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Dynamic behavior for a model of coupled limit cycle oscillators with delays

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ABSTRACT: In this paper, the stability and oscillatory behavior of the solutions for a model of coupled limit cycle oscillators with delays are investigated. By means of mathematical analysis method, some sufficient conditions to guarantee the stability and oscillation of the solutions are obtained. Computer simulations are provided to demonstrate our results.
Keywords: a coupled limit cycle system, delay, oscillation, stability

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I. INTRODUCTION

It is well known that the coupled system is a research topic of science studying how parts of a system lead to the collective behavior. Since last decade, a lot of researches about coupled systems have appeared in a wide areas, such as engineering, biology, social science, neural networks and so on [1- 12]. The following delay-coupled limit cycle oscillators have been investigated by many researchers [13-19]:

$$\begin{cases} Z_1'(t) = (1 + i\omega_1 - |Z_1(t)|^2)Z_1(t) + k[Z_2(t-\tau)) - Z_1(t)], \\ Z_2'(t) = (1 + i\omega_2 - |Z_2(t)|^2)Z_2(t) + k[Z_1(t-\tau)) - Z_2(t)]. \end{cases} \tag{1}$$

where $Z_1(t)$ and $Z_2(t)$ are the complex amplitude of the oscillator 1 and oscillator 2, respectively. Each oscillator has a stable limit cycle of unit amplitude $|Z_j(t)| = 1$ with angular frequencies ω_1 and ω_2, $k \geq 0$ is the coupling strength and $\tau \geq 0$ is a measure of time delay. In [13], Li et. al have studied the double Hopf bifurcation at zero equilibrium point for system (1). The authors not only gave the critical values of Hopf and double Hopf bifurcations, but also derive the universal unfolding and a complete bifurcation diagram of the system. The normal forms of several strong resonant cases were listed. Recently, Thounaojam and Shrimali have considered the following coupled system [20]:

$$\begin{cases} Z_{1,3}'(t) = (A + i\omega - |Z_{1,3}(t)|^2)Z_{1,3}(t) + \varepsilon[Z_2(t-\tau)) - Z_{1,3}(t)], \\ Z_2'(t) = (A + i\omega - |Z_2(t)|^2)Z_2(t) + \varepsilon[Z_1(t-\tau) + Z_3(t-\tau) - Z_2(t)] + \varepsilon_U U \\ U'(t) = -kU - \varepsilon_U Re(Z_2). \end{cases} \tag{2}$$

where A is a parameter, $Re(Z_2)$ is the real part of $Z_2(t)$, and k is the rate of decay of the linear system U . System (2) has been employed in a system of Hindmarsh-Rose neurons to incorporate an interaction with external environment. For the cases $Z_1 = Z_3$, and $Z_1 = -Z_3$, the authors have demonstrated that a relay system of time delay coupled limit cycle oscillators supports a dynamical regime where two spatially separated oscillators operate in anti-phase when middle oscillator is driven to its fixed point. However, the authors did not provide the theoretical analysis. Motivated by the above models, in this paper we consider the following general coupled time delay model:

$$\begin{cases} Z_1'(t) = (a + i\omega_1 - |Z_1(t)|^2)Z_1(t) - k_{11}[Z_1(t-\tau_1)) - Z_1(t)] \\ \qquad +k_{12}[Z_2(t-\tau_2)) - Z_1(t)] + k_{13}[Z_3(t-\tau_3)) - Z_1(t)], \\ Z_2'(t) = (b + i\omega_2 - |Z_2(t)|^2)Z_2(t) + k_{21}[Z_1(t-\tau_2)) - Z_2(t)] \\ \qquad -k_{22}[Z_2(t-\tau_2)) - Z_2(t)] + k_{23}[Z_3(t-\tau_3)) - Z_2(t)], \\ Z_3'(t) = (c + i\omega_3 - |Z_3(t)|^2)Z_3(t) + k_{31}[Z_1(t-\tau_1)) - Z_3(t)] \\ \qquad +k_{32}[Z_2(t-\tau_2)) - Z_3(t)] - k_{33}[Z_3(t-\tau_3)) - Z_3(t)]. \end{cases} \tag{3}$$

where $0 < a, b, c, k_{ij}, \omega_i$; $0 \le \tau_i (1 \le i, j \le 3)$ are parameters. By means of mathematical analysis method, the dynamical behavior of system (3) has been discussed.Preliminaries

II. PRELIMINARIES

For convenience, writing $Z_j(t) = x_j(t) + iy_j(t)(j = 1, 2, 3)$, then system (3) can be written as a real form of six dimensional system:

$$\begin{cases} x_1'(t) = (a + k_{11} - k_{12} - k_{13})x_1(t) - \omega_1 y_1(t) - k_{11}x_1(t-\tau_1) \\ \qquad +k_{12}x_2(t-\tau_2) + k_{13}x_3(t-\tau_3) - x_1^3(t) - x_1(t)y_1^2(t), \\ y_1'(t) = \omega_1 x_1(t) + (a + k_{11} - k_{12} - k_{13})y_1(t) - k_{11}y_1(t-\tau_1) \\ \qquad +k_{12}y_2(t-\tau_2) + k_{13}y_3(t-\tau_3) - x_1^2(t)y_1(t) - y_1^3(t), \\ x_2'(t) = (b + k_{22} - k_{21} - k_{23})x_2(t) - \omega_2 y_2(t) + k_{21}x_1(t-\tau_1) \\ \qquad -k_{22}x_2(t-\tau_2) + k_{23}x_3(t-\tau_3) - x_2^3(t) - x_2(t)y_2^2(t), \\ y_2'(t) = \omega_2 x_2(t) + (b + k_{22} - k_{21} - k_{23})y_2(t) + k_{21}y_1(t-\tau_1) \\ \qquad -k_{22}y_2(t-\tau_2) + k_{23}y_3(t-\tau_3) - x_2^2(t)y_2(t) - y_2^3(t), \\ x_3'(t) = (c + k_{33} - k_{31} - k_{32})x_3(t) - \omega_3 y_3(t) + k_{31}x_1(t-\tau_1) \\ \qquad +k_{32}x_2(t-\tau_2) - k_{33}x_3(t-\tau_3) - x_3^3(t) - x_3(t)y_3^2(t), \\ y_3'(t) = \omega_3 x_3(t) + (c + k_{33} - k_{31} - k_{32})y_3(t) + k_{31}y_1(t-\tau_1) \\ \qquad +k_{32}y_2(t-\tau_2) - k_{33}y_3(t-\tau_3) - x_3^2(t)y_3(t) - y_3^3(t). \\ . \end{cases} \tag{4}$$

$$u'(t) = Au(t) + Bu(t-\tau) + f(u(t)) \tag{5}$$

where $u = (x_1, y_1, x_2, y_2, x_3, y_3)^T$, $u(t-\tau) = (x_1(t-\tau_1), y_1(t-\tau_1), x_2(t-\tau_2), y_2(t-\tau_2), x_3(t-\tau_3), y_3(t-\tau_3))^T$, A and B both are 6 by 6 matrices, and $f(u(t))$ is a 6 by 1 vector:

$$A = (a_{ij})_{6\times 6} = \begin{pmatrix} a_{11} & -\omega_1 & 0 & 0 & 0 & 0 \\ \omega_1 & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & -\omega_2 & 0 & 0 \\ 0 & 0 & \omega_2 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & -\omega_3 \\ 0 & 0 & 0 & 0 & \omega_3 & a_{66} \end{pmatrix},$$

where $a_{11} = a_{22} = a + k_{11} - k_{12} - k_{13}$, $a_{33} = a_{44} = b + k_{22} - k_{21} - k_{23}$, $a_{55} = a_{66} = c + k_{33} - k_{31} - k_{32}$.

$$B = (b_{ij})_{6\times 6} = \begin{pmatrix} -k_{11} & 0 & k_{12} & 0 & k_{13} & 0 \\ 0 & -k_{11} & 0 & k_{12} & 0 & k_{13} \\ k_{21} & 0 & -k_{22} & 0 & k_{23} & 0 \\ 0 & k_{21} & 0 & -k_{22} & 0 & k_{23} \\ k_{31} & 0 & k_{32} & 0 & -k_{33} & 0 \\ 0 & k_{31} & 0 & k_{32} & 0 & -k_{33} \end{pmatrix},$$

$$f(u) = [-x_1^3 - x_1 y_1^2, -y_1^3 - x_1^2 y_1, -x_2^3 - x_2 y_2^2, -y_2^3 - x_2^2 y_2, -x_3^3 - x_3 y_3^2, -y_3^3 - x_3^2 y_3]^T.$$

The linearized system of (5) is

$$u'(t) = Au(t) + Bu(t - \tau) \tag{6}$$

Lemma 1 If matrix R (= A + B) is a nonsingular matrix, then there exists a unique equilibrium point for system (4) (or (5)).

Proof Noting that f (u) is a continuous function and only f (0) = 0. Obviously, zero is an equilibrium point of system (4) (or (5)).

Now we prove that zero is a unique equilibrium point of system (4) (or (5)). Assume that $u^* = [x^*, y^*, x^*, y^*, x^*, y^*]^T$ *f*= 0 is an equilibrium point of system (5), then we have

$$Au^* + Bu^* + f(u^*) = (A + B)(u^*) + f(u^*) = Ru^* + f(u^*) = \mathbf{0} \tag{7}$$

From (7) we get

$$Ru^* = -f(u^*) \tag{8}$$

Let M_i be the matrix obtained from R by replacing column i of R by $-\mathbf{f}(u^*)$. Since R is a nonsingular matrix, then the determinant of R is not equal zero. By the Cramer's Rule of linear algebra, $x^* =$det M_1/det R, y^*=det M_2/det R, · · · , y^*=det M_6/det3 R. This implies that we must have $x^* = y^* \cdot = y^* = 0$, and zero is a unique equilibrium point of system (4) (or (5)). The proof is completed.

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Lemma 2 All solutions of system (4) are bounded.

Proof To prove the boundedness of the solutions in system (4), we construct a Lyapunov function

$$\begin{aligned} V'(t)|_{(4)} &= \sum_{i=1}^{3}[x_i'(t)x_i(t) + y_i'(t)y_i(t)] \\ &\le \sum_{i=1}^{3}(|a_{ii} + k_{ii}|x_i^2 + |\omega_i||x_i y_i|) + \sum_{i=1}^{3}|k_{i2}||x_i x_2| + \sum_{i=1}^{3}|k_{i3}||x_i x_3| \\ &\quad - \sum_{i=1}^{3}(x_i^4 + 2x_i^2 y_i^2 + y_i^4) \end{aligned} \tag{9}$$

Obviously, when $x_i(t)$, $y_i(t)(1 \le i \le 3)$ tend to infinity, $x^4(t)$, $x^2(t)y^2(t)$, $y^4(t)$, are higher order infinity than $x^2(t)$, $|x_i y_i|$, $|x_i x_j|$, respectively. Therefore, there exists suitably large L > 0 such that $V'(t)|_{(4)} < 0$ as $|x_i| > L$, $|y_i| > L$. This means that all solutions of system (4) are bounded.

III. OSCILLATION AND STABILITY OF THE SOLUTIONS

Theorem 1 Assume that zero is the unique equilibrium point of system (4) (or(5)) for selecting parameter values. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ be characteristic values of matrix A and B, respectively. If there exists some $\alpha_k > 0$, or $|Re(\alpha_k)| < Re(\beta_k)$, then the unique equilibrium point of system (6) is unstable. System (4) generates an oscillatory solution.

Proof Obviously, if the trivial solution of system (6) is unstable, then the trivial solution of system (5) is unstable. Therefore, we only need to prove the instability of the trivial solution of system (6). Since α_i and β_i (i = 1, 2, · · · , 6) are characteristic values of matrix A and B, respectively, then the characteristic equation corresponding to system (6) is the following:

$$\Pi_{i=1}^{6}(\lambda - \alpha_i - \beta_i e^{-\lambda\tau_j}) = 0 \quad (10)$$

So, we are led to an investigation of the nature of the roots for some $k, k \in \{1, 2, \cdots, 6\}$

$$\lambda - \alpha_k - \beta_k e^{-\lambda\tau_j} = 0 \quad (11)$$

System (11) is a transcendental equation which is hard to find all solutions for the equation. However, we show that there exists a positive real part eigenvalue of equation (11) under the assumption of Theorem 1. If $|Re(\alpha_k)| < Re(\beta_k)$. Let $\lambda = \sigma + i\theta$, $\alpha_k = \alpha_{k1} + i\alpha_{k2}$, $\beta_k = \beta_{k1} + i\beta_{k2}$. Separating the real and imaginary parts from equation (11), we have

$$\sigma = \alpha_{k1} + \beta_{k1}e^{-\sigma\tau_j}\cos(\theta\tau_j) + \beta_{k2}e^{-\sigma\tau_j}\sin(\theta\tau_j). \quad (12)$$

$$\omega = \alpha_{k2} - \beta_{k1}e^{-\sigma\tau_j}\sin(\theta\tau_j) + \beta_{k2}e^{-\sigma\tau_j}\cos(\theta\tau_j). \quad (13)$$

We show that equation (11) has a positive real part root. Let

$$\varphi(\sigma) = \sigma - \alpha_{k1} - \beta_{k1}e^{-\sigma\tau_j}\cos(\theta\tau_j) - \beta_{k2}e^{-\sigma\tau_j}\sin(\theta\tau_j). \quad (14)$$

Obviously, $\phi(\sigma)$ is a continuous function of σ. When $\sigma = 0$ we have $\phi(0) = -\alpha_{k1} - \beta_{k1}\cos(\theta\tau_j) - \beta_{k2}\sin(\theta\tau_j)$. If τ_j is suitably small, we have $\sin(\theta\tau_j) \to 0$, $\cos(\theta\tau_j) \to 1$, and $\phi(0) \le |Re(\alpha_k)| - Re(\beta_k) < 0$. Noting that $\lim_{\sigma\to+\infty} e^{-\sigma\tau_j} = 0$, so there exists a suitably large $\tilde{\sigma}(> 0)$ such that $\phi(\tilde{\sigma}) = \tilde{\sigma} - \alpha_{k1} - \beta_{k1}e^{-\tilde{\sigma}\tau_j}\cos(\theta\tau_j) - \beta_{k2}e^{-\tilde{\sigma}\tau_j}\sin(\theta\tau_j) > 0$. By means of the Intermediate ValueTheorem, there exists a $\bar{\sigma} \in (0, \tilde{\sigma})$ such that $\phi(\bar{\sigma}) = \bar{\sigma} - \alpha_{k1} - \beta_{k1}e^{-\bar{\sigma}\tau_j}\cos(\theta\tau_j) - \beta_{k2}e^{-\bar{\sigma}\tau_j}\sin(\theta\tau_j) = 0$. This means that the characteristic value λ has a positive real part when τ_j is suitably small. In a time delayed system, it was emphasized that if the trivial solution is unstable for small time delay, then the instability of trivial solution will maintain as delay increases. Therefore, the trivial solution of system

(5) is unstable for any time delays. Since all solutions of system (5) are bounded, the instability of the trivial solution and the boundedness of the solutions will force system (5) to generate an oscillatory solution. For the case of $\alpha_k > 0$, equation (11) will have a positive root. The proof is similar and we omit it.

Theorem 2 Assume that zero is the unique equilibrium point of system (5) for selecting parameter values. Let $m = \max\{a_{11} + |-\omega_1|, a_{33} + |-\omega_2|, a_{55} + |-\omega_3|\}$, $n = \max\{|-k_{11}| + k_{21} + k_{31}, |-k_{22}| + k_{12} + k_{32}, |-k_{33}| + k_{13} + k_{23}\}$. If the following inequality holds:

$$m + n > 0 \quad (15)$$

then system (5) has an oscillatory solution.

Proof Let $v(t) = \sum_{i=1}^{3}|x_i(t)| + |y_i(t)|$, from (6) we have

$$v'(t) \le mv(t) + nv(t-\tau) \quad (16)$$

Consider the scalar differential equation

$$z'(t) = mz(t) + nz(t-\tau) \quad (17)$$

According to the comparison theorem of differential equation, we have $v(t) \le z(t)$. For equation (16), the characteristic equation associated with (16) is given by

$$\lambda = m + ne^{-\lambda\tau} \quad (18)$$

We claim that there exists a positive characteristic root of equation (18). Indeed, let $g(\lambda) = \lambda - m - ne^{-\lambda\tau}$. Then $g(\lambda)$ is a continuous function of λ. From condition (15), we have $g(0) = -m - n < 0$. On the other hand, $\lim_{\lambda\to+\infty} e^{-\lambda\tau} \to 0$. Thus, there exists a suitably large positive λ, say λ_1 such that $g(\lambda_1) = \lambda_1 - m - ne^{-\lambda_1\tau} > 0$. It means that there exists a λ', where $\lambda' \in (0, \lambda_1)$ such that $g(\lambda') = 0$ from the Intermediate Value Theorem. In other words, λ' is a positive characteristic root of equation (18), implying that the trivial solution of equation (17) is unstable. Since $v(t) \le z(t)$, this means that the trivial solution of equation (16) is unstable. It suggested that system (4) (or (5)) has an oscillatory solution.

Theorem 3 Assume that zero is the unique equilibrium point of system (5) for selecting parameter values. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ be characteristic values of matrix A and B, respectively. If each $\alpha_i < 0$, or $Re(\alpha_i) < 0$ and $|Re(\alpha_i)| > Re(\beta_i)$, then the trivial solution of system
(4) is stable.

Proof According to the basic theory of functional differential equations [21], the unique trivialsolution of system (6) is stable. Noting that f (u) is a higher order infinitesimal when $x_i \to 0$, and $y_i \to 0$. Thus, the stability of the trivial solution of system (6) implies the stability of the trivial solution of system (4). The proof is completed.

IV. SIMULATION RESULTS

The simulation is based on the system (4), first the parameters are selected as follows: a = 0.5, b = 0.6, c = 0.8; ω_1 = 0.75, ω_2 = 0.85, ω_3 = 0.65; k_{11} = 0.35, k_{12} = 0.45, k_{13} = 0.5, k_{21} = 0.15, k_{22} = 0.25, k_{23} = 0.32, k_{31} = 0.15, k_{32} = 0.24, k_{33} = 0.18. The time delays τ_1 = 1.5, τ_2 = 1.2, τ_3 = 1.6. Then the characteristic values of A are −0.1000±0.7500i, 0.3800±0.8500i, 0.5900±0.6500i, the characteristic values of B are 0.2879, 0.2879, −0.4953, −0.4953, −0.5726, −0.5726. Since all characteristic value of matrix B are real numbers, so $Re(\beta_k) = \beta_k$, and $|Re(\alpha_1)| = |-0.1000| < \beta_1 = 0.2879$, the condition of Theorem 1 are satisfied. There exists an oscillatory solution for system (4) (see Fig.1). In order to see the effect of time delays, we increase delays as τ_1 = 5.5, τ_2 = 6.2, τ_3 = 7.6. The other parameters are the same as Fig. 1, we see the oscillatory behavior is still maintained (see Fig. 2). Then we change the parameters as a = 0.8, b = 0.6, c = 0.5; ω_1 = 0.25, ω_2 = 0.55, ω_3 = 0.35; k_{11} = 1.35, k_{12} = 0.75, k_{13} = 0.5, k_{21} = 0.85, k_{22} = 1.25, k_{23} = 0.62, k_{31} = 0.55, k_{32} = 0.64, k_{33} = 1.18. The time delays τ_1 = 0.5, τ_2 = 0.8, τ_3 = 0.6. Then m = 1 and n = 2.75, so the conditions of Theorem 2 are satisfied, system (4) generates an oscillatory solution (see Fig. 3). Finally we select the parameters as a = 0.12, b = 0.15, c = 0.18; ω_1 = 1.75, ω_2 = 1.55, ω_3 = 1.65; k_{11} = 0.85, k_{12} = 0.72, k_{13} = 0.5, k_{21} = 0.85, k_{22} = 0.55, k_{23} = 0.64, k_{31} = 0.56, k_{32} = 0.68, k_{33} = 0.58. The time delays τ_1 = 0.65, τ_2 = 0.7, τ_3 = 0.68. Then the characteristic values of A are −0.8000 ± 1.7500i, −0.78500 ± 1.5500i, −0.7800 ± 1.6500i, and the characteristic values of B are 0.6693, 0.6693, −1.4957, −1.4957, −1.1536, −1.1536. One can see that each $Re(\alpha_k) < 0$ and $|Re(\alpha_k)| > 0.6693$, the conditions of Theorem 3 are satisfied, the solutions of system 4 are convergent (see Fig. 4). We pointed out that our criterion only is a sufficient condition.

V. CONCLUSION

In this paper, we have discussed the oscillatory behavior of the solutions for a model of coupled limit cycle oscillators with delays. Based on mathematical analysis method, we provided some sufficient conditions to guarantee the oscillation and stability of the solutions. Some simulations are provided to indicate the effectness of the criteria.

Competing Interests
Author has declared that no competing interests exist.

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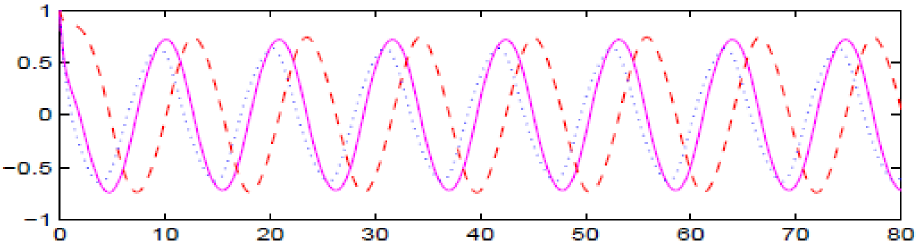
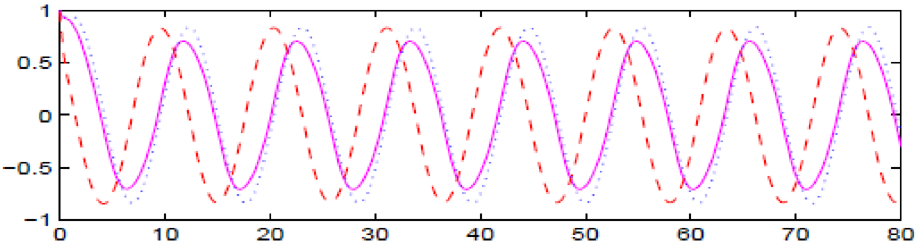
Fig. 1 Oscillation of the solutions, delays: 1.5, 1.2, 1.6.

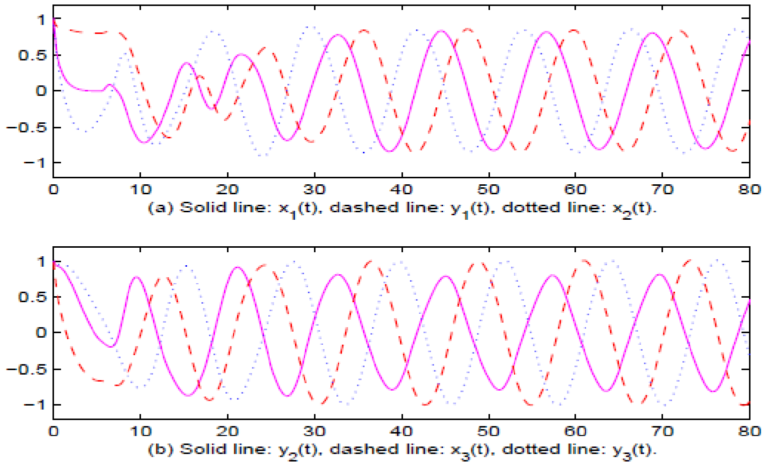


(a) Solid line: $x_1(t)$, dashed line: $y_1(t)$, dotted line: $x_2(t)$.



(b) Solid line:$y_2(t)$, dashed line: $x_3(t)$, dotted line: $y_3(t)$.

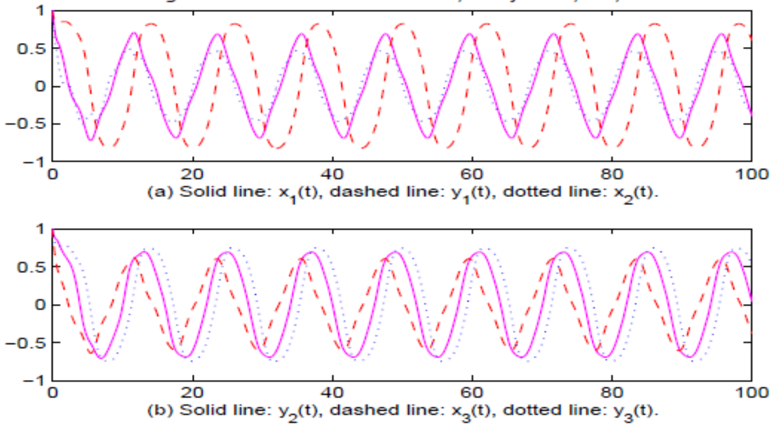
Fig. 2 Oscillation of the solutions, delays: 5.5, 6.2, 7.6.



(a) Solid line: $x_1(t)$, dashed line: $y_1(t)$, dotted line: $x_2(t)$.

(b) Solid line: $y_2(t)$, dashed line: $x_3(t)$, dotted line: $y_3(t)$.

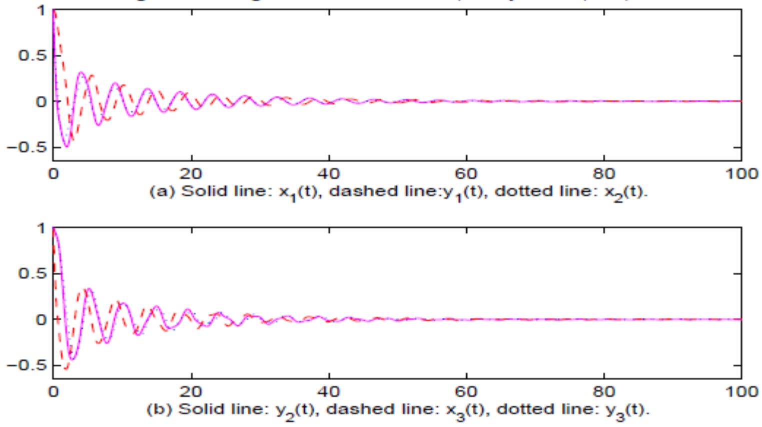
Fig. 3 Oscillation of the solutions, delays: 0.5, 0.8, 0.6.



(a) Solid line: $x_1(t)$, dashed line: $y_1(t)$, dotted line: $x_2(t)$.

(b) Solid line: $y_2(t)$, dashed line: $x_3(t)$, dotted line: $y_3(t)$.

Fig. 4 Convergence of the solutions, delays: 0.65, 0.7, 0.68.



(a) Solid line: $x_1(t)$, dashed line:$y_1(t)$, dotted line: $x_2(t)$.

(b) Solid line: $y_2(t)$, dashed line: $x_3(t)$, dotted line: $y_3(t)$.

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