ ; \ i = 1,2,....., N \}$ on y, $\{ X_{1i} \ ; \ i = 1,2,....., N \}$ on x_1 and $\{ X_{2i} \ ; \ i = 1,2,....., N \}$ on x_2, let the population means of y, x_1 and x_2 are respectively

$$\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i ,$$

$$\bar{X}_1 = \frac{1}{N}\sum_{i=1}^{N} X_{1i} ,$$

and $$\bar{X}_2 = \frac{1}{N}\sum_{i=1}^{N} X_{2i} .$$

The population ratio (R) of the population means $\bar{Y}$ and $\bar{X}_1$ and their product (P) are $R = \frac{\bar{Y}}{\bar{X}_1}$ and $P = \bar{Y}\,\bar{X}_1$.

Further, let ρ_{01}, ρ_{02} and ρ_{12} be the correlation coefficients between (y, x_1), (y, x_2) and (x_1, x_2) respectively, and

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$$S_y^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}\left(Y_i - \overline{Y}\right)^2 ,$$

$$S_{x_1}^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}\left(X_{1i} - \overline{X}_1\right)^2 ,$$

$$S_{x_2}^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}\left(X_{2i} - \overline{X}_2\right)^2 ,$$

$$C_0 = \frac{S_y}{\overline{Y}} , \quad C_1 = \frac{S_{x_1}}{\overline{X}_1} , \quad C_2 = \frac{S_{x_2}}{\overline{X}_2} .$$

Also, for first phase sample values $\left\{ x'_{2i} \ ; \ i = 1,2,..., n' \right\}$ on x_2, second phase sample values $\left\{ y_i \ ; \ i = 1,2,..., n \right\}$ on y, $\left\{ x_{1i} \ ; \ i = 1,2,..., n \right\}$ on x_1 and $\left\{ x_{2i} \ ; \ i = 1,2,..., n \right\}$ on x_2, let

$$\overline{x}'_2 = \frac{1}{n'}\sum_{i=1}^{n'} x'_{2i} , \ \overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i , \ \overline{x}_1 = \frac{1}{n}\sum_{i=1}^{n} x_{1i} , \ \overline{x}_2 = \frac{1}{n}\sum_{i=1}^{n} x_{2i} ,$$

$$s'^2_{x_2} = \frac{1}{(n'-1)}\sum_{i=1}^{n'}\left(x'_{2i} - \overline{x}'_2\right)^2 \quad \text{and} \quad s^2_{x_2} = \frac{1}{(n-1)}\sum_{i=1}^{n}\left(x_{2i} - \overline{x}_2\right)^2 .$$

Using auxiliary information on x_2, the double sampling estimators of Singh (1965) for the ratio $R = \frac{\overline{Y}}{\overline{X}_1}$ and the product $P = \overline{Y}\,\overline{X}_1$ are respectively

$$\hat{R}_{1d} = \frac{\overline{y}}{\overline{x}_1}\frac{\overline{x}_2}{\overline{x}'_2} = \hat{R}\left(\frac{\overline{x}_2}{\overline{x}'_2}\right) ,$$

$$\hat{R}_{2d} = \frac{\overline{y}}{\overline{x}_1}\frac{\overline{x}'_2}{\overline{x}_2} = \hat{R}\left(\frac{\overline{x}'_2}{\overline{x}_2}\right) ,$$

and

$$\hat{P}_{1d} = \overline{y}\ \overline{x}_1 \frac{\overline{x}_2}{\overline{x}'_2} = \hat{P}\left(\frac{\overline{x}_2}{\overline{x}'_2}\right) ,$$

$$\hat{P}_{2d} = \overline{y}\ \overline{x}_1 \frac{\overline{x}'_2}{\overline{x}_2} = \hat{P}\left(\frac{\overline{x}'_2}{\overline{x}_2}\right)$$

where $\hat{R} = \frac{\overline{y}}{\overline{x}_1}$ and $\hat{P} = \overline{y}\ \overline{x}_1$.

Further, Singh, R. Karan, Rizvi, S.A.H and Rizvi S.A.M. (2012) proposed the following generalized double sampling estimators:

$$\hat{R}_{gd} = g\left(\hat{R}, \overline{x}_2, \overline{x}'_2\right) \tag{A}$$

and $$\hat{P}_{gd} = h\left(\hat{P}, \overline{x}_2, \overline{x}'_2\right) \tag{B}$$

Also, g_1 and h_1 are the first order partial derivatives. The optimum value of g_1 minimizing $MSE\ (\hat{R}_{gd})$ is

$$g_{1*} = -\left(\frac{R}{\overline{X}_2}\right)C \quad \text{(C)}$$

and the minimum mean square error is given by

$$MSE\ (\hat{R}_{gd})_{min} = MSE\ (\hat{R}) - \left(\frac{1}{n} - \frac{1}{n'}\right)R^2 C_2^{\,2} C^2\ . \quad \text{(D)}$$

Proceeding on same lines, the optimum value of h_1 minimizing $MSE\ (\hat{P}_{gd})$ is

$$h_{1*} = -\left(\frac{P}{\overline{X}_2}\right)C^* \quad \text{(E)}$$

for which the minimum mean square error to the first degree of approximation, is

$$MSE\ (\hat{P}_{gd})_{min} = MSE\ (\hat{P}) - \left(\frac{1}{n} - \frac{1}{n'}\right)P^2 C_2^{\,2} C^{*2}\ , \quad \text{(F)}$$

The optimum value $g_{1*} = -\left(\frac{R}{\overline{X}_2}\right)C$ in (C) may not be known always in practice, hence the alternative is to replace the parameters involved in the optimum value by their unbiased or consistent estimators to get the estimated optimum value depending upon sample observations as

$$\hat{g}_1 = -\frac{\hat{R}}{s_{x_2}^2}\left(\frac{s_{yx_2}}{\overline{y}} - \frac{s_{x_1x_2}}{\overline{x}_1}\right)$$

$$= -\left(\frac{\hat{R}}{\overline{x}'_2}\right)\hat{C}\ . \quad \text{(G)}$$

Using the estimated optimum value in (G), the estimator depending upon estimated optimum value is given as a function

$$R^*_{d(est)} = g^{**}\left(\hat{R}, \overline{x}_2, \overline{x}'_2, \hat{C}\right) \quad \text{(H)}$$

attains the minimum mean square error given by (D).

Similarly, replacing the parameters involved in $h_{1*} = -\left(\frac{P}{\overline{X}_2}\right)C^*$ by their unbiased or consistent estimators, the estimated optimum value is

$$\hat{h}_1 = -\left(\frac{\hat{P}}{\overline{x}'_2}\right)\hat{C}^* \quad \text{(I)}$$

which gives the estimator depending upon estimated optimum value as a function

$$P^*_{d(est)} = h^{**}\left(\hat{P}, \overline{x}_2, \overline{x}'_2, \hat{C}\right) \quad \text{(J)}$$

and $MSE\ (\hat{P}^*_{d(est)})$ equals the minimum mean square error given by (F).

The Proposed Estimators

The proposed estimators for ratio (R) and product (P) are:

$$\hat{R}_g = g\left(\hat{R}, \overline{x}_2, \overline{x}'_2, s_{x_2}^2, s'^2_{x_2}\right) \quad \text{(1.1)}$$

and

$$\hat{P}_g = h\left(\hat{P}, \overline{x}_2, \overline{x}_2', s_{x_2}^2, s_{x_2}'^2\right) \quad (1.2)$$

For estimating R, the proposed generalized double sampling estimator is

$$\hat{R}_g = g\left(\hat{R}, \overline{x}_2, \overline{x}_2', s_{x_2}^2, s_{x_2}'^2\right)$$
$$= g(t)$$

where $t = \left(\hat{R}, \overline{x}_2, \overline{x}_2', s_{x_2}^2, s_{x_2}'^2\right)$ and $g(t)$ satisfying the validity conditions of Taylor's series expansion is a bounded function of t such that at the point

$T = \left(R, \overline{X}_2, \overline{X}_2, S_{x_2}^2, S_{x_2}^2\right)$,

(i) $g(t = T) = R$ (1.3)

and, for first order partial derivatives

$$g_0 = \left.\frac{\partial g}{\partial \hat{R}}\right]_T, g_1 = \left.\frac{\partial g}{\partial \overline{x}_2}\right]_T, g_2 = \left.\frac{\partial g}{\partial \overline{x}_2'}\right]_T, g_3 = \left.\frac{\partial g}{\partial s_{x_2}^2}\right]_T, g_4 = \left.\frac{\partial g}{\partial s_{x_2}'^2}\right]_T$$

of $g(t)$ with respect to $\hat{R}, \overline{x}_2, \overline{x}_2', s_{x_2}^2, s_{x_2}'^2$ respectively at the point $t = T$ and second order partial derivatives

$$g_{00} = \left.\frac{\partial^2 g}{\partial \hat{R}^2}\right]_T, g_{01} = \left.\frac{\partial^2 g}{\partial \hat{R}\partial \overline{x}_2}\right]_T,$$

$$g_{02} = \left.\frac{\partial^2 g}{\partial \hat{R}\partial \overline{x}_2'}\right]_T, g_{03} = \left.\frac{\partial^2 g}{\partial \hat{R}\partial s_{x_2}^2}\right]_T, g_{04} = \left.\frac{\partial^2 g}{\partial \hat{R}\partial s_{x_2}'^2}\right]_T$$

of $g(t)$ with respect to $\hat{R}, \left(\hat{R}, \overline{x}_2\right), \left(\hat{R}, \overline{x}_2'\right), \left(\hat{R}, s_{x_2}^2\right), \left(\hat{R}, s_{x_2}'^2\right)$ respectively at the point $t = T$,

(ii) $g_0 = 1$ (1.4)

(iii) $g_1 = -g_2$ (1.5)

(iv) $g_3 = -g_4$ (1.6)

(v) $g_{00} = 0$ (1.7)

(vi) $g_{01} = -g_{02}$ (1.8)

(vii) $g_{03} = -g_{04}$. (1.9)

Some particular members belonging to the generalized double sampling estimator $\hat{R}_g$ are

(i) $\hat{R}_1 = \hat{R}\left(\dfrac{\overline{x}_2}{\overline{x}_2'}\right)\left(\dfrac{s_{x_2}^2}{s_{x_2}'^2}\right)$,

(ii) $\hat{R}_2 = \hat{R}\left(\dfrac{\overline{x}_2'}{\overline{x}_2}\right)\left(\dfrac{s_{x_2}'^2}{s_{x_2}^2}\right)$,

(iii) $\hat{R}_3 = \hat{R} + k_1\left(\overline{x}_2 - \overline{x}_2'\right) + k_2\left(s_{x_2}^2 - s_{x_2}'^2\right)$, and (iv) $\hat{R}_4 = \hat{R}\left(\frac{\bar{x}_2}{\bar{x}'_2}\right)^{k_1}\left(\frac{s^2_{x_2}}{s'^2_{x_2}}\right)^{k_2}$

where k_1 and k_2 are the characterizing scalars to be chosen suitably.

It may be easily verified that conditions (1.3) to (1.9) are satisfied for all the estimators $\hat{R}_i, i = 1,2,3,4$.

For example, considering the estimator $\hat{R}_5$, the value of $\hat{R}_5$ at the point $t = \left(R, \bar{X}_2, \bar{X}_2, S^2_{x_2}, S^2_{x_2}\right)$ is

R and $g_0 = \left.\frac{\partial \hat{R}_5}{\partial \hat{R}}\right]_T = (1-k) + k\left.\left(\frac{\bar{x}_2}{\bar{x}'_2}\right)\left(\frac{s^2_{x_2}}{s'^2_{x_2}}\right)\right]_T = (1-k) + k = 1$ satisfying (1.3) and (1.4);

and further

$$g_1 = \left.\frac{\partial \hat{R}_5}{\partial \bar{x}_2}\right]_T = k\hat{R}\left.\left(\frac{1}{\bar{x}'_2}\right)\left(\frac{s^2_{x_2}}{s'^2_{x_2}}\right)\right]_T = \frac{kR}{\bar{X}_2},$$

$$g_2 = \left.\frac{\partial \hat{R}_5}{\partial \bar{x}'_2}\right]_T = k\hat{R}\bar{x}_2\left.\left(-\frac{1}{\bar{x}'^2_2}\right)\left(\frac{s^2_{x_2}}{s'^2_{x_2}}\right)\right]_T = -\frac{kR}{\bar{X}_2},$$

$$g_3 = \left.\frac{\partial \hat{R}_5}{\partial s^2_{x_2}}\right]_T = k\hat{R}\left.\left(\frac{\bar{x}_2}{\bar{x}'_2}\right)\left(\frac{1}{s'^2_{x_2}}\right)\right]_T = \frac{kR}{S^2_{x_2}},$$

$$g_4 = \left.\frac{\partial \hat{R}_5}{\partial s'^2_{x_2}}\right]_T = k\hat{R}\left.\left(\frac{\bar{x}_2}{\bar{x}'_2}\right)\left\{-\frac{s^2_{x_2}}{\left(s'^2_{x_2}\right)^2}\right\}\right]_T = -\frac{kR}{S^2_{x_2}},$$

$$g_{00} = \left.\frac{\partial^2 \hat{R}_5}{\partial \hat{R}^2}\right]_T = 0,$$

$$g_{01} = \left.\frac{\partial^2 \hat{R}_5}{\partial \hat{R}\,\partial \bar{x}_2}\right]_T = k\left.\left(\frac{1}{\bar{x}'_2}\right)\left(\frac{s^2_{x_2}}{s'^2_{x_2}}\right)\right]_T = \frac{k}{\bar{X}_2},$$

$$g_{02} = \left.\frac{\partial^2 \hat{R}_5}{\partial \hat{R}\,\partial \bar{x}'_2}\right]_T = k\left.\left(-\frac{\bar{x}_2}{\bar{x}'^2_2}\right)\left(\frac{s^2_{x_2}}{s'^2_{x_2}}\right)\right]_T = -\frac{k}{\bar{X}_2},$$

$$g_{03} = \left.\frac{\partial^2 \hat{R}_5}{\partial \hat{R}\,\partial s^2_{x_2}}\right]_T = k\left.\left(\frac{\bar{x}_2}{\bar{x}'_2}\right)\left(\frac{1}{s'^2_{x_2}}\right)\right]_T = \frac{k}{S^2_{x_2}},$$

$$g_{04} = \left.\frac{\partial^2 \hat{R}_5}{\partial \hat{R}\,\partial s'^2_{x_2}}\right]_T = k\left.\left(\frac{\bar{x}_2}{\bar{x}'_2}\right)\left\{-\frac{s^2_{x_2}}{\left(s'^2_{x_2}\right)^2}\right\}\right]_T = -\frac{k}{S^2_{x_2}}$$

show that the conditions (1.5) to (1.9) are also satisfied for the estimator $\hat{R}_5$. The regularity conditions from (1.3) to (1.9) may easily be verified for the estimators $\hat{R}_1, \hat{R}_2, \hat{R}_3$ and $\hat{R}_4$ also, and some more double sampling estimators in the literature [see Cochran (1977), Sukhatme et.al. (1984) and Murthy (1967) for further details].

II. BIAS AND MEAN SQUARE ERROR

Let

$$\mu_{rsl} = \frac{1}{N}\sum_{i=1}^{N}\left(Y_i - \overline{Y}\right)^r\left(X_{1i} - \overline{X}_1\right)^s\left(X_{2i} - \overline{X}_2\right)^l, \text{ for } r,s,l = 0,1,2,3,4;$$

$$e_0 = \frac{\overline{y} - \overline{Y}}{\overline{Y}}, \quad e_1 = \frac{\overline{x}_1 - \overline{X}_1}{\overline{X}_1}, \quad e_2 = \frac{\overline{x}_2 - \overline{X}_2}{\overline{X}_2}, \quad e_2' = \frac{\overline{x}_2' - \overline{X}_2}{\overline{X}_2},$$

$$e_3 = s_{x_2}^2 - S_{x_2}^2, \quad e_3' = s_{x_2}'^2 - S_{x_2}^2$$

so that ignoring fpc (finite population correction) for simplicity,

$$E(e_0) = E(e_1) = E(e_2) = E(e_2') = E(e_3) = E(e_3') = 0,$$

$$E(e_0^2) = \frac{\mu_{200}}{n\overline{Y}^2}, \quad E(e_1^2) = \frac{\mu_{020}}{n\overline{X}_1^2}, \quad E(e_2^2) = \frac{\mu_{002}}{n\overline{X}_2^2},$$

$$E(e_2'^2) = \frac{\mu_{002}}{n'\overline{X}_2^2}, \quad E(e_3^2) = \frac{\mu_{002}^2\left(\beta_{2x_2} - 1\right)}{n},$$

$$E(e_3'^2) = \frac{\mu_{002}^2\left(\beta_{2x_2} - 1\right)}{n'}, \quad E(e_0e_1) = \frac{\mu_{110}}{n\overline{Y}\,\overline{X}_1}, \quad E(e_0e_2) = \frac{\mu_{101}}{n\overline{Y}\,\overline{X}_2}$$

$$, \quad E(e_1e_2) = \frac{\mu_{011}}{n\overline{X}_1\overline{X}_2}, \quad E(e_0e_2') = \frac{\mu_{101}}{n'\overline{Y}\,\overline{X}_2},$$

$$E(e_1e_2') = \frac{\mu_{011}}{n'\overline{X}_1\overline{X}_2}, \quad E(e_2e_2') = \frac{\mu_{002}}{n'\overline{X}_2^2},$$

$$E(e_1e_3) = \frac{\mu_{012}}{n\overline{X}_1}, \quad E(e_2'e_3) = \frac{\mu_{003}}{n'\overline{X}_2},$$

$$E(e_0e_3) = \frac{\mu_{102}}{n\overline{Y}},$$

$$E(e_2'e_3') = \frac{\mu_{003}}{n'\overline{X}_2}, \quad E(e_1e_3') = \frac{\mu_{012}}{n'\overline{X}_1},$$

$$E(e_0e_3') = \frac{\mu_{102}}{n'\overline{Y}}, \quad E(e_2e_3) = \frac{\mu_{003}}{n\overline{X}_2},$$

$$E(e_2e_3') = \frac{\mu_{003}}{n'\overline{X}_2} \quad \text{and} \quad E(e_3e_3') = \frac{\mu_{002}^2\left(\beta_{2x_2} - 1\right)}{n'},$$

where $\beta_{2x_2} = \dfrac{\mu_{004}}{\mu_{002}^2}$ is the coefficient of kurtosis of x_2 .

Further, it is assumed that the sample is large enough to ignore terms involving e_0 , e_1 , e_2 , e_2' , e_3 , e_3' of degree greater than two, to justify the first degree approximation [see Murthy (1967) for more details].

Expanding $\hat{R}_g = g(t)$ about the point $T = \left(R, \overline{X}_2, \overline{X}_2, S_{x_2}^2, S_{x_2}^2\right)$ in third order Taylor's series, we have

$$
\begin{aligned}
\hat{R}_g &= g(T) + \left(\hat{R} - R\right)g_0 + \left(\overline{x}_2 - \overline{X}_2\right)g_1 + \left(\overline{x}_2' - \overline{X}_2\right)g_2 + \left(s_{x_2}^2 - S_{x_2}^2\right)g_3 \\
&\quad + \left(s_{x_2}'^2 - S_{x_2}^2\right)g_4 + \frac{1}{2!}\Big\{ \left(\hat{R} - R\right)^2 g_{00} + \left(\overline{x}_2 - \overline{X}_2\right)^2 g_{11} \\
&\quad + \left(\overline{x}_2' - \overline{X}_2\right)^2 g_{22} + \left(s_{x_2}^2 - S_{x_2}^2\right)^2 g_{33} + \left(s_{x_2}'^2 - S_{x_2}^2\right)^2 g_{44} \\
&+ 2\left(\hat{R} - R\right)\left(\overline{x}_2 - \overline{X}_2\right)g_{01} + 2\left(\hat{R} - R\right)\left(\overline{x}_2' - \overline{X}_2\right)g_{02} \\
&+ 2\left(\hat{R} - R\right)\left(s_{x_2}^2 - S_{x_2}^2\right)g_{03} + 2\left(\hat{R} - R\right)\left(s_{x_2}'^2 - S_{x_2}^2\right)g_{04} \\
&+ 2\left(\overline{x}_2 - \overline{X}_2\right)\left(\overline{x}_2' - \overline{X}_2\right)g_{12} + 2\left(\overline{x}_2 - \overline{X}_2\right)\left(s_{x_2}^2 - S_{x_2}^2\right)g_{13} \\
&+ 2\left(\overline{x}_2 - \overline{X}_2\right)\left(s_{x_2}'^2 - S_{x_2}^2\right)g_{14} + 2\left(\overline{x}_2' - \overline{X}_2\right)\left(s_{x_2}^2 - S_{x_2}^2\right)g_{23} \\
&+ 2\left(\overline{x}_2' - \overline{X}_2\right)\left(s_{x_2}'^2 - S_{x_2}^2\right)g_{24} + 2\left(s_{x_2}^2 - S_{x_2}^2\right)\left(s_{x_2}'^2 - S_{x_2}^2\right)g_{34} \Big\} \\
&+ \frac{1}{3!}\Big\{ \left(\hat{R} - R\right)\frac{\partial}{\partial \hat{R}} + \left(\overline{x}_2 - \overline{X}_2\right)\frac{\partial}{\partial \overline{x}_2} \\
&+ \left(\overline{x}_2' - \overline{X}_2\right)\frac{\partial}{\partial \overline{x}_2'} + \left(s_{x_2}^2 - S_{x_2}^2\right)\frac{\partial}{\partial s_{x_2}^2} \\
&+ \left(s_{x_2}'^2 - S_{x_2}^2\right)\frac{\partial}{\partial s_{x_2}'^2} \Big\}^3 g\left(\hat{R}_*, \overline{x}_{2*}, \overline{x}_{2*}', s_{x_{2*}}^2, s_{x_{2*}}'^2\right)
\end{aligned}
\tag{2.1}
$$

where g_0 , g_1 , g_2 , g_3 , g_4 , g_{00} , g_{01} , g_{02} , g_{03} , g_{04} are already defined, g_{11} , g_{22} , g_{33} , g_{44} , g_{12} , g_{13} , g_{14} , g_{23} , g_{24} , g_{34} are the second order partial derivatives with respect to $\overline{x}_2, \overline{x}_2'$, $s_{x_2}^2, s_{x_2}'^2$, $(\overline{x}_2, \overline{x}_2'), (\overline{x}_2, s_{x_2}^2), (\overline{x}_2, s_{x_2}'^2), (\overline{x}_2', s_{x_2}^2)$, $(\overline{x}_2', s_{x_2}'^2), (s_{x_2}^2, s_{x_2}'^2)$ respectively,

and $\hat{R}_* = R + \theta\left(\hat{R} - R\right)$, $\overline{x}_{2*} = \overline{X}_2 + \theta\left(\overline{x}_2 - \overline{X}_2\right)$,
$\overline{x}_{2*}' = \overline{X}_2 + \theta\left(\overline{x}_2' - \overline{X}_2\right)$, $s_{x_{2*}}^2 = S_{x_2}^2 + \theta\left(s_{x_2}^2 - S_{x_2}^2\right)$,
$s_{x_{2*}}'^2 = S_{x_2}^2 + \theta\left(s_{x_2}'^2 - S_{x_2}^2\right)$ for $0 < \theta < 1$.

From regularity conditions (1.3) to (1.9), substituting $g(T)=R$, $g_0=1$, $g_1=-g_2$, $g_3=-g_4$, $g_{00}=0$, $g_{01}=-g_{02}$, and $g_{03}=-g_{04}$ in (2.1), we have

$$\begin{aligned}\hat{R}_g-R&=(\hat{R}-R)+(\bar{x}_2-\bar{X}_2)g_1-(\bar{x}_2'-\bar{X}_2)g_1+(s_{x_2}^2-S_{x_2}^2)g_3\\&-(s_{x_2}'^2-S_{x_2}^2)g_3+\frac{1}{2}\Big\{(\bar{x}_2-\bar{X}_2)^2g_{11}+(\bar{x}_2'-\bar{X}_2)^2g_{22}\\&+(s_{x_2}^2-S_{x_2}^2)^2g_{33}+(s_{x_2}'^2-S_{x_2}^2)^2g_{44}+2(\hat{R}-R)(\bar{x}_2-\bar{X}_2)g_{01}\\&-2(\hat{R}-R)(\bar{x}_2'-\bar{X}_2)g_{01}+2(\hat{R}-R)(s_{x_2}^2-S_{x_2}^2)g_{03}\\&-2(\hat{R}-R)(s_{x_2}'^2-S_{x_2}^2)g_{03}+2(\bar{x}_2-\bar{X}_2)(\bar{x}_2'-\bar{X}_2)g_{12}\\&+2(\bar{x}_2-\bar{X}_2)(s_{x_2}^2-S_{x_2}^2)g_{13}+2(\bar{x}_2-\bar{X}_2)(s_{x_2}'^2-S_{x_2}^2)g_{14}\\&+2(\bar{x}_2'-\bar{X}_2)(s_{x_2}^2-S_{x_2}^2)g_{23}+2(\bar{x}_2'-\bar{X}_2)(s_{x_2}'^2-S_{x_2}^2)g_{24}\\&+2(s_{x_2}^2-S_{x_2}^2)(s_{x_2}'^2-S_{x_2}^2)g_{34}\Big\}+\frac{1}{3!}\Big\{(\hat{R}-R)\frac{\partial}{\partial\hat{R}}\\&+(\bar{x}_2-\bar{X}_2)\frac{\partial}{\partial\bar{x}_2}+(\bar{x}_2'-\bar{X}_2)\frac{\partial}{\partial\bar{x}_2'}+(s_{x_2}^2-S_{x_2}^2)\frac{\partial}{\partial s_{x_2}^2}\\&+(s_{x_2}'^2-S_{x_2}^2)\frac{\partial}{\partial s_{x_2}'^2}\Big\}^3 g(\hat{R}_*,\bar{x}_{2*},\bar{x}_{2*}',s_{x_{2*}}^2,s_{x_{2*}}'^2).\end{aligned}\tag{2.2}$$

Noting that $\hat{R}=R(1+e_0)(1+e_1)^{-1}=R(1+e_0)(1-e_1+e_1^2-...)$, we have from (2.2)

$$\begin{aligned}\hat{R}_g-R&=R(e_0-e_1+e_1^2-e_0e_1+...)+\bar{X}_2(e_2-e_2')g_1+(e_3-e_3')g_3\\&+\frac{1}{2!}\Big\{\bar{X}_2^2e_2^2g_{11}+\bar{X}_2^2e_2'^2g_{22}+e_3^2g_{33}+e_3'^2g_{44}+2R\bar{X}_2(e_0e_2\\&-e_1e_2-e_0e_2'+e_1e_2'-...)g_{01}+2R(e_0e_3-e_1e_3-e_0e_3'\\&+e_1e_3'-...)g_{03}+2\bar{X}_2^2e_2e_2'g_{12}+2\bar{X}_2e_2e_3g_{13}+2\bar{X}_2e_2e_3'g_{14}\\&+2\bar{X}_2e_2'e_3g_{23}+2\bar{X}_2e_2'e_3'g_{24}+2e_3e_3'g_{34}\Big\}\\&+\frac{1}{3!}\Big\{(\hat{R}-R)\frac{\partial}{\partial\hat{R}}+(\bar{x}_2-\bar{X}_2)\frac{\partial}{\partial\bar{x}_2}+(\bar{x}_2'-\bar{X}_2)\frac{\partial}{\partial\bar{x}_2'}\\&+(s_{x_2}^2-S_{x_2}^2)\frac{\partial}{\partial s_{x_2}^2}+(s_{x_2}'^2-S_{x_2}^2)\frac{\partial}{\partial s_{x_2}'^2}\Big\}^3\\&g(\hat{R}_*,\bar{x}_{2*},\bar{x}_{2*}',s_{x_{2*}}^2,s_{x_{2*}}'^2).\end{aligned}\tag{2.3}$$

Taking expectation on both sides of (2.3), to the first degree of approximation

$$E\left(\hat{R}_g\right)-R=\Big[R\ E\left(e_1^{\,2}-e_0e_1\right)+\frac{1}{2}\Big\{\overline{X}_2^{\,2}\mathrm{E}\left(e_2^{\,2}\right)g_{11}+\overline{X}_2^{\,2}E\left(e_2'^{\,2}\right)g_{22}$$
$$+E\left(e_3^{\,2}\right)g_{33}+E\left(e_3'^{\,2}\right)g_{44}+2R\overline{X}_2E\left(e_0e_2-e_1e_2-e_0e_2'\right.$$
$$\left.+e_1e_2'\right)g_{01}+2R\ E\left(e_0e_3-e_1e_3-e_0e_3'+e_1e_3'\right)g_{03}$$
$$+2\overline{X}_2^{\,2}E\left(e_2e_2'\right)g_{12}+2\overline{X}_2E\left(e_2e_3\right)g_{13}+2\overline{X}_2E\left(e_2e_3'\right)g_{14}$$
$$+2\overline{X}_2E\left(e_2'e_3\right)g_{23}+2\overline{X}_2E\left(e_2'e_3'\right)g_{24}+2E\left(e_3e_3'\right)g_{34}\Big\}\Big]$$

or
$$Bias\ \left(\hat{R}_g\right)=\frac{R}{n}\left(\frac{\mu_{020}}{\overline{X}_1^{\,2}}-\frac{\mu_{110}}{\overline{Y}\ \overline{X}_1}\right)+\frac{1}{2}\Bigg\{\frac{\mu_{002}}{n}g_{11}+\frac{\mu_{002}}{n'}\left(g_{22}+2g_{12}\right)$$
$$+\frac{\mu_{002}^{2}}{n}\left(\beta_{2x_2}-1\right)g_{33}+\frac{\mu_{002}^{2}}{n'}\left(\beta_{2x_2}-1\right)g_{44}$$
$$+2\left(\frac{1}{n}-\frac{1}{n'}\right)R\overline{X}_2\left(\frac{\mu_{101}}{\overline{Y}\ \overline{X}_2}-\frac{\mu_{011}}{\overline{X}_1\overline{X}_2}\right)g_{01}$$
$$+2\left(\frac{1}{n}-\frac{1}{n'}\right)R\left(\frac{\mu_{102}}{\overline{Y}}-\frac{\mu_{012}}{\overline{X}_1}\right)g_{03}+\frac{2\overline{X}_2}{n}\left(\frac{\mu_{003}}{\overline{X}_2}\right)g_{13}$$
$$+\frac{2\overline{X}_2}{n'}\left(\frac{\mu_{003}}{\overline{X}_2}\right)\left(g_{14}+g_{23}+g_{24}\right)$$
$$+\frac{2}{n'}\mu_{002}^{2}\left(\beta_{2x_x}-1\right)g_{34}\Bigg\}\ . \qquad (2.4)$$

Squaring both sides of (2.3) and taking expectation, $MSE\ \left(\hat{R}_g\right)=E\left(\hat{R}_g-R\right)^2$ to the first degree of approximation, is

$$MSE\ \left(\hat{R}_g\right)=E\Big[R^2\ \left(e_0-e_1\right)^2+\overline{X}_2^{\,2}\left(e_2-e_2'\right)^2g_1^{\,2}+\left(e_3-e_3'\right)^2g_3^{\,2}$$
$$+2R\overline{X}_2\left(e_0-e_1\right)\left(e_2-e_2'\right)g_1+2R\left(e_0-e_1\right)\left(e_3-e_3'\right)g_3$$
$$+2\overline{X}_2\left(e_2-e_2'\right)\left(e_3-e_3'\right)g_1g_3\Big]$$
$$=\frac{1}{n}R^2\left(\frac{\mu_{200}}{\overline{Y}^{2}}-\frac{2\mu_{110}}{\overline{Y}\ \overline{X}_1}+\frac{\mu_{020}}{\overline{X}_1^{\,2}}\right)+\left(\frac{1}{n}-\frac{1}{n'}\right)\overline{X}_2^{\,2}\left(\frac{\mu_{002}}{\overline{X}_2^{\,2}}\right)g_1^{\,2}$$
$$+\left(\frac{1}{n}-\frac{1}{n'}\right)\mu_{002}^{2}\left(\beta_{2x_2}-1\right)g_3^{\,2}+2R\overline{X}_2\left(\frac{1}{n}-\frac{1}{n'}\right)\left(\frac{\mu_{101}}{\overline{Y}\ \overline{X}_2}-\frac{\mu_{011}}{\overline{X}_1\overline{X}_2}\right)g_1$$
$$+2R\left(\frac{1}{n}-\frac{1}{n'}\right)\left(\frac{\mu_{102}}{\overline{Y}}-\frac{\mu_{012}}{\overline{X}_1}\right)g_3+2\overline{X}_2\left(\frac{1}{n}-\frac{1}{n'}\right)\left(\frac{\mu_{003}}{\overline{X}_2}\right)g_1g_3$$

$$= \frac{1}{n} R^2 \left(C_y^2 - 2\rho_{01} C_0 C_1 + C_1^2\right) + \left(\frac{1}{n} - \frac{1}{n'}\right)\left\{\overline{X}_2^2 C_2^2 g_1^2 + \mu_{002}^2 \left(\beta_{2x_2} - 1\right) g_3^2 + 2R\overline{X}_2 \left(\rho_{02} C_0 C_2 - \rho_{12} C_1 C_2\right) g_1 + 2R\mu_{002}\left(\lambda_{02} - \lambda_{12}\right) g_3 + 2\mu_{003} g_1 g_3\right\}$$

$$= MSE\ (\hat{R}) + \left(\frac{1}{n} - \frac{1}{n'}\right)\overline{X}_2^2 C_2^2 \left\{ g_1^2 + 2\left(\frac{R}{\overline{X}_2}\right) C g_1 \right\} + \left(\frac{1}{n} - \frac{1}{n'}\right)\left\{ \mu_{002}^2 \left(\beta_{2x_2} - 1\right) g_3^2 + 2R\mu_{002}\left(\lambda_{02} - \lambda_{12}\right) g_3 + 2\mu_{003} g_1 g_3 \right\} \tag{2.5}$$

where $C_0^2 = \frac{\mu_{200}}{\overline{Y}^2}$, $C_1^2 = \frac{\mu_{020}}{\overline{X}_1^2}$, $C_2^2 = \frac{\mu_{002}}{\overline{X}_2^2}$,

$$C = \rho_{02}\left(\frac{C_0}{C_2}\right) - \rho_{12}\left(\frac{C_1}{C_2}\right) = \frac{1}{\overline{X}_2 C_2^2}\left(\frac{\mu_{101}}{\overline{Y}} - \frac{\mu_{011}}{\overline{X}_1}\right),$$

$\mu_{110} = \rho_{01}\sqrt{\mu_{200}}\sqrt{\mu_{020}}$, $\mu_{101} = \rho_{02}\sqrt{\mu_{200}}\sqrt{\mu_{002}}$, $\mu_{011} = \rho_{12}\sqrt{\mu_{020}}\sqrt{\mu_{002}}$,

ρ_{01}, ρ_{02}, ρ_{12} are the correlation coefficients between $(y, x_1), (y, x_2), (x_1, x_2)$ respectively,

and $\lambda_{02} = \frac{\mu_{102}}{\overline{Y}\mu_{002}}$, $\lambda_{12} = \frac{\mu_{012}}{\overline{X}_1 \mu_{002}}$.

III. OPTIMUM AND ESTIMATED OPTIMUM VALUES

From (2.5), the optimum values of g_1 and g_3 minimizing $MSE\ (\hat{R}_g)$ are

$$D_1 = \frac{R\left\{ \mu_{003}\left(\lambda_{02} - \lambda_{12}\right) - \overline{X}_2 C_2^2 C \mu_{002}\left(\beta_{2x_2} - 1\right)\right\}}{\left(\beta_{2x_2} - \beta_{1x_2} - 1\right)\mu_{002}^2} = RB_1 \tag{3.1}$$

and

$$D_2 = \frac{R\left\{\overline{X}_2 C_2^2 C \mu_{003} - C\mu_{002}^2\left(\lambda_{02} - \lambda_{12}\right)\right\}}{\left(\beta_{2x_2} - \beta_{1x_2} - 1\right)\mu_{002}^3} = RB_2 \tag{3.2}$$

where $B_1 = \frac{\left\{ \mu_{003}\left(\lambda_{02} - \lambda_{12}\right) - \overline{X}_2 C_2^2 C \mu_{002}\left(\beta_{2x_2} - 1\right)\right\}}{\left(\beta_{2x_2} - \beta_{1x_2} - 1\right)\mu_{002}^2}$

and $$B_2 = \frac{\left\{\overline{X}_2 C_2^{\,2} C \mu_{003} - C \mu_{002}^2 (\lambda_{02} - \lambda_{12})\right\}}{\left(\beta_{2x_2} - \beta_{1x_2} - 1\right)\mu_{002}^3}.$$

The minimum mean square error of $MSE\left(\hat{R}_g\right)$ for the optimum values D_1 and D_2 given in (3.1) and (3.2) is

$$MSE(\hat{R}_g)_{min} = MSE(\hat{R}) - \left(\frac{1}{n} - \frac{1}{n'}\right)\left[R^2 C_2^{\,2} C^2 + \frac{R^2\left\{(\lambda_{02} - \lambda_{12}) - C_2 C r_{1x_2}\right\}^2}{\left(\beta_{2x_2} - \beta_{1x_2} - 1\right)}\right] \tag{3.3}$$

where $r_{1x_2} = \frac{\mu_{003}}{(\mu_{002})^{3/2}}$ and $\beta_{1x_2} = r_{1x_2}^2$.

The optimum values D_1 and D_2 in (3.1) and (3.2) may not be known in practice, hence the alternative is to replace the parameters involved therein by their unbiased or consistent estimators which result in the estimated optimum values $\hat{D}_1$ and $\hat{D}_2$ given by

$$\hat{D}_1 = \frac{\hat{R}\left\{\hat{\mu}_{003}\left(\hat{\lambda}_{02} - \hat{\lambda}_{12}\right) - \overline{x}_2 \hat{C}_2^{\,2} \hat{C} \hat{\mu}_{002}\left(\hat{\beta}_{2x_2} - 1\right)\right\}}{\left(\hat{\beta}_{2x_2} - \hat{\beta}_{1x_2} - 1\right)\hat{\mu}_{002}^2} = \hat{R}\hat{B}_1 \tag{3.4}$$

$$\hat{D}_2 = \frac{\hat{R}\left\{\overline{x}_2 \hat{C}_2^{\,2} \hat{C} \hat{\mu}_{003} - \hat{\mu}_{002}^2\left(\hat{\lambda}_{02} - \hat{\lambda}_{12}\right)\right\}}{\left(\hat{\beta}_{2x_2} - \hat{\beta}_{1x_2} - 1\right)\hat{\mu}_{002}^3} = \hat{R}\hat{B}_2 \tag{3.5}$$

where, defining $m_{rsl} = \frac{1}{(n-1)}\sum_{i=1}^{n}(y_i - \overline{y})^r (x_{1i} - \overline{x}_1)^s (x_{2i} - \overline{x}_2)^l$, we have

$$\hat{\mu}_{003} = m_{003},\ \hat{\mu}_{002} = m_{002},\ \hat{\lambda}_{02} = \frac{m_{102}}{\overline{y} m_{002}},\ \hat{\lambda}_{12} = \frac{m_{012}}{\overline{x}_1 m_{002}},\ \hat{C}_2^{\,2} = \frac{m_{002}}{\overline{x}_2^{\,2}},$$

$$\hat{C} = \frac{1}{\overline{x}_2 \hat{C}_2^{\,2}}\left(\frac{m_{101}}{\overline{y}} - \frac{m_{011}}{\overline{x}_1}\right),\ \hat{\beta}_{2x_2} = \frac{m_{004}}{m_{002}^2},\ \hat{\beta}_{1x_2} = \frac{m_{003}^2}{m_{002}^3},$$

$$\hat{B}_1 = \frac{\left\{\hat{\mu}_{003}\left(\hat{\lambda}_{02} - \hat{\lambda}_{12}\right) - \overline{x}_2 \hat{C}_2^{\,2} \hat{C} \hat{\mu}_{002}\left(\hat{\beta}_{2x_2} - 1\right)\right\}}{\left(\hat{\beta}_{2x_2} - \hat{\beta}_{1x_2} - 1\right)\hat{\mu}_{002}^2}$$

and $$\hat{B}_2 = \frac{\left\{\overline{x}_2 \hat{C}_2^{\,2} \hat{C} \hat{\mu}_{003} - \hat{\mu}_{002}^2\left(\hat{\lambda}_{02} - \hat{\lambda}_{12}\right)\right\}}{\left(\hat{\beta}_{2x_2} - \hat{\beta}_{1x_2} - 1\right)\hat{\mu}_{002}^3}.$$

The generalized double sampling estimator $\hat{R}_g$ attains the minimum mean square error in (3.3) if the conditions from (1.3) to (1.9), (3.1) and (3.2) are satisfied for the estimator $\hat{R}_g$.

This means that the function $\hat{R}_g = g\left(\hat{R}, \bar{x}_2, \bar{x}_2', s_{x_2}^2, s_{x_2}'^2\right)$ as an estimator of R should not involve only $\left(\hat{R}, \bar{x}_2, \bar{x}_2', s_{x_2}^2, s_{x_2}'^2\right)$ but also D_1 and D_2 for the conditions (3.1) and (3.2) to be satisfied. Thus, we get the resulting estimator as a function $g\left(\hat{R}, \bar{x}_2, \bar{x}_2', s_{x_2}^2, s_{x_2}'^2, D_1, D_2\right)$ satisfying the conditions from (1.3) to (1.9) along with the conditions (3.1) and (3.2) to attain the minimum mean square error in (3.3). Replacing unknowns D_1 and D_2 in $g\left(\hat{R}, \bar{x}_2, \bar{x}_2', s_{x_2}^2, s_{x_2}'^2, D_1, D_2\right)$, we get the estimator as a function $\hat{R}_g^* = g\left(\hat{R}, \bar{x}_2, \bar{x}_2', s_{x_2}^2, s_{x_2}'^2, \hat{D}_1, \hat{D}_2\right)$ or equivalently the estimator as the function $g^*\left(\hat{R}, \bar{x}_2, \bar{x}_2', s_{x_2}^2, s_{x_2}'^2, \hat{B}_1, \hat{B}_2\right)$ depending upon estimated optimum values. Now expanding $g^*\left(\hat{R}, \bar{x}_2, \bar{x}_2', s_{x_2}^2, s_{x_2}'^2, \hat{B}_1, \hat{B}_2\right)$ about the point $T^* = \left(R, \bar{X}_2, \bar{X}_2, S_{x_2}^2, S_{x_2}^2, B_1, B_2\right)$ in Taylor's series, we have

$$\begin{aligned} g^*\left(\hat{R}, \bar{x}_2, \bar{x}_2', s_{x_2}^2, s_{x_2}'^2, \hat{B}_1, \hat{B}_2\right) = {} & g^*\left(T^*\right) + \left(\hat{R} - R\right)\left.\frac{\partial g^*}{\partial \hat{R}}\right|_{T^*} + \left(\bar{x}_2 - \bar{X}_2\right)g_1^* \\ & + \left(\bar{x}_2' - \bar{X}_2\right)g_2^* + \left(s_{x_2}^2 - S_{x_2}^2\right)g_3^* \\ & + \left(s_{x_2}'^2 - S_{x_2}^2\right)g_4^* + \left(\hat{B}_1 - B_1\right)g_5^* \\ & + \left(\hat{B}_2 - B_2\right)g_6^* + \ldots \end{aligned} \tag{3.6}$$

where $g^*\left(T^*\right) = R$, $g_1^* = \left.\frac{\partial g^*}{\partial \bar{x}_2}\right|_{T^*} = 1$, $g_2^* = \left.\frac{\partial g^*}{\partial \bar{x}_2'}\right|_{T^*}$, $g_3^* = \left.\frac{\partial g^*}{\partial s_{x_2}^2}\right|_{T^*}$,

$g_4^* = \left.\frac{\partial g^*}{\partial s_{x_2}'^2}\right|_{T^*}$, $g_5^* = \left.\frac{\partial g^*}{\partial \hat{B}_1}\right|_{T^*}$ and $g_6^* = \left.\frac{\partial g^*}{\partial \hat{B}_2}\right|_{T^*}$

or

$$\begin{aligned} g^*\left(\hat{R}, \bar{x}_2, \bar{x}_2', s_{x_2}^2, s_{x_2}'^2, \hat{B}_1, \hat{B}_2\right) - R = {} & \left(\hat{R} - R\right) + \left(\bar{x}_2 - \bar{X}_2\right)g_1^* \\ & + \left(\bar{x}_2' - \bar{X}_2\right)g_2^* + \left(s_{x_2}^2 - S_{x_2}^2\right)g_3^* \\ & + \left(s_{x_2}'^2 - S_{x_2}^2\right)g_4^* + \left(\hat{B}_1 - B_1\right)g_5^* \\ & + \left(\hat{B}_2 - B_2\right)g_6^* + \ldots \end{aligned} \tag{3.7}$$

Squaring both the sides of (3.7) and taking expectation, we see that the mean square error $E\left[g^*\left(\hat{R}, \bar{x}_2, \bar{x}_2', s_{x_2}^2, s_{x_2}'^2, \hat{B}_1, \hat{B}_2\right) - R\right]^2$ to the first degree of approximation, becomes equal to $MSE\left(\hat{R}_g\right)_{min}$ given by (3.3) if $g_5^* = g_6^* = 0$, and thus the estimator taken as a function

$R^*_{ge} = g^*\left(\hat{R}, \bar{x}_2, \bar{x}'_2, s^2_{x_2}, s'^2_{x_2}, \hat{B}_1, \hat{B}_2\right)$ depending upon estimated optimum values attains the minimum mean square error given by (3.3) if

$$\left.\begin{array}{l}
g^*\left(\hat{R}, \bar{x}_2, \bar{x}'_2, s^2_{x_2}, s'^2_{x_2}, \hat{B}_1, \hat{B}_2\right)\Big]_{T^*} = R, \quad \left.\dfrac{\partial g^*}{\partial \hat{R}}\right]_{T^*} = 1, \\
g^*_1 = \left.\dfrac{\partial g^*}{\partial \bar{x}_2}\right]_{T^*} = -\left.\dfrac{\partial g^*}{\partial \bar{x}'_2}\right]_{T^*} = -g^*_2, \\
g^*_3 = \left.\dfrac{\partial g^*}{\partial s^2_{x_2}}\right]_{T^*} = -\left.\dfrac{\partial g^*}{\partial s'^2_{x_2}}\right]_{T^*} = -g^*_4, \quad \left.\dfrac{\partial^2 g^*}{\partial \hat{R}^2}\right]_{T^*} = 0, \\
g^*_{01} = \left.\dfrac{\partial^2 g^*}{\partial \hat{R}\,\partial \bar{x}_2}\right]_{T^*} = -\left.\dfrac{\partial^2 g^*}{\partial \hat{R}\,\partial \bar{x}'_2}\right]_{T^*} = -g^*_{02}, \\
g^*_{03} = \left.\dfrac{\partial^2 g^*}{\partial \hat{R}\,\partial s^2_{x_2}}\right]_{T^*} = -\left.\dfrac{\partial^2 g^*}{\partial \hat{R}\,\partial s'^2_{x_2}}\right]_{T^*} = -g^*_{04}, \\
\left.\dfrac{\partial g^*}{\partial \bar{x}_2}\right]_{T^*} = D_1, \quad \left.\dfrac{\partial g^*}{\partial s^2_{x_2}}\right]_{T^*} = D_2, \\
g^*_5 = 0 \quad and \quad g^*_6 = 0 \; .
\end{array}\right\} \qquad (3.8)$$

Satisfying the conditions in (3.8), some particular estimators depending on estimated optimum values $\hat{D}_1, \hat{D}_2$ and attaining the minimum mean square error in (3.3), are given in the following section 4 (Conclusions).

IV. CONCLSIONS

(a) The optimum values D_1 and D_2 of g_1 and g_3 respectively minimize the mean square error of $\hat{R}_g = g\left(\hat{R}, \bar{x}_2, \bar{x}'_2, s^2_{x_2}, s'^2_{x_2}\right)$ and the resulting mean square error is given by

$$MSE\left(\hat{R}_g\right)_{min} = MSE\ (\hat{R}) - \left(\frac{1}{n} - \frac{1}{n'}\right)\left[R^2 C_2{}^2 C^2 + \frac{R^2\left\{(\lambda_{02} - \lambda_{12}) - C_2 C r_{1x_2}\right\}^2}{\left(\beta_{2x_2} - \beta_{1x_2} - 1\right)}\right] \qquad (4.1)$$

$$= MSE\ (\hat{R}) - \left(\frac{1}{n} - \frac{1}{n'}\right) R^2 C_2{}^2 C^2$$

$$-\left(\frac{1}{n}-\frac{1}{n'}\right)\frac{R^2\left\{\left(\lambda_{02}-\lambda_{12}\right)-C_2 Cr_{1x_2}\right\}^2}{\left(\beta_{2x_2}-\beta_{1x_2}-1\right)}\ . \tag{4.2}$$

Further, the minimum mean square error of the generalized estimator proposed by Singh, R. Karan, Rizvi, S.A.H and Rizvi S.A.M. (2012) $\hat{R}_{gd} = g\left(\hat{R}, \bar{x}_2, \bar{x}'_2\right)$ in (A) is given by

$$MSE\left(\hat{R}_{gd}\right)_{min} = MSE\left(\hat{R}\right)-\left(\frac{1}{n}-\frac{1}{n'}\right)R^2 C_2{}^2 C^2\ . \tag{4.3}$$

From (4.1) and (4.2) we have

$MSE\left(\hat{R}_g\right)_{min} = MSE\left(\hat{R}_{gd}\right)-\left(\frac{1}{n}-\frac{1}{n'}\right)\frac{R^2\left\{\left(\lambda_{02}-\lambda_{12}\right)-C_2 Cr_{1x_2}\right\}^2}{\left(\beta_{2x_2}-\beta_{1x_2}-1\right)}$, showing that the proposed class of estimators represented by $\hat{R}_g$ contains more efficient estimators than those in the class of estimators earlier represented by $\hat{R}_{gd}$ in (A) in the sense of having lesser mean square error.

(b) From the class of estimators represented by $\hat{R}_g$, considering the estimators $\hat{R}_3 = \hat{R} + k_1\left(\bar{x}_2 - \bar{x}'_2\right) + k_2\left(s^2_{x_2} - s'^2_{x_2}\right)$ and $\hat{R}_4 = \hat{R}\left(\frac{\bar{x}_2}{\bar{x}'_2}\right)^{k_1}\left(\frac{s^2_{x_2}}{s'^2_{x_2}}\right)^{k_2}$, we find that $g_1 = k_1$ and $g_3 = k_2$ for $\hat{R}_3$, and $g_1 = k_1\frac{R}{\bar{X}_2}$ and $g_3 = k_2\frac{R}{S^2_{x_2}}$ for $\hat{R}_4$. By equating these values of g_1 and g_3 for $\hat{R}_3$ and $\hat{R}_4$ to $D_1 = RB_1$ and $D_2 = RB_2$ in (3.1) and (3.2) respectively, we see that the estimator $\hat{R}_3 = \hat{R} + k_1\left(\bar{x}_2 - \bar{x}'_2\right) + k_2\left(s^2_{x_2} - s'^2_{x_2}\right)$ for $k_1 = RB_1$, $k_2 = RB_2$; and the estimator $\hat{R}_4 = \hat{R}\left(\frac{\bar{x}_2}{\bar{x}'_2}\right)^{k_1}\left(\frac{s^2_{x_2}}{s'^2_{x_2}}\right)^{k_2}$ for $k_1 = \bar{X}_2 B_1$ and $k_2 = S^2_{x_2} B_2$, attain the minimum mean square error given in (3.3) or (4.1).

(c) The optimum values of $k_1 = RB_1$, $k_2 = RB_2$ for $\hat{R}_3$ and $k_1 = \bar{X}_2 B_1$, $k_2 = S^2_{x_2} B_2$ for $\hat{R}_4$ may be rarely known, hence replacing the unknown parameters involved in RB_1, RB_2, $\bar{X}_2 B_1$ and $S^2_{x_2} B_2$ by their consistent or unbiased estimators, we get the estimated optimum values $\hat{k}_1 = \hat{R}\hat{B}_1$, $\hat{k}_2 = \hat{R}\hat{B}_2$ for $\hat{R}_3$ and $\hat{k}_1 = \bar{x}_2\hat{B}_1$, $\hat{k}_2 = s^2_{x_2}\hat{B}_2$ for $\hat{R}_4$ so that the estimators depending upon the estimated optimum values corresponding to $\hat{R}_3$ and $\hat{R}_4$ become

$$\hat{R}_{3e} = \hat{R} + \bar{x}_2\hat{B}_1\left(\bar{x}_2 - \bar{x}'_2\right) + \hat{R}\hat{B}_2\left(s^2_{x_2} - s'^2_{x_2}\right) \tag{4.4}$$

and
$$\hat{R}_{4e} = \hat{R}\left(\frac{\bar{x}_2}{\bar{x}'_2}\right)^{\bar{x}_2\hat{B}_1}\left(\frac{s^2_{x_2}}{s'^2_{x_2}}\right)^{\hat{R}\hat{B}_2} \tag{4.5}$$

which satisfying all the regularity conditions in (3.8) for the generalized estimator $\hat{R}^*_{ge}$ depending upon estimated optimum values, attain the minimum mean square error in (3.3) or (4.1).

(d) We may easily derive the similar results for the estimators developing for P also on the same lines of $\hat{R}_g$ or $\hat{R}_{ge}$.

(e) Single sampling results may be easily found as the special cases of this study for $n' = N$.

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