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# Controlling Bifurcation in Fractional Order Delayed Predator-Prey System

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**ABSTRACT:** In this paper, we investigate hybrid control in fractional order predator-prey system attained with Hopf bifurcation of time delay as a bifurcation parameter. Oscillatory and periodical nature of solutions and dynamical value of complex system for both commensurate and incommensurate order are examined. With the help of proposed controller, it is showed that the occurrence of tremendous stability, Hopf bifurcation occurs in advance if the controller is removed. Atlast the numerical examples are used to validate the effectiveness of derived theoretical results.

**Keywords:** Commensurate order, Discrete delay, Fractional order, Hopf bifurcation, Hybrid control, Incommensurate order, Prey-Predator.
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## I. INTRODUCTION

In mathematical biology, population dynamics has retained more attentive field for numerous scholars. Time delay in ecological system has an important influence in dynamical behaviour. The stability concept of dynamical system diluted by time delay was exposed. Result of impact of particular delay can preserve the concept of stability in prey-predator system [2, 5].

Fractional order differential equations in modelling dynamical system consist of more benefits compared with classical integer order due to the memory and hereditary properties of fractional calculus [4, 8, 9, 13]. Comparison and review indicates more importance of fractional order delayed prey-predator system rather than integer order. With the gradual development of fractional calculus, Hopf bifurcation in fractional order has attained the superior level [6].

Firstly, the authors intend to deal with the controlling bifurcation in a delayed fractional predator-prey system with incommensurate orders and to apply the feedback control to the delayed fractional order chaotic systems, bifurcation as parameter [1, 3]. In [16], the problem based on the time delay is a bifurcation parameter and applying the hybrid tactics of control strategy for controlling bifurcation for a fractional delayed predator-prey system and achieved the delay-induced bifurcation conditions of Hopf bifurcation. Motivated by this, we are interested to study the problem of stabilizing bifurcation for a delayed fractional predator-prey system stated in [16] with commensurate and incommensurate orders through hybrid control technique. The highlights of this paper are listed as follows:

i) Bifurcation control in a delayed predator-prey model.
ii) The stability performance uncontrolled model in commensurate order can be enormously exalted on account of the proposed controller.
iii) The property of bifurcation control is more remarkable in the proposed system than the corresponding integer-order if choosing the same proposed feedback gain.

## II. PRELIMINARIES

### 2.1 Fractional Order Derivative

There are many definitions of fractional derivatives. The definitions given by Riemann-Liouville and Caputo are widely used. Caputo derivative with integer order are frequently used in many researches and has understandable features of physical concepts. We adopt Caputo derivative concepts in this paper.

**Definition 1:** [12] The fractional integral of order $\varepsilon > 0$ for a function y(t) is defined as function $y : (0, \infty) \to R$ is given by

$$I_{0+}^{\varepsilon} y(t) = \frac{1}{\Gamma\varepsilon} \int_{t_0}^{t} (t-s)^{\varepsilon-1}\, y(s) ds$$

where $t_0 \leq t$ and the right hand side is point-wise defined on (0, ∞), $\Gamma(.)$ is a gamma function defined as

$$\Gamma(s) = \int_0^{\infty} t^{s-1}\, e^{-t}\, dt.$$

**Definition 2:** [12] The order $\varepsilon > 0$ of Caputo's fractional derivative for a function $y(t) \in C^m([t_{0,}\infty], R)$ is defined by

$$\frac{d^{\varepsilon}y(t)}{dt^{\varepsilon}} = \frac{1}{\Gamma(n-\varepsilon)} \frac{d^n}{dt^n} \int_a^t \frac{y(\tau)}{(t-\tau)^{\varepsilon-n+1}} d\tau$$

for $n-1 \leq \varepsilon < n$ . Furthermore, the fractional integral of order $\varepsilon > 0$ of a function $y : (0, \infty) \to R$ is given by

$$\frac{d^{\varepsilon}y(t)}{dt^{\varepsilon}} = \frac{1}{\Gamma(1-\varepsilon)} \frac{d}{dt} \int_a^t \frac{y(\tau)}{(t-\tau)^{\varepsilon}} d\tau$$

where the right hand side is point-wise defined on (0, ∞).

Hypothesis ($H_1$): The transfer function $f_i, g_j$(i, j = 1, 2...nsatisfies Lipschitz condition, i.e., there exists positive constants $F_{j,} G_j$ such that $|f_i(x) - f_j(y)| \leq F_j|x-y|$, $|g_i(x) - g_j(y)| \leq G_j|x-y|$, for all x,y ∈ R.

## III. MATHEMATICAL MODEL

Along with the above definitions, method of new modelling provide us the importance of fractional differential equations in the interaction of biological multiple species in ecological system. Predator-prey models are more significant system in the multi species population interactions and these interactions through integer order models have been studied by many authors [5].

The mathematical model mentioned below seems to the issue of bifurcation for the predator ratio-dependent predator-prey model with Holling type III functional response with two time delays and stage structure [15]. The system in [15] has restructured by the time delay applied in the density of mature predator at time t mentioned in [10, 11] is given as

$$\frac{dx}{dt} = x(t)\left(r - ax(t) - \frac{a_1 y_2(t-\tau)}{1+mx(t)}\right)$$

$$\frac{dy_1}{dt} = \frac{a_2 x(t) y_2(t-\tau)}{1+mx(t-\tau)} - r_1 y_1(t) - \beta y_1(t)$$

(A) $$\frac{dy_2}{dt} = \beta y_1(t) - r_2 y_2(t)$$

**Table 1:** The particulars of significant variables and parameters of system (A)

| Variables (Parameter) | Descriptions |
|---|---|
| $x(t)$ | Densities of the prey with respect to time t |
| $y_1(t)$ | Densities of the immature with respect to time t |
| $y_2(t)$ | Densities of the mature predator with respect to time t |
| $r$ | The intrinsic growth rate of the prey |
| $r_1$ | Death rate of the immature predator |
| $r_2$ | Death rate of the mature predator |
| $a$ | The intraspecific constant of the prey |
| $a_2$ | Capturing rate |
| $\frac{a_2}{a_1}$ | Conversion rate of mature predator |
| $m$ | Half-capturing saturation constant |
| $\beta$ | Rate of proportional to the density of the immature predator |
| $\tau$ | The time delay that gestation of mature adult predators |

According to the system (A), we investigate a fractional order prey-predator interaction along with time delay $\tau$is described by

$$D^{q_1}x = x(t)\left(r - ax(t) - \frac{a_1 z(t-\tau)}{1+mx(t)}\right)$$

$$D^{q_2}x = \frac{a_2 x(t) z(t-\tau)}{1+mx(t)} - r_1 y(t) - \beta y(t)$$

(B)

$$D^{q_3}x = \beta y(t) - r_2 z(t)$$

where the variables and parameters in (B) are defined in Table 1.

With the initial conditions x (0), $y(0) > 0$ and $z(t) = \rho(t)$, the smooth function is considered as $\rho \in [-\tau, 0]$. If the orders are selected as $q_1 = q_2 = q_3 = 1$, then the system (B) is designated as an integer order system.

In fact, the system (B) has a unique positive equilibrium point $E^* = (x^*, y^*, z^*)$ provided that

$$x^* = \frac{r_2(\beta + r_1)}{a_2\beta - mr_2(\beta + r_1)}$$

$$y^* = \frac{r_2}{\beta} z$$

$$z^* = \frac{a_2\beta[a_2 r\beta - r_2(a+mr)(\beta + r_1)]}{a_1[a_2\beta - mr_2(\beta + r_1)]^2}$$

On the basis of model (B), we propose the hybrid control to control the onset of the Hopf bifurcation in the following fractional order.

$D^{q_1}x = x(t)\big(r -$
axt−a1z(t−τ)1+mx(t)

$$D^{q_2}x = \frac{a_2 x(t) z(t-\tau)}{1+mx(t)} - r_1 y(t) - \beta y(t)$$

(C)

$D^{q_3}x = (1-\alpha)[\beta y(t) -$
r2zt+αzt−τ−z∗
where α is the negative feedback gain.

In this study, our aim is to describe the conditions of Hopf bifurcation for system (C) by choosing time delay as a bifurcation parameter based on the approach of stability analysis [17], the collision of feedback gain on bifurcation in given domain is released.

## IV. MAIN RESULTS

### 4.1 Stability and bifurcation analysis of controlled system

In this section, we proceed with stability and bifurcation theory of fractional order shall be investigate with the system (C) with proposed control. At first, we consider the proposed system is derived for the fractional incommensurate order $(q_1, q_2, q_3) \in (0,1)$.

The characteristic equation of the system (C) at equilibrium state at $(q_1, q_2, q_3) \in (0,1)$ is

$$\begin{vmatrix} \lambda^{q_1} - r + 2ax + \frac{a_1 z}{(1+mx)} - \frac{a_1 xzm}{(1+mx)^2} & 0 & \frac{a_1 x e^{-\lambda\tau}}{(1+mx)} \\ -\frac{a_2 z}{(1+mx)^2} & \lambda^{q_2} + r_1 + \beta & \frac{a_2 x e^{-\lambda\tau}}{(1+mx)} \\ 0 & -\beta(1-\alpha) & \lambda^{q_3} + r_2(1-\alpha) - \alpha e^{-\lambda\tau} \end{vmatrix} = 0 \quad (1)$$

Then the characteristic polynomial is simplified as
$\lambda^{q_1+q_2+q_3} + A_1\lambda^{q_1+q_2} + A_2\lambda^{q_2+q_3} + A_3\lambda^{q_1+q_3} + A_4\lambda^{q_1+q_2}e^{-\lambda\tau} + A_5\lambda^{q_1} + A_6\lambda^{q_2} + A_7\lambda^{q_3} +$
A8λq1e−λτ+A9λq3e−λτ+A10λq3e−λτ+A11=0

(2)

where $A_1 = r_2(1-\alpha)$ ; $A_2 = -r + 2ax + \frac{a_1 z}{(1+mx)} - \frac{a_1 xzm}{(1+mx)^2}$ ; $A_3 = r_1 + \beta$; $A_4 = -\alpha$ ;
$A_5 = r_2(r_1 + \beta)(1-\alpha)$ ; $A_6 = r_2(1-\alpha)(-r + 2ax + \frac{a_1 z}{(1+mx)} - \frac{a_1 xzm}{(1+mx)^2})$ ;
$A_7 = (r_1 + \beta)(-r + 2ax + \frac{a_1 z}{(1+mx)} - \frac{a_1 xzm}{(1+mx)^2})$
; $A_8 = -\alpha(r_1 + \beta) - (1-\alpha)\frac{a_2 x\beta}{(1+mx)})$;
$A_9 = -\alpha(-r + 2ax + \frac{a_1 z}{(1+mx)} - \frac{a_1 xzm}{(1+mx)^2})$;
$A_{10} = -\alpha(r_1 + \beta) - (1-\alpha)\frac{a_2 x\beta}{(1+mx)})(-r + 2ax + \frac{a_1 z}{(1+mx)} - \frac{a_1 xzm}{(1+mx)^2}) + \frac{a_1 a_2 \beta xz(1-\alpha)}{(1+mx)^3}$;
$A_{11} = (1-\alpha)r_2(r_1 + \beta)$

The bifurcation point and critical frequency of the system (C) has been calculated by substitute $\lambda = i\omega$ in equation (2) where ω is the critical frequency, $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ and proceed the steps to separate the real and imaginary parts of the observations then we can obtain the following results

$$\mu + \phi\cos\omega\tau + \psi\sin\omega\tau = 0 \quad (3)$$
$$\eta + \psi\cos\omega\tau - \phi\sin\omega\tau = 0$$

Solve the linear equations (3) and getting the bifurcation points of both the trigonometric functions with respect to the critical value $\omega_0$

$$\sin\omega_0\tau_1 = \frac{\eta\phi - \mu\psi}{\phi^2 + \psi^2}$$

$$\cos\omega_0\tau_2 = \frac{-(\phi\mu + \eta\psi)}{\phi^2 + \psi^2} \quad (4)$$

The bifurcation points are

$$\tau_1 = \frac{1}{\omega_0}\left(\sin^{-1}(\frac{\eta\phi - \mu\psi}{\phi^2 + \psi^2}) + 2n\pi\right),\ n = 0,1,2 \ldots$$

$$\tau_2 = \frac{1}{\omega_0}\left(\cos^{-1}(\frac{-(\phi\mu + \eta\psi}{\phi^2 + \psi^2}) + 2n\pi\right),\ n = 0,1,2 \ldots \quad (5)$$

where the functions are defined as follows:

$$\mu = \omega^{q_1+q_2+q_3}\cos\frac{(q_1+q_2+q_3)\pi}{2} + A_1\omega^{q_1+q_2}\cos\frac{(q_1+q_2)\pi}{2} + A_2\omega^{q_2+q_3}\cos\frac{(q_2+q_3)\pi}{2} + A_3\omega^{q_1+q_3}\cos\frac{(q_1+q_3)\pi}{2} + A_5\omega^{q_1}\cos\frac{q_1\pi}{2} + A_6\omega^{q_2}\cos\frac{q_2\pi}{2} + +A_7\omega^{q_3}\cos\frac{q_3\pi}{2} + A_{11}$$

$$\eta = \omega^{q_1+q_2+q_3}\sin\frac{(q_1+q_2+q_3)\pi}{2} + A_1\omega^{q_1+q_2}\sin\frac{(q_1+q_2)\pi}{2} + A_2\omega^{q_2+q_3}\sin\frac{(q_2+q_3)\pi}{2} + A_3\omega^{q_1+q_3}\sin\frac{(q_1+q_3)\pi}{2} + A_5\omega^{q_1}\sin\frac{q_1\pi}{2} + A_6\omega^{q_2}\sin\frac{q_2\pi}{2} + +A_7\omega^{q_3}\sin\frac{q_3\pi}{2}$$

$$\phi = A_4\omega^{q_1+q_2}\cos\frac{(q_1+q_2)\pi}{2} + A_8\omega^{q_1}\cos\frac{q_1\pi}{2} + A_9\omega^{q_2}\cos\frac{q_2\pi}{2} + A_{10}$$

$$\psi = A_4\omega^{q_1+q_2}\sin\frac{(q_1+q_2)\pi}{2} + A_8\omega^{q_1}\sin\frac{q_1\pi}{2} + A_9\omega^{q_2}\sin\frac{q_2\pi}{2}$$

Squaring and adding (4), we assured that the trigonometric identity $\cos^2\omega\tau_1 + \sin^2\omega\tau_1 = 1$ of LHS has at least one positive real root. Substitute $\mu, \eta, \phi$ and $\psi$ in the RHS of our observations, we can find the critical value by using the following parameterised equation

$$\omega^{2(q_1+q_2+q_3)} + Q_1\omega^{2(q_1+q_2)+q_3} + Q_2\omega^{2q_1+q_2+2q_3} + Q_3\omega^{q_1+2q_2+2q_3} + Q_4\omega^{q_1+2q_2+q_3} + +Q_5\omega^{q_1+q_2+2q_3} + Q_6\omega^{2q_1+q_2+q_3} + Q_7\omega^{q_1+q_2+q_3} + Q_8\omega^{2(q_1+q_2)} + Q_9\omega^{2(q_2+q_3)} + Q_{10}\omega^{2(q_1+q_3)} + Q_{11}\omega^{2q_1+q_2} + Q_{12}\omega^{q_1+2q_2} + Q_{13}\omega^{2q_1+q_3} + Q_{14}\omega^{q_1+2q_3} + Q_{15}\omega^{2q_2+q_3} + Q_{16}\omega^{q_2+2q_3} + +Q_{17}\omega^{q_1+q_2} + Q_{18}\omega^{q_1+q_3} + Q_{19}\omega^{q_2+q_3} + Q_{20}\omega^{2q_1} + Q_{21}\omega^{2q_2} + Q_{22}\omega^{2q_3} + Q_{23}\omega^{q_1} + Q_{24}\omega^{q_2} + Q_{25}\omega^{q_3} = 0 \quad (6)$$

where

$Q_1 = 2A_1\cos\left(\frac{q_3\pi}{2}\right)$; $Q_2 = 2A_3\cos\left(\frac{q_2\pi}{2}\right)$; $Q_3 = 2A_3\cos\left(\frac{q_1\pi}{2}\right)$ ;

$Q_4 = 2(A_6\cos\left(\frac{(q_1+q_3)\pi}{2}\right) + A_1A_2\cos\left(\frac{(q_1-q_3)\pi}{2}\right)$; $Q_5 = 2(A_7\cos\left(\frac{(q_1+q_2)\pi}{2}\right) + A_3A_2\cos\left(\frac{(q_1-q_2)\pi}{2}\right))$;

$Q_6 = 2(A_5\cos\left(\frac{(q_3+q_2)\pi}{2}\right) + A_3A_1\cos\left(\frac{(q_2-q_3)\pi}{2}\right))$

$Q_7 = 2(A_1A_7\cos\left(\frac{(q_1+q_2-q_3)\pi}{2}\right) + A_2A_5\cos\left(\frac{(q_3+q_2-q_1)\pi}{2}\right) + A_3A_6\cos\left(\frac{(q_1+q_3-q_2)\pi}{2}\right)$;

$Q_8 = A_1^2 - A_4^2$; $Q_9 = A_2^2$ ; $Q_{10} = A_3^2$ ; $Q_{11} = 2(A_1A_5\cos\left(\frac{q_2\pi}{2}\right) - A_4A_8\cos\left(\frac{q_2\pi}{2}\right))$ ;

$Q_{12} = 2(A_1A_6\cos\left(\frac{q_1\pi}{2}\right) - A_4A_9\cos\left(\frac{q_1\pi}{2}\right))$; $Q_{13} = 2\left(A_3A_5\left(\cos\left(\frac{(q_1+q_2)\pi}{2}\right) + \sin\left(\frac{(q_1+q_2)\pi}{2}\right)\right)\right)$;

$Q_{14} = 2A_3A_7\cos\left(\frac{q_1\pi}{2}\right)$; $Q_{15} = 2A_2A_6\cos\left(\frac{q_3\pi}{2}\right)$; $Q_{16} = 2A_2A_7\cos\left(\frac{q_2\pi}{2}\right)$;

$Q_{17} = 2(A_1A_{11}\cos\left(\frac{(q_1+q_2)\pi}{2}\right) - A_4A_{10}\cos\left(\frac{(q_1+q_2)\pi}{2}\right) - A_8A_9\cos\left(\frac{(q_1-q_2)\pi}{2}\right))$;

$Q_{18} = 2(A_5A_7\cos\left(\frac{(q_1-q_3)\pi}{2}\right) - A_3A_{11}\cos\left(\frac{(q_1+q_3)\pi}{2}\right))$;

$Q_{19} = 2(A_2A_{11}\cos\left(\frac{(q_2+q_3)\pi}{2}\right) + A_6A_7\cos\left(\frac{(q_2-q_3)\pi}{2}\right))$; $Q_{20} = A_5^2 - A_8^2$ ; $Q_{21} = A_6^2 - A_9^2$ ;

$Q_{22} = A_7^2$ ; $Q_{23} = 2A_5A_{11}\cos\left(\frac{q_1\pi}{2}\right)$ ; $Q_{24} = 2A_{11}A_7\cos\left(\frac{q_3\pi}{2}\right)$;

$Q_{25} = 2(A_6A_{11}\cos\left(\frac{q_2\pi}{2}\right) - A_9A_{10}\cos\left(\frac{q_2\pi}{2}\right))$; $Q_{26} = A_{11}^2 - A_{10}^2$

Define the bifurcation point

$$\tau^*_0 = \min\left(\tau_1^{(n)}, \tau_2^{(n)}\right), \quad n = 0,1,2,\ldots \quad (7)$$

From (2), it is defined as

$P_1 = A_1 + A_2 + A_3 + A_4$ , $P_2 = A_5 + A_6 + A_7 + A_8 + A_9 + A_{10}$ and $P_3 = A_{11}$

To establish the stability of system (A) when $\tau = 0$ we address the following assumption: Hypothesis ($H_2$): $P_1 > 0$, $P_1P_2 > P_3$, $P_3 > 0$.

**Lemma 1:** When $\tau = 0$, the positive equilibrium point (x*, y*, z*) of the fractional order system with (A) is asymptotically stable if ($H_2$) holds.

**Proof:** When time delay dissolves then the characteristic equation (2) becomes

$$\lambda^{q_1+q_2+q_3} + P_1(\lambda^{q_1+q_2} + \lambda^{q_2+q_3} + \lambda^{q_1+q_3}) + P_2(\lambda^{q_1} + \lambda^{q_2} + \lambda^{q_3}) + P_3 = 0 \quad (8)$$

According to the hypothesis ($H_2$), it is easy to verify from Routh – Hurwitz criterion that the two characteristic roots of equation (8) have negative real parts. Hence, the positive equilibrium point (x*, y*, z*) of the fractional system (C) is asymptotically stable when $\tau = 0$.

The conditions obtained in Lemma 1 are only sufficient conditions, not necessarily one. It is necessary to assure that all the roots of equation (8) satisfy $|\arg(\lambda_i)| > \frac{q_i\pi}{2}$ where i= 1, 2, 3 then the Lemma 2 may hold.

We need the following additional assumption to get the transversal condition of the existence for Hopf bifurcation is useful and necessary:

Hypothesis ($H_3$) $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)|_{\tau=\tau_0,\ \omega=\omega_0} = \frac{\alpha_1+i\beta_1}{\alpha_2+i\beta_2} \neq 0.$

**Lemma 2:**

Let $s(\tau) = \rho(\tau) + i\theta(\tau)$ be the roots of equation (2) satisfying $\rho(\tau_i) = 0$, $\theta(\tau_i) = \omega_0$ , then the transversality condition holds.

$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)|_{\tau=\tau_0,\ \omega=\omega_0} \neq 0,$

where $\tau_0$, $\omega_0$ are the bifurcation point and critical frequency.

**Proof:** Differentiate (2) with respect to $\lambda$ and $\tau$ and put the eigen value $\lambda = i\omega$ then we get

$$\frac{\lambda'}{\tau'} = \frac{\alpha_1+i\beta_1}{\alpha_2+i\beta_2} \quad (9)$$

Where

**Remark 1:**

$\alpha_1 = \omega^{q_1+q_2+1}\left(A_4\cos\left(\frac{(q_1+q_2-1)\pi}{2} - \omega\tau_0\right) + \omega^{q_1+1}A_8\cos\left(\frac{(q_1+1)\pi}{2} - \omega\tau_0\right)\right) + \omega^{q_2+1}A_9\cos\left(\frac{(q_2+1)\pi}{2} - \omega\tau_0\right)$

$\beta_1 = \omega^{q_1+q_2+1}\left(A_4\sin\left(\frac{(q_1+q_2-1)\pi}{2} - \omega\tau_0\right) + \omega^{q_1+1}A_8\sin\left(\frac{(q_1+1)\pi}{2} - \omega\tau_0\right)\right) + \omega^{q_2+1}A_9\sin\left(\frac{(q_2+1)\pi}{2} - \omega\tau_0\right) + A_9\sin\omega\tau_0$

$\alpha_2 = (q_1 + q_2 + q_3)\omega^{q_1+q_2+q_3-1}\cos\left(\frac{(q_1+q_2+q_3-1)}{2}\right)\pi + A_1(q_1 + q_2)\omega^{q_1+q_2-1}\cos\left(\frac{q_1+q_2-1}{2}\right)\pi + A_2(q_2 + q_3)\omega^{q_2+q_3-1}\cos\left(\frac{(q_2+q_3-1)\pi}{2}\right) + A_3(q_1 + q_3)\omega^{q_1+q_3-1}\cos\left(\frac{(q_1+q_3-1)\pi}{2}\right) + A_4(q_1 + q_2)\omega^{q_1+q_2-1}(\cos\left(\frac{(q_1+q_2-1)\pi}{2}\right) - \omega\tau_0) + A_4(q_1 + q_2)\tau\omega^{q_1+q_2}(\cos\left(\frac{(q_1+q_2-1)\pi}{2}\right) - \omega\tau_0) + A_5q_1\omega^{q_1-1}\cos\left(\frac{(q_1-1)\pi}{2}\right) + A_6q_2\omega^{q_2-1}\cos\left(\frac{(q_2-1)\pi}{2}\right) + A_7q_3\omega^{q_3-1}\cos\left(\frac{(q_3-1)\pi}{2}\right) + A_8q_1\omega^{q_1-1}(\cos\left(\frac{(q_1-1)\pi}{2}\right) -$

$\omega\tau_0) + A_8\tau q_1\omega^{q_1}(\cos\left(\frac{q_1\pi}{2}\right) - \omega\tau_0) + A_9 q_2\omega^{q_2-1}(\cos\left(\frac{(q_2-1)\pi}{2}\right) - \omega\tau_0) + A_2\tau q_2\omega^{q_2}(\cos\left(\frac{q_2\pi}{2}\right) -$
$\omega\tau_0) - A_{10}\tau\cos\omega\tau_0$

$\beta_2 \;= (q_1 + q_2 + q_3)\omega^{q_1+q_2+q_3-1}\sin\left(\frac{(q_1+q_2+q_3-1}{2}\right)\pi + A_1(q_1 + q_2)\omega^{q_1+q_2-1}\sin\left(\frac{q_1+q_2-1}{2}\right)\pi +$
A2(q2+q3)ωq2+q3−1sin(q2+q3−1)π2+A3(q1+q3)ωq1+q3−1sin(q1+q3−1)π2+A4(q1+q2)ωq1+q2−1(sin(q1+q2−1)π2−ωτ0)+
A4(q1+q2)τωq1+q2(sin(q1+q2−1)π2−ωτ0)+A5q1ωq1−1sin(q1−1)π2+A6q2ωq2−1sin(q2−1)π2+A7q3ωq3−1sin(q3−1)π2+A8q1ωq1−1(sin(q1−1)π2−ωτ0)+A8τq1ωq1(sinq1π2−ωτ0)+A9q2ωq2−1(sin(q2−1)π2−ωτ0)+A2τq2ωq2(sinq2π2−ωτ0)+A10τsinωτ0

Then $\quad \mathrm{Re}(\frac{d\lambda}{d\tau})|_{\tau=\tau_0,\ \omega=\omega_0} = \frac{\alpha_1\alpha_2+\beta_1\beta_2}{{\alpha_2}^2+{\beta_2}^2} \neq 0.$
Hence $(H_3)$ states the transversality condition holds.
As the above mentioned Lemma 1 and Lemma 2, the following theorem can be as concluded.

**Theorem 1:**
If $(H_1)$ - $(H_3)$ holds for system (C), the following results can be determined.
(1) The zero equilibrium point is asymptotically stable for $\tau \in [0, \tau_0)$.
(2) The system (C) undergoes a Hopf bifurcation at the origin when $\tau = \tau_0$, i.e., system (C) has a branch of periodic solution bifurcating from the zero equilibrium point near $\tau = \tau_0$.

**Remark 2.**
It needs to be underlined that the impact of time delay as bifurcation point on the stability, bifurcation phenomenon of proposed predator-prey system (C) was discussed in [14]. It is clear that the control scheme applied on bifurcation did not be introduced in fractional order delayed system. In this work, a hybrid controller is designed to be an effective stabilizer of the commencement of delayed fractional order predator-prey model with different orders.

**Remark 3.**
It is well-known that a linear controller was introduced to control the Hopf bifurcation and improve the stability for the fractional incommensurate order of delayed predator prey system [5]. The usage of hybrid controller to postpone the onset of Hopf bifurcation for fractional order delayed regulatory network [7] and it is the first time the hybrid controller applied to control the Hopf bifurcation in efficient manner of fractional delayed predator-prey system with incommensurate order.

**Remark 4.**
Compare with the existing integer order in the effect of bifurcation control, the proposed fractional order delayed predator-prey system are more preferable.

## V. NUMERICAL SIMULATION

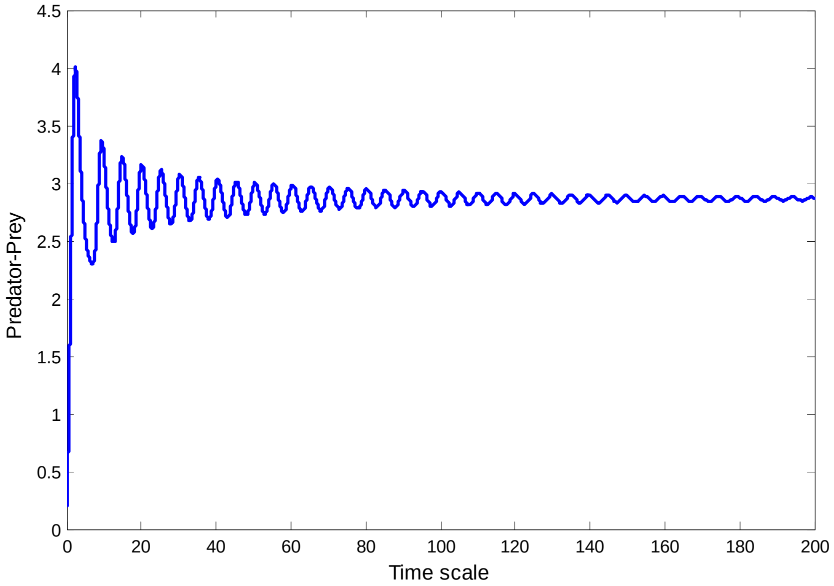
In this section, we provide a numerical example to examine the efficiency and optimization of our theoretical results using Adama-Bashforth-Moulton predictor corrector method followed by the formulation stated in a predictor-corrector scheme for solving nonlinear delay differential equations of fractional order by the authors S.Bhalekor and D.Varsha and select the step length h = 0.01. The parameters represented to the following predator-prey model (C) with different order mentioned as $q_1, q_2$ and $q_3$.

$$D^{q_1}x = x(t)\left(15 - 16x(t) - \frac{5z(t-\tau)}{1+0.1x(t)}\right)$$
$$D^{q_2}x = \frac{3x(t)z(t-\tau)}{1+0.1x(t)} - (\tfrac{1}{8})y(t) - y(t) \quad (10)$$
$$D^{q_3}x = (1-\alpha)\left[y(t) - (\tfrac{1}{8})z(t)\right] + \alpha(z(t-\tau) - z^*)$$

At first, the positive equilibrium point for the system (C) can be calculated as (x*, y*, z*) = (0.0471, 0.3578, 2.8627). In integer order system, we are selected at $q_1 = q_2 = q_3 = 1$, then the system (C) has been computed for the interval $\alpha \in [-0.8, 0)$. We exhibit the impact of $\alpha$ on $\tau$ in system (10) which is exposed in Table 2.The origin of bifurcation can be postponed when the feedback gain $\alpha$ increases is displayed in Table 2. At integer order delayed predator-prey system, we suggest at $\alpha = -0.8$ in Table 2, we obtain the critical frequency $\omega = 2.582$ then the bifurcation point $\tau = 0.2821$. As stated in Theorem 1, the positive equilibrium point (x*, y*, z*) is asymptotically stable when $\tau = 0.23 < \tau_0 = 0.2821$ is shown in the Fig 1. The Hopf bifurcation occurs when $\tau = 0.3 > \tau_0 = 0.2821$ replicated in Fig 2. at integer order system.

**Table 2**: The proportion values between feedback gain $\alpha = [-0.8,0)$ and Bifurcation point $\tau_0$.

| Fractional order $(q_1, q_2, q_3)$ | Feedback gain $(\alpha)$ | Critical frequency $(\omega_0)$ | Bifurcation point $(\tau_0)$ |
|---|---|---|---|
| (1,1,1) | -0.70<br>-0.80 | 2.7495<br>2.5820 | 0.4604<br>0.2821 |



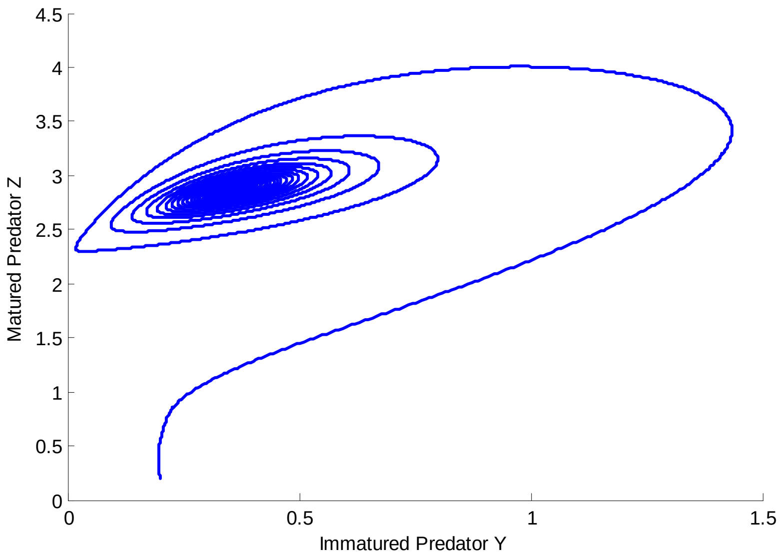




**Fig 1:** State and Phase diagrams represent the stability concept of Predator-Prey system in integer order system when control α=-0.8 at $\tau = 0.23 < \tau_0 = 0.2821$

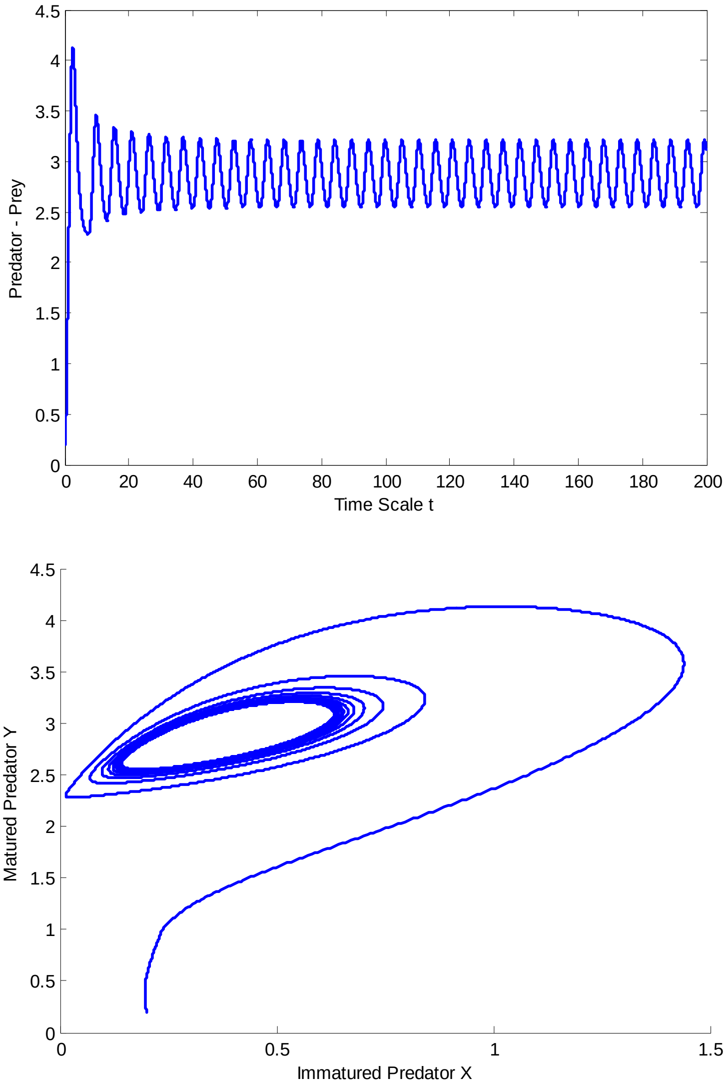



**Fig 2:** State and Phase diagrams represent the Hopf Bifurcation of Predator-Prey system in integer order system when control α=-0.8 at $\tau = 0.295 > \tau_0 = 0.2821$

**Bifurcation analysis in incommensurate order**

In fractional delayed predator-prey system, we select incommensurate order such as $q_1 = 0.92$, $q_2 = 0.9$ and $q_3 = 1$. As framed like integer order system, we set the feedback gain $\alpha \in [-0.8,0)$. In Table 3 we exhibit the proportions of the impact of α on τ in system (10). Onset of bifurcation should be postponed when the feedback gain α increases is displayed in Table 3. In Table 3,it shows at$\alpha = -0.8$, we obtain the bifurcation point $\tau_0 = 0.4326$ corresponding to the critical value $\omega = 1.5679$. By Theorem 1, the positive equilibrium point (x*, y*, z*) is asymptotically stable when$\tau = 0.395 < \tau_0 = 0.4326$ is shown in the Fig 3. The Hopf bifurcation occurs when $\tau = 0.44 > \tau_0 = 0.4326$ replicated in Fig 4. At last Fig 5 and 6 represents the uncontrollable Predator-Prey system with integer and fractional order.

**Table 3:** The proportion values between feedback gain $\alpha = [-0.8, 0)$ and Bifurcation point τ.

| Fractional order $(q_1, q_2, q_3)$ | Feedback gain (α) | Critical frequency (ω) | Bifurcation point (τ) |
|---|---|---|---|
| (0.92,0.93,1) | -0.70<br>-0.80 | 1.5004<br>1.4723 | 0.2871<br>0.4326 |

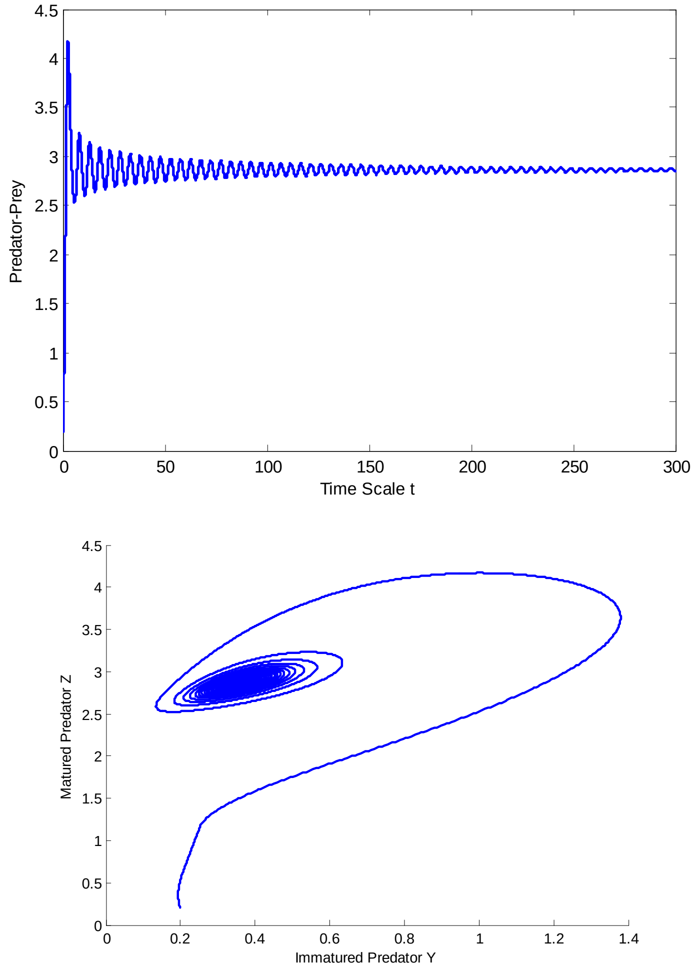
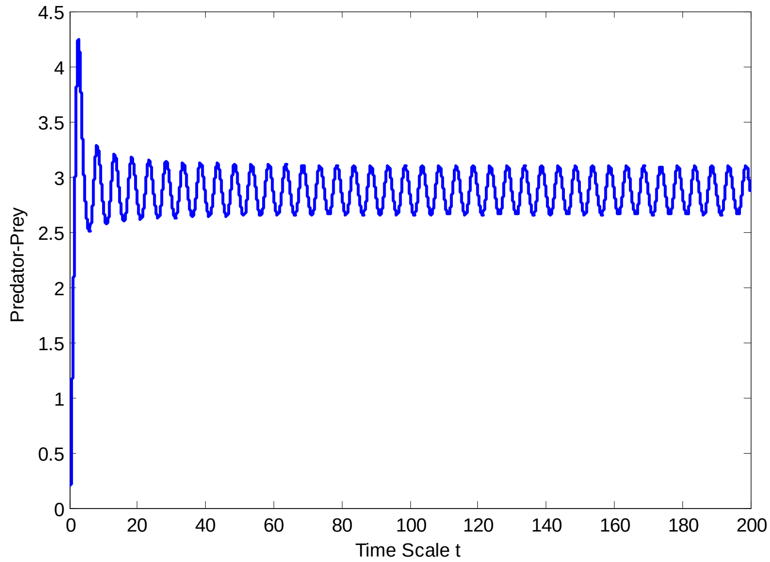



**Fig 3:** State and Phase diagrams represent the Stability analysis of controlled Predator-Prey system in Fractional order system (0.92, 0.93, 1) when α=-0.8 at $\tau = 0.395 < \tau_0 = 0.4326$



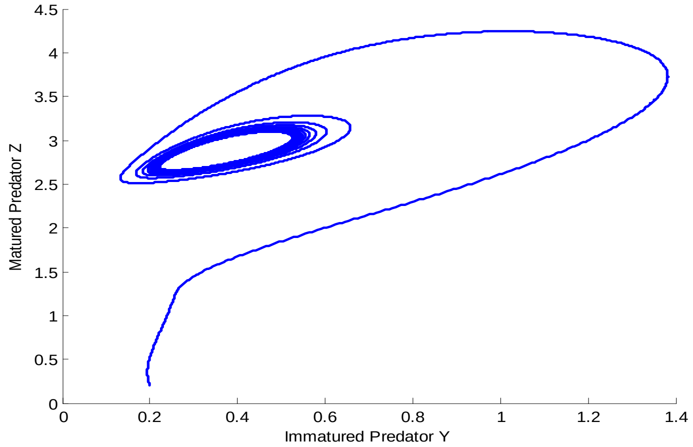



**Fig 4:** State Phase diagrams represent the Hopf Bifurcation of Predator-Prey system in Fractional order system (0.92, 0.93, 1) when α = -0.8 at $\tau = 0.44 > \tau_0 = 0.4321$

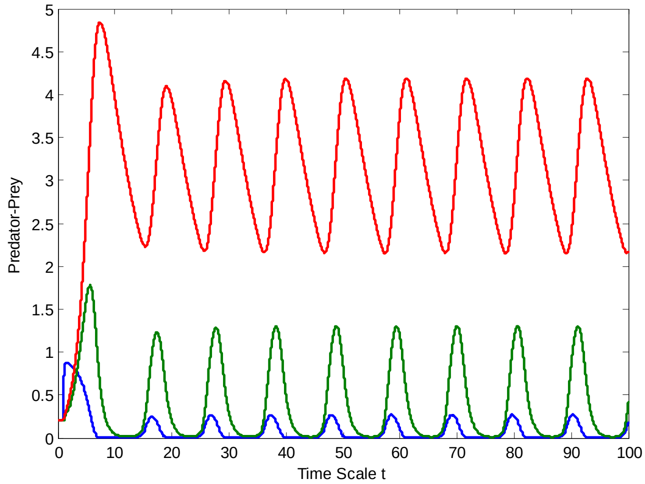



**Fig 5:** State diagram represent the Hopf Bifurcation of uncontrolled Predator-Prey system in integer order when $\tau = 0.9 > 0.8057$

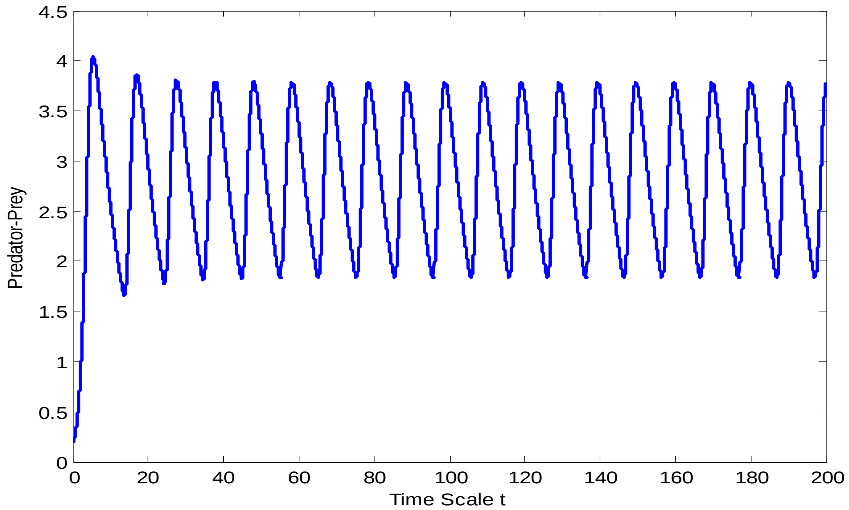



**Fig 6:** State diagram represent the Hopf Bifurcation of uncontrolled Predator-Prey system in Fractional order (0.92, 0.93, 1) when $\tau = 0.2 > 0.1963$

## VI. CONCLUSION

We have developed a theoretical framework that includes accurate description of the dynamical contribution of multi-species interaction. In fractional order delayed predator-prey system, the Hopf bifurcation has been controlled applied by active hybrid control. The conditions for emergence of Hopf bifurcation have been derived with the help of hybrid controller that control bifurcation of uncontrolled system efficiently. The stability concepts of predator-prey system with active control in a delayed fractional order for both commensurate and incommensurate order are examined. It has established that feedback gain has significant influence on dynamical behaviours. The obtained constraints are crispy, accuracy and easy to be validated simultaneously. Finally, some numerical examples have been addressed to validate the efficiency of our obtained theoretical results.

## REFERENCE

[1]. Chengdai Huang, Jinde Cao, Min Xiao, Ahmed Alsaedi and Fuad E.Alsaadi, Controlling bifurcation in a delayed fractional predator-prey system with incommensurate orders, Applied Mathematics and Computation, 293(2017), 293-310.
[2]. Deng.L, Wang. X and Peng. M, Hopf bifurcation analysis for a ratio-dependent predator prey system with two delays and stage structure for the predator, Applied Mathematics and Computation, 231(2014), 214-230.
[3]. Gjurchinovski.A, Sandev.T and Urumov.V, Delayed feedback control of fractional-order chaotic systems, Journal of Physics A: Mathematical and Theoretical, 43(2010), 1-17.
[4]. Hilfer. R, Applications of fractional calculus in physics, World Scientific, Singapore, 2000.
[5]. Hu. H and Huang. L, Stability and Hopf bifurcation in a delayed predator-prey system with stage structure for prey, Nonlinear Analysis: Real World Applications, 11(2010), 2757-2769.
[6]. Huang C.D, Cao J.D, Xiao M., Alsaedi A and Alsaadi F.E, Dynamical analysis of a tri-neuron fractional network, Asian journal of Control, 19(2017), 293–310.
[7]. Huang C.D, Cao J.D and Xiao M, Hybrid control on bifurcation for a delayed fractional gene regulatory network, Chaos, Solitons and Fractals, 87(2016), 19-29.
[8]. Laskin.N, Fractional quantum mechanics, Physical Review E, 62(2000), 3135–3145.
[9]. Lingzhi Zhao, Beibei Shi and Min Xiao, Hopf Bifurcation in a delayed two-neuron fractional network with incommensurate-order, Intelligent Computing, Networked Control and their Engineering Applications, 762(2017), 477-487.
[10]. Sivakumar.D, Loganathan.C and Prakash.M, Dynamical analysis of a fractional order delayed Prey-Predator model with stage structure, International Journal of Mathematics and its Applications, 2(2018), 261–271.
[11]. Sivakumar.D and Loganathan.C, Fractional order delayed Predator-Prey model with stage structure, Annals of Pure and Applied Mathematics, 1(2018), 113-124.
[12]. Podlubny.A, Fractional differential Equation, Academic Press, Cambridge, 1999.
[13]. Reyes Melo.E, Martinez Vega. J, Guerrero-Salazar. C and Otiz-Mendez.U, Application of fractional calculus to the modelling of dielectric relaxation phenomena in polymeric materials, Journal of Application in Polymer Science, 98(2005), 923-935.
[14]. Rihan.F.A, Lakshmanan.S, Hashish.A.H, Rakkiyappan.R and Ahmed.E, Fractional-order delayed predator-prey system with Holling type II functional response, Nonlinear Dynamics, 80(2015), 409-416.
[15]. Wang.X.D, Peng.M and Lin.X.Y, Stability and bifurcation analysis of a ratio-dependent predator prey with two time delays and Holling type III functional response, Applied Mathematics and Computation, 268(2015), 495-508.
[16]. Weigang Zhou, Chengdai Huang, Min Xiao and Jinde Cao, Hybrid tactics for bifurcation control in a fractional-order delayed predator–prey model, Physica A: Statistical Mechanics and its Applications, 515(2019), 183-191.
[17]. Weihua Deng, Changpin Li and Jinhu Lu, Stability analysis of linear fractional differential system with multiple time delays, Nonlinear Dynamics, 48 (2007), 409-416.

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