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Formulation of Direct Force Method

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ABSTRACT
In this paper basis of direct force method is developed and presented in a concise and step by step way. Though direct force method has no edge over direct stiffness method, yet it is of a great academic interest for the scholars working on matrix methods of analysis. There can be three ways of selection of redundants to analyze the indeterminate structures by force method. A structure may have all the external redundants or it may have all the internal redundants. Still there is third probability that it may have mixed redundants i.e., external as well as internal redundants. These three cases require separate formulations. This paper presents all the three formulations.
Keywords–analysis, force method, formulation, matrix methods, redundants

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I. INTRODUCTION

W. K. Nukulchai [1] has given the two governing equations which are used to determine redundants and associated displacements in a joint loaded indeterminate structure.

$$B_X^t f .B_X \ X = x + D_X^t q - B_X^t V_* - B_X^t f .B_R \ R \qquad 1.1$$

$$r = B_R^t .V_* - D_R^t .q + B_R^t .f .B_R .R + B_R^t .f .B_X .X \qquad 1.2$$

The indeterminate structure is rendered to a basic stable and determinate structure in which X denotes the unknown redundants, f is structure flexibility matrix composed of member flexibilities; and x, q and r are kinematics respectively corresponding to the statics X the redundants, Q the reactions and R the other nodal loads. The member deformations due to temperature changes and misfits are denoted by V_*. The determinate structure is analyzed for each redundant applied as a unit load and each nodal load applied again as a unit load. The four equilibrium matrices B_X, B_R, D_X and D_R in above equations are thus obtained from this analysis which have been defined by the following relationships.

$$S_X = B_X .X \qquad 1.3$$

$$S_R = B_R .R \qquad 1.4$$

$$Q_X = D_X .X \qquad 1.5$$

$$Q_R = D_R .R \qquad 1.6$$

In (1.3) to (1.6) S_X denotes member forces due to redundants X in the basic determinate structure and S_R is vector of forces due to applied joint loads R, whereas Q_X and Q_R respectively denote reactions in basic determinate structure due to X and R. It may be noted that the four matrices B_R, B_X, D_R and D_X are determined by analyzing the basic determinate structure subjected to unit nodal loads and unit redundants. The details of the method can be found elsewhere [2]. Three types of strategies are available in analysis of indeterminate framed structures, depending upon the prevailing combinations of nodes, elements and support restraints; first being a solution where all the redundants are external loads that is reactions, second where all the redundants are internal member forces and third where a mixture of external reactions and internal forces is taken as redundants In the forthcoming sections the three approaches of solution by direct force method are described.

II. ANALYSIS OF INDETERMINATE STRUCTURES

Let S^e denote the vector of Element Actions or element forces in its local coordinates. It may be noted that axial force in a truss element is element action/force for this type of element and it is considered positive if it is tensile force. Similarly for a beam element by neglecting axial deformation the two end moments may be taken as element actions. These moments are considered positive if according to the right hand rule the double headed arrow follows the positive direction of the axis the moment is acting about. The element actions/ forces rise to three in case of general flexure element in a plane and the set of forces is combination of the element forces of truss and beam elements. The element actions for a grid element may consist of 3 actions namely one axial torsion, and two end moments. Similarly for a flexure element in space, it may be a set of 6 actions consisting of axial force, axial torsion, and four end moments.

Let $\underset{\sim}{p}^e$ and $\underset{\sim}{d}^e$ respectively denote the nodal forces and displacements of an element in its local coordinate system, and $\underset{\sim}{P}^e$, $\underset{\sim}{D}^e$ respectively denote the nodal forces and displacement of the element in structure or global coordinate system. If the two displacements are related by the following relationship where $\underset{\sim}{a}$ is a transformation matrix.

$$\underset{\sim}{d}^e = \underset{\sim}{a} \, . \, \underset{\sim}{D}^e \qquad 2.1$$

Then the element forces in the two coordinates are can be related using contra-gradient law.

$$\underset{\sim}{P}^e = \underset{\sim}{a}^t . \underset{\sim}{p}^e \qquad 2.2$$

Let $\underset{\sim}{h}^e$ be an equilibrium matrix which associates the element forces $\underset{\sim}{S}_e$ with nodal forces $\underset{\sim}{p}^e$.

$$\underset{\sim}{p}^e = \underset{\sim}{h}^e . \underset{\sim}{S}^e \qquad 2.3$$

Similarly the relationship between $\underset{\sim}{P}^e$ and $\underset{\sim}{S}^e$ can be setup using another equilibrium matrix $\underset{\sim}{H}^e$ of the same element in global coordinate system.

$$\underset{\sim}{P}^e = \underset{\sim}{H}^e . \underset{\sim}{S}^e \qquad 2.4$$

It can be shown that the two element equilibrium matrices $\underset{\sim}{H}^e$ and $\underset{\sim}{h}^e$ are associated to each other through the transformation matrix $\underset{\sim}{a}$.

$$\underset{\sim}{H}^e = \underset{\sim}{a}^t \, \underset{\sim}{h}^e \qquad 2.5$$

The nodal load-element force relationship for a structure shall be essentially extension of (2.4) and can be written as:

$$\underset{\sim}{P} = \underset{\sim}{H} . \underset{\sim}{S} \qquad (2.6)$$

The symbol $\underset{\sim}{P}$ in (2.6) is vector of nodal loads including all loads corresponding to free nodal coordinates and reactions corresponding to restrained nodal coordinates of a structure; $\underset{\sim}{H}$ is structure equilibrium matrix and $\underset{\sim}{S}$ is vector of element forces. The equilibrium matrix $\underset{\sim}{H}$ can be set up by assembling element equilibrium matrices $\underset{\sim}{H}^e$

Three types of structures may be encountered during analysis of indeterminate structures depending upon the prevailing combinations of nodes, elements and support restraints; first being a solution where all the redundants are external in nature or reactions, second where all the redundants are internal element forces and third type where a mixture of external reactions and internal element forces have to be taken as redundants In the forthcoming sections the formulation of three types of solutions by direct force method is described.

III. ALL EXTERNAL REDUNDANTS

The nodal forces in this case consist of three independent sources; the loads at free nodal directions $\underset{\sim}{R}$, the reactions selected as redundants $\underset{\sim}{X}$ at the chosen external restraints which are released to make the structure determinate, and the reactions $\underset{\sim}{Q}$ corresponding to the remaining external restraints which are just necessary and sufficient to suppress the rigid body motion of the released structure. The nodal force vector $\underset{\sim}{P}$ is partitioned properly so that the three type of nodal loads are separated and ordered as given by (3.1):

$$\underset{\sim}{P} = \begin{bmatrix} \underset{\sim}{R} \\ \underset{\sim}{X} \\ \underset{\sim}{Q} \end{bmatrix} \qquad 3.1$$

It may be noted from statics that number of the combined components in the first two vectors, $\underset{\sim}{R}$ and $\underset{\sim}{X}$ is just equal to the element forces $\underset{\sim}{S}$. Let the two vectors $\underset{\sim}{R}$ and $\underset{\sim}{X}$ be combined into a single one denoted by $\underset{\sim}{P}_{RX}$ and the basic relationship given by (2.6) be written as follows

$$\begin{bmatrix} \underset{\sim}{P}_{RX} \\ \underset{\sim}{Q} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{H}_{RX} \\ \underset{\sim}{H}_{Q} \end{bmatrix} . \underset{\sim}{S} \qquad 3.2$$

$\underset{\sim}{H}_{RX}$ in (3.2) is square matrix of the order equal to the number of components in $\underset{\sim}{S}$ and it relates combined vector $\underset{\sim}{P}_{RX}$ (combination of $\underset{\sim}{R}$ and $\underset{\sim}{X}$) to member forces $\underset{\sim}{S}$. The first part of the partitioned equation (3.2) may further be expanded to (3.3) where $\underset{\sim}{H}_R$ is equilibrium matrix that relates $\underset{\sim}{R}$ and $\underset{\sim}{S}$ whereas $\underset{\sim}{H}_X$ is equilibrium matrix that relates $\underset{\sim}{X}$ and $\underset{\sim}{S}$.

$$\begin{bmatrix} \underset{\sim}{R} \\ \underset{\sim}{X} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{H}_{R} \\ \underset{\sim}{H}_{X} \end{bmatrix} . \underset{\sim}{S} \qquad 3.3$$

Before proceeding further, let us refresh the method of analysis of indeterminate structures where all the redundants in this particular formulation are external reactions at the released restraints. In the method a basic determinate structure is selected after removing the restrains which render a stable and determinate structure. This structure will be termed as base structure in this paper. Then unit loads are applied one at a time in turn at all the free nodal coordinates and element forces are obtained, the forces in elements of base structure due to unit forces applied at free nodes in the basic structure are termed as $\underset{\sim}{B}_R$ and the forces produced by unit redundants are denoted by $\underset{\sim}{B}_X$. Similarly the reactions at supports of base structure are denoted by $\underset{\sim}{D}_R$ and $\underset{\sim}{D}_X$ respectively due to two sets of unit forces applied at originally free nodal coordinates and at released nodal directions.

Let us now write the two subsets of partitioned (3.2) independently as follows:

$$\underset{\sim}{P}_{RX} = \underset{\sim}{H}_{RX}.\underset{\sim}{S} \qquad 3.4$$

$$\underset{\sim}{Q} = \underset{\sim}{H}_{Q}.\underset{\sim}{S} \qquad 3.5$$

The matrix $\underset{\sim}{H}_{RX}$ which relates nodal forces combined at originally free nodes and released nodal restraints in basic structure is a non-singular matrix if a proper stable and determinate base structure is selected. Hence inverse of $\underset{\sim}{H}_{RX}$ exists and (3.4) may be rewritten as (3.6)

$$\underset{\sim}{S} = \underset{\sim}{H}^{-1}_{RX}.\underset{\sim}{P}_{RX} \qquad 3.6$$

It is essential to note that $\underset{\sim}{P}_{RX}$ is consisting of forces duly applied at the free nodal coordinates of the base structure and is composed of two parts; $\underset{\sim}{R}$ which is a vector of nodal forces corresponding to originally free nodal coordinates all applied in a proper order and $\underset{\sim}{X}$ which is consisting of nodal reactions corresponding to the released restraints of the structure. Thus another form of (3.6) may be written after partitioning as follows:

$$\underset{\sim}{S} = \begin{bmatrix} \underset{\sim}{H}^{-1}_{*R} & \underset{\sim}{H}^{-1}_{*X} \end{bmatrix} . \begin{bmatrix} \underset{\sim}{R} \\ \underset{\sim}{X} \end{bmatrix} \qquad 3.7$$

$$\begin{bmatrix} \underset{\sim}{H}^{-1}_{*R} & \underset{\sim}{H}^{-1}_{*X} \end{bmatrix} . \begin{bmatrix} \underset{\sim}{H}_{R} \\ \underset{\sim}{H}_{X} \end{bmatrix} = \underset{\sim}{I} \qquad 3.8$$

It may be noted from (3.7) that

$$\underset{\sim}{S} = \underset{\sim}{S}_R + \underset{\sim}{S}_X \qquad 3.9$$

It may be depicted from (3.7) and (3.8) $\underset{\sim}{H}^{-1}_{*R}$ and $\underset{\sim}{H}^{-1}_{*X}$ are not direct inverses of the two matrices $\underset{\sim}{H}_R$ and $\underset{\sim}{H}_X$, rather they are sub matrices obtained after proper partitioning of $\underset{\sim}{H}^{-1}_{RX}$ that is

$$\underset{\sim}{H}^{-1}_{RX} = \begin{bmatrix} \underset{\sim}{H}^{-1}_{*R} & \underset{\sim}{H}^{-1}_{*X} \end{bmatrix} \qquad 3.10$$

Now rewriting the first of the two parts of (3.7):

$$\underset{\sim}{S}_R = \underset{\sim}{H}^{-1}_{*R}.\underset{\sim}{R} \qquad 3.11$$

It may be noted from (3.4) that $\underset{\sim}{S}_R = \underset{\sim}{B}_R.\underset{\sim}{R}$ therefore, $\underset{\sim}{B}_R$ can be obtained from (3.11).

$$\underset{\sim}{B}_R = \underset{\sim}{H}^{-1}_{*R} \qquad 3.12$$

And working on the same lines, the second part of (3.6) is rewritten in order to obtain $\underset{\sim}{B}_X$.

$$\underset{\sim}{S}_X = \underset{\sim}{H}^{-1}_{*X}.\underset{\sim}{X} \qquad 3.13$$

Again it is noted from (1.3) that $\underset{\sim}{S}_X = \underset{\sim}{B}_X.\underset{\sim}{X}$, therefore, $\underset{\sim}{B}_X$ can be obtained from (3.13)

$$\underset{\sim}{B}_X = \underset{\sim}{H}^{-1}_{*X} \qquad 3.14$$

The remaining two equilibrium matrices $\underset{\sim}{D}_R$ and $\underset{\sim}{D}_X$ can be obtained by writing the second part of (3.2)

$$\underset{\sim}{Q} = \underset{\sim}{H}_{Q}.\underset{\sim}{S} \qquad 3.15$$

By substitute the value of element forces from (3.6) into (3.15), (3.16) is obtained.

$$\underset{\sim}{Q} = \underset{\sim}{H}_{Q}.\begin{bmatrix} \underset{\sim}{H}^{-1}_{*R} & \underset{\sim}{H}^{-1}_{*X} \end{bmatrix} . \begin{bmatrix} \underset{\sim}{R} \\ \underset{\sim}{X} \end{bmatrix} \qquad 3.16$$

$$\underset{\sim}{Q} = \underset{\sim}{H}_{Q}.\underset{\sim}{H}^{-1}_{*R}.\underset{\sim}{R} + \underset{\sim}{H}_{Q}\underset{\sim}{H}^{-1}_{*X}\underset{\sim}{X} \qquad 3.17$$

The two matrices $\underset{\sim}{D}_X$ and $\underset{\sim}{D}_R$ can be obtained from (3.17) in view of (1.5) and (1.6) respectively.

$$\underset{\sim}{D}_X = \underset{\sim}{H}_{Q}\underset{\sim}{H}^{-1}_{*X} \qquad 3.18$$

$$\underset{\sim}{D}_R = \underset{\sim}{H}_{Q}\underset{\sim}{H}^{-1}_{*R} \qquad 3.19$$

The force method for all external redundants can now be programmed using any computer language as the four equilibrium matrices have been obtained as given by (3.12), (3.14), 3.18) and (3.19).

IV. ALL INTERNAL REDUNDANTS

This approach is adoptable only in the cases where external restrains are just necessary and sufficient to have a stable and externally determinate structure, and the redundancy is only due to internal forces of the structure.

Let us rewrite the basic relationship between nodal forces and element forces of a structure

$$\underset{\sim}{P} = \underset{\sim}{H}.\underset{\sim}{S} \qquad 4.1$$

The nodal force vector $\underset{\sim}{P}$ in this case is partitioned into two groups, $\underset{\sim}{R}$ the loads belonging to free nodal coordinates and $\underset{\sim}{Q}$ the reactions at the restrained nodal coordinates.

$$\underset{\sim}{P} = \begin{bmatrix} \underset{\sim}{R} \\ \underset{\sim}{Q} \end{bmatrix} \qquad 4.2$$

The base structure, in this case is obtained by releasing the internal restraints as required but in a way to get a stable and determinate base structure. The element forces are subdivided into two classes, one being essential forces $\underset{\sim}{S}^E$ the forces present in the released structure, and the other class is termed as redundant forces $\underset{\sim}{S}^X$. The term essential is coined due to the reason that these element forces are essentially required to give an internally stable base structure. The element forces shall be ordered according to node numbers and coordinate directions but essential forces $\underset{\sim}{S}^E$ shall precede the redundant forces $\underset{\sim}{S}^X$ as shown by (4.3).

$$\underset{\sim}{S} = \begin{bmatrix} \underset{\sim}{S}^E \\ \underset{\sim}{S}^X \end{bmatrix} \qquad 4.3$$

The relationship described by (4.1) can now be rewritten in view of (4.2) and (4.3)

$$\begin{bmatrix} \underset{\sim}{R} \\ \underset{\sim}{Q} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{H}_{RE} & \underset{\sim}{H}_{RX} \\ \underset{\sim}{H}_{QE} & \underset{\sim}{H}_{QX} \end{bmatrix} . \begin{bmatrix} \underset{\sim}{S}^E \\ \underset{\sim}{S}^X \end{bmatrix} \qquad 4.4$$

The convention adopted in notation of equilibrium matrices $\underset{\sim}{H}_{JK}$ is so that it relates loads $\underset{\sim}{J}$ with forces $\underset{\sim}{S}^K$. There will be two sources contributing to each essential force in base structure; one due to applied loads R denoted as S_R^E and the other due to internal redundant forces $X = S^X$ which are denoted by S_X^E. The relationship (4.3) takes the following form in view of the two sources inducing the element forces.

$$S^E = S_R^E + S_X^E \qquad 4.5$$

The two forces on the right hand side of (4.5) are S_R^E essential forces due to applied loads R and S_X^E essential forces due to redundant forces S^X. The first of the two equations in (4.4) is re-written as (4.6).

$$R = H_{RE}.S^E + H_{RX}, S^X \qquad 4.6$$

Substituting (4.5) into (4.6),

$$R = H_{RE}.S_R^E + H_{RE}.S_X^E + H_{RX}, S^X \qquad 4.7$$

It is pertinent to note that there will be no contribution by acting loads to redundant forces when the internal restraints have been released to obtain a base structure. Therefore, the contribution of the last two terms in (4.7) is zero.

$$H_{RE}.S_X^E + H_{RX}, S^X = 0 \qquad 4.8$$

Hence (4.7) is simplified in view of (4.8)

$$R = H_{RE}.S_R^E \qquad 4.9a$$

S_R^E can be obtained from (4.9a).

$$S_R^E = H_{RE}^{-1}.R \qquad 4.9b$$

The internal forces in all members due to nodal loads R in view of (4.3) can be written as:

$$S_R = \begin{bmatrix} S_R^E \\ S_R^X \end{bmatrix} \qquad 4.10$$

The first vector on right hand side of (4.10) consists of essential forces and second consists of redundant forces both due to applied loading R in the base structure. It is a fact that there can be no force in the released restraints S_R^X due to nodal loads R in base structure; hence

$$S_R^X = 0 \qquad 4.11$$

The relationships given by (4.10) in view of (4.9b) and (4.11) can be written as given by (4.12).

$$S_R = \begin{bmatrix} S_R^E \\ S_R^X \end{bmatrix} = \begin{bmatrix} H_{RE}^{-1} \\ 0 \end{bmatrix}.R \qquad 4.12$$

Comparison of (4.12) with (2.4) provides the equilibrium matrix B_R.

$$B_R = \begin{bmatrix} H_{RE}^{-1} \\ 0 \end{bmatrix} \qquad 4.13$$

The total force due to redundants S_X can be divided into two categories on the basis already used in (4.3) and (4.10).

$$S_X = \begin{bmatrix} S_X^E \\ S_X^X \end{bmatrix} \qquad 4.14$$

The first category S_X^E on right hand side of (4.14) represents the essential forces and the second one S_X^X represents the redundant forces due to redundants S^X. The force vector S_X^E can be obtained from (4.8)

$$H_{RE}.S_X^E + H_{RX}.S^X = 0 \qquad 4.8$$

$$S_X^E = -H_{RE}^{-1}.H_{RX}.S^X \qquad 4.15$$

It can be written using the basic definition (2.3) that

$$S_X^E = B_X^E.S^X \qquad 4.16$$

Therefore comparing (4.15) and (4.16), B_X^E can be obtained.

$$B_X^E = -H_{RE}^{-1}.H_{RX} \qquad 4.17$$

When a redundant force is applied to base structure it induces the force equal to itself in that redundant, however, it does not induce any force in any one of the other redundants which have been released to make the structure determinate. Therefore, the force in redundant due to redundant itself can be written by following relationship

$$S_X^X = I.S^X \qquad 4.18$$

I in (4.18) is a square unit matrix of the order of number of redundants. Using the basic definition (2.3), S_X^X can be written as given by (4.19a).

$$S_X^X = B_X^X.S^X \qquad 4.19a$$

Therefore B_X^X can be obtained from comparison of (4.18) and (4.19a).

$$B_X^X = I \qquad 4.19b$$

It is known from (2.3) that $S_X = B_X.X$ and for the solution scheme consisting of all internal redundants $S^X = X$. Using this definition in view of (4.15), (4.17), (4.18), (4.19) along with (4.14), B_X can be obtained.

$$B_X = \begin{bmatrix} B_X^E \\ B_X^X \end{bmatrix} = \begin{bmatrix} -H_{RE}^{-1}.H_{RX} \\ I \end{bmatrix} \qquad 4.20$$

The focus is next turned towards the second set of equations given by (4.4) so that D_X and D_R can be obtained.

$$\begin{bmatrix} R \\ Q \end{bmatrix} = \begin{bmatrix} H_{RE} & H_{RX} \\ H_{QE} & H_{QX} \end{bmatrix} . \begin{bmatrix} S^E \\ S^X \end{bmatrix} \qquad 4.2$$

$$Q = H_{QE}.S^E + H_{QX}.S^X \qquad 4.21$$

The relationship (4.21) in view of (4.5), i.e., $S^E = S_R^E + S_X^E$ can be written as (4.22).

$$Q = H_{QE}.S_R^E + H_{QE}.S_X^E + H_{QX}.S^X \qquad 4.22$$

Substituting (4.9) and (4.15) into (4.22) and regrouping separately for the terms containing $\underset{\sim}{R}$ and $\underset{\sim}{S}^X$, (4.22) takes the following shape.

$$\underset{\sim}{Q} = \underset{\sim}{H}_{QE}.\underset{\sim}{H}_{RE}^{-1}.\underset{\sim}{R} + (\underset{\sim}{H}_{QX} - \underset{\sim}{H}_{QE}.\underset{\sim}{H}_{RE}^{-1}.\underset{\sim}{H}_{RX}).\underset{\sim}{S}^X \quad 4.23$$

However, the total reaction $\underset{\sim}{Q}$ is sum of two reaction vectors $\underset{\sim}{Q}_R$ and $\underset{\sim}{Q}_X$, therefore:

$$\underset{\sim}{Q} = \underset{\sim}{Q}_R + \underset{\sim}{Q}_X \quad 4.24$$

Moreover, $\underset{\sim}{Q}_X$ and $\underset{\sim}{Q}_R$ have already been defined by (1.5) and (1.6) which are reproduced for ready reference.

$$\underset{\sim}{Q}_X = \underset{\sim}{D}_X.\underset{\sim}{X} \quad 1.5$$

$$\underset{\sim}{Q}_R = \underset{\sim}{D}_R.\underset{\sim}{R} \quad 1.6$$

It may be noted that while all the redundants have been taken from internal member forces, the two vector $\underset{\sim}{S}^X$ and $\underset{\sim}{X}$ are identical.

$$\underset{\sim}{S}^X = \underset{\sim}{X} \quad 4.25$$

Comparison of (4.23) and (4.24) in view of (1.5), (1.6) and (4.25) reveals that

$$\underset{\sim}{D}_R = \underset{\sim}{H}_{QE}.\underset{\sim}{H}_{RE}^{-1} \quad 4.26$$

$$\underset{\sim}{D}_X = \underset{\sim}{H}_{QX} - \underset{\sim}{H}_{QE}.\underset{\sim}{H}_{RE}^{-1}.\underset{\sim}{H}_{RX} \quad 4.27$$

The four equilibrium matrices $\underset{\sim}{B}_R$, $\underset{\sim}{B}_X$, $\underset{\sim}{D}_R$ and $\underset{\sim}{D}_X$ have been obtained for the formulation with all internal redundants.

V. GENERAL FORMULATION

This approach is termed as General Formulation and here redundants are a combination of external reactions $\underset{\sim}{X}_e$ and internal forces $\underset{\sim}{S}^X$. Let us rewrite the basic relationship between nodal forces $\underset{\sim}{P}$ and element forces $\underset{\sim}{S}$:

$$\underset{\sim}{P} = \underset{\sim}{H}.\underset{\sim}{S} \quad 5.1$$

The nodal loads shall be consisting of three categories; loads $\underset{\sim}{R}$ corresponding to free nodal coordinates, the external redundants $\underset{\sim}{X}_e$ acting at released external restraints and reactions $\underset{\sim}{Q}$ at the remaining necessary and sufficient restraints to stop the rigid body movement of the structure.

The load vector $\underset{\sim}{P}$ will be arranged in a proper order so that loads $\underset{\sim}{R}$ corresponding to free nodal coordinates are grouped at top, the external redundants X_e are placed next and the reactions in base structure $\underset{\sim}{Q}$ are placed last as given by (5.2).

$$\underset{\sim}{P} = \begin{bmatrix} \underset{\sim}{R} \\ \underset{\sim}{X}_e \\ \underset{\sim}{Q} \end{bmatrix} \quad 5.2$$

The element force vector $\underset{\sim}{S}$ shall be having two categories same as in the case of all internal redundant formulation; the essential element forces $\underset{\sim}{S}^E$ and redundant element forces. $\underset{\sim}{S}^X$.

$$\underset{\sim}{S} = \begin{bmatrix} \underset{\sim}{S}^E \\ \underset{\sim}{S}^X \end{bmatrix} \quad 5.3$$

The relationship (5.1) in view of (5.2) and (5.3) can be expanded into (5.4).

$$\begin{bmatrix} \underset{\sim}{R} \\ \underset{\sim}{X}_e \\ \underset{\sim}{Q} \end{bmatrix} = \begin{bmatrix} H_{RE} & \underset{\sim}{H}_{RX} \\ H_{XeE} & \underset{\sim}{H}_{XeX} \\ H_{QE} & \underset{\sim}{H}_{QX} \end{bmatrix} . \begin{bmatrix} \underset{\sim}{S}^E \\ \underset{\sim}{S}^X \end{bmatrix} \quad 5.4$$

Let the two vectors $\underset{\sim}{R}$ and $\underset{\sim}{X}_e$ be combined temporarily into a single vector denoted by $\underset{\sim}{R}_x$ as shown by (5.5).

$$\underset{\sim}{R}_x = \begin{bmatrix} \underset{\sim}{R} \\ \underset{\sim}{X}_e \end{bmatrix} \quad 5.5$$

Rewrite (5.4) in view of (5.5).

$$\begin{bmatrix} \underset{\sim}{R}_x \\ \underset{\sim}{Q} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{H}_{RxE} & \underset{\sim}{H}_{RxX} \\ \underset{\sim}{H}_{QE} & \underset{\sim}{H}_{QX} \end{bmatrix} . \begin{bmatrix} \underset{\sim}{S}^E \\ \underset{\sim}{S}^X \end{bmatrix} \quad 5.6$$

A comparison of (5.4) with (5.6) in view of (5.5) reveals that

$$\underset{\sim}{H}_{RxE} = \begin{bmatrix} \underset{\sim}{H}_{RE} \\ \underset{\sim}{H}_{Xe} \end{bmatrix} \quad 5.7a$$

$$\underset{\sim}{H}_{RxX} = \begin{bmatrix} \underset{\sim}{H}_{RX} \\ \underset{\sim}{H}_{XeX} \end{bmatrix} \quad 5.7b$$

The two equilibrium matrices on right side of 5.7a relate essential element forces $\underset{\sim}{S}^E$ to the loads $\underset{\sim}{R}$ and external redundants $\underset{\sim}{X}_e$ respectively and those two on right side of 5.7b relate redundant element forces $\underset{\sim}{S}^X$ to the loads $\underset{\sim}{R}$ and external redundants $\underset{\sim}{X}_e$ respectively. When all types of loads are acting at nodes, the following equation holds for essential element forces.

$$\underset{\sim}{S}^E = \underset{\sim}{S}_R^E + \underset{\sim}{S}_{Xe}^E + \underset{\sim}{S}_X^E \quad 5.8$$

The three components on right side of the (5.8) are all essential forces; $\underset{\sim}{S}_R^E$ due to free nodal loads $\underset{\sim}{R}$, $\underset{\sim}{S}_{Xe}^E$ due to external redundants $\underset{\sim}{X}_e$ and $\underset{\sim}{S}_X^E$ due to internal redundant $\underset{\sim}{S}^X$ respectively. The first set of equations given by (5.6) relates nodal loads other than reactions in base structure to all the element forces:

$$\underset{\sim}{R}_x = \underset{\sim}{H}_{RxE}.\underset{\sim}{S}^E + \underset{\sim}{H}_{RxX}.\underset{\sim}{S}^X \quad 5.9$$

Substitution of (5.8) into (5.9) yields

$$\underset{\sim}{R}_x = \underset{\sim}{H}_{RxE}.(\underset{\sim}{S}_R^E + \underset{\sim}{S}_{Xe}^E + \underset{\sim}{S}_X^E) + \underset{\sim}{H}_{RxX}.\underset{\sim}{S}^X$$

The terms on right hand side of above relationship are grouped on the basis of forces due to external and internal sources.

$$\underset{\sim}{R}_x = \underset{\sim}{H}_{RxE}.(\underset{\sim}{S}_R^E + \underset{\sim}{S}_{Xe}^E) + (\underset{\sim}{H}_{RxE}.\underset{\sim}{S}_X^E + \underset{\sim}{H}_{RxX}.\underset{\sim}{S}^X) \quad 5.10$$

After application of release to internal restraints corresponding to the internal redundants $\underset{\sim}{S}^X$, we get

$$H_{RxE}.S_X^E + H_{RxX}.S^X = 0 \qquad 5.11$$

Solving (5.11) for S_X^E

$$S_X^E = -H_{RxE}^{-1}.H_{RxX}.S^X \qquad 5.12$$

Next (5.10) is re-written in view of (5.11).

$$R_x = H_{RxE}.(S_R^E + S_{Xe}^E) \qquad 5.13a$$

(5.13a) can be solved for essential element forces due to loads from external sources.

$$(S_R^E + S_{Xe}^E) = H_{RxE}^{-1}.R_x \qquad 5.13b$$

The above relationship in view of (5.5) takes the following form.

$$S_R^E + S_{Xe}^E = \begin{bmatrix} H_{RE}^{-1*} & H_{Xe}^{-1*} \end{bmatrix}.\begin{bmatrix} R \\ X_e \end{bmatrix} \qquad 5.13c$$

The two categories of essential element forces S_R^E and S_{Xe}^E can be written separately from (5.13c).

$$S_R^E = H_{RE}^{-1*}.R \qquad 5.14a$$

$$S_{Xe}^E = H_{Xe}^{-1*}.X_e \qquad 5.14b$$

The two starred matrices H_{RE}^{-1*} and H_{Xe}^{-1*} in (5.14a) and (5.14b) are not direct inverses of H_{RE} and H_{Xe} referred by (5.7a), rather they are properly partitioned sub-matrices of H_{RxE}^{-1} given by (5.13b) in view of (5.14a), (5.14b). It may be noted that after releasing internal redundants, there will no forces in the redundant elements S^X due to nodal loads R.

$$S_R^X = 0.R \qquad 5.15$$

The forces S_R in all elements whether essential or redundant due to external loads R can be written.

$$S_R = \begin{bmatrix} S_R^E \\ S_R^X \end{bmatrix} = \begin{bmatrix} H_{RE}^{-1*} \\ 0 \end{bmatrix}.R \qquad 5.16$$

It may be noted from (1.4) that $S_R = B_R.R$ therefore the equilibrium matrix B_R can be obtained from (5.16).

$$B_R = \begin{bmatrix} H_{RE}^{-1*} \\ 0 \end{bmatrix} \qquad 5.17$$

It may further be noted from (1.3) that $S_X = B_X.X$, so in order to establish B_X one have to combine all the redundants whether external X_e or internal S^X into a single redundant vector X as follows:

$$X = \begin{bmatrix} S^X \\ X_e \end{bmatrix} \qquad 5.18$$

Let S_X be element forces due to all types of redundants whether internal or external, i.e.

$$S_X = S^X + S_{Xe} = \begin{bmatrix} S_X^E \\ S_X^X \end{bmatrix} + \begin{bmatrix} S_{Xe}^E \\ S_{Xe}^X \end{bmatrix} \qquad 5.19$$

While internal redundants have been released, and unit redundants are applied in turn we have the following relationship between redundant element forces due to internal redundants.

$$S_X^X = I.S^X \qquad 5.20$$

It may be noted that there will be zero force S_{Xe}^X in all internal redundants due to applied external redundants X_e, or

$$S_{Xe}^X = 0.X_e \qquad 5.21$$

Rewriting (5.19)

$$S_X = \begin{bmatrix} S_X^E \\ S_X^X \end{bmatrix} + \begin{bmatrix} S_{Xe}^E \\ S_{Xe}^X \end{bmatrix} \qquad 5.19$$

Substituting S_X^E from (5.12), S_{Xe}^E from (5.14b), S_X^X from 5.20 and S_{Xe}^X from (5.21) into (5.19), the following is obtained.

$$S_X = \begin{bmatrix} -H_{RxE}^{-1}.H_{RxX} \\ I \end{bmatrix}.S^X + \begin{bmatrix} H_{Xe}^{-1*} \\ 0 \end{bmatrix}.X_e$$

The above relationship can be written in the following matrix format as well.

$$S_X = \begin{bmatrix} -H_{RxE}^{-1}.H_{RxX} & H_{Xe}^{-1*} \\ I & 0 \end{bmatrix}.\begin{bmatrix} S^X \\ X_e \end{bmatrix} \qquad 5.22$$

The relationship (5.22) in view of (1.3) i.e., $S_X = B_X.X$ and X as defined by (5.18) provides B_X.

$$B_X = \begin{bmatrix} -H_{RxE}^{-1}.H_{RxX} & H_{Xe}^{-1*} \\ I & 0 \end{bmatrix} \qquad 5.23$$

After obtaining B_R and B_X the focus now should be concentrated on the 2nd set of equations given by (5.6) so that D_R and D_X can be obtained.

$$\begin{bmatrix} R_x \\ Q \end{bmatrix} = \begin{bmatrix} H_{RxE} & H_{RxX} \\ H_{QE} & H_{QX} \end{bmatrix}.\begin{bmatrix} S^E \\ S^X \end{bmatrix} \qquad 5.6$$

$$Q = H_{QE}.S^E + H_{QX}.S^X$$

Substituting S^E from (5.8) into above relationship, reactions Q are determined.

$$Q = H_{QE}.(S_R^E + S_{Xe}^E + S_X^E) + H_{QX}.S^X \qquad 5.24$$

In the second round substitutions are made into (5.24) for S_X^E, S_R^E and S_{Xe}^E from (5.12), (5.14a) and (5.14b) respectively.

$$Q = H_{QE}.(H_{RE}^{-1*}.R + H_{Xe}^{-1*}.X_e - H_{RxE}^{-1}.H_{RxX}.S^X) + H_{QX}.S^X$$

Which, after rearrangement of the terms can be written as

$$Q = \{H_{QE}.H_{RE}^{-1*}.R\} + \{H_{QE}.H_{Xe}^{-1*}.X_e + (H_{QX} - H_{QE}.H_{RxE}^{-1}.H_{RxX}.)S^X\} \qquad 5.25$$

The two parts of (5.25) grouped by {} are obvious and can be written as

$$Q = Q_R + Q_X = D_R.R + D_X.X \qquad 5.26$$

Therefore from comparison of 5.25 and 5.26

$$\underset{\sim}{Q}_R = \underset{\sim}{D}_R.\underset{\sim}{R} = \underset{\sim}{H}_{QE}.\underset{\sim}{H}_{RE}^{-1*}.\underset{\sim}{R} \qquad 5.27$$

$$\underset{\sim}{Q}_X = (\underset{\sim}{H}_{QX} - \underset{\sim}{H}_{QE}.\underset{\sim}{H}_{RxE}^{-1}.\underset{\sim}{H}_{RxX}.)\underset{\sim}{S}^X + \underset{\sim}{H}_{QE}.\underset{\sim}{H}_{Xe}^{-1*}.\underset{\sim}{X}_e \qquad 5.28$$

The relationship given by (5.28) can be written in the following format as well.

$$\underset{\sim}{Q}_X = \left[\underset{\sim}{H}_{QX} - \underset{\sim}{H}_{QE}.\underset{\sim}{H}_{RxE}^{-1}.\underset{\sim}{H}_{RxX} \quad \underset{\sim}{H}_{QE}.\underset{\sim}{H}_{Xe}^{-1*} \right].\begin{bmatrix} \underset{\sim}{S}^X \\ \underset{\sim}{X}_e \end{bmatrix} \qquad 5.29$$

$\underset{\sim}{D}_R$ and $\underset{\sim}{D}_X$ be obtained from (5.27) and (5.29) respectively in view of (1.6) and (1.5).

$$\underset{\sim}{D}_R = \underset{\sim}{H}_{QE}.\underset{\sim}{H}_{RE}^{-1*} \qquad 5.30$$

$$\underset{\sim}{D}_X = \left[\underset{\sim}{H}_{QX} - \underset{\sim}{H}_{QE}.\underset{\sim}{H}_{RxE}^{-1}.\underset{\sim}{H}_{RxX} \quad \underset{\sim}{H}_{QE}.\underset{\sim}{H}_{Xe}^{-1*} \right] \qquad 5.31$$

The formulation for three approaches has been completed. A computer program in any high level language may be developed to implement the formulation. However it would be much easy to implement this formulation if a symbolic language like AIT [3] or MAIL [4] is used that have been developed for matrix operations especially.

VI. CONCLUSION

The basic objective of this piece of research was to determine the four equilibrium matrices $\underset{\sim}{B}_R$, $\underset{\sim}{B}_X$, $\underset{\sim}{D}_R$ and $\underset{\sim}{D}_X$ automatically from the geometric data of a structure after selection of the redundants which have been achieved successfully for three approaches namely 'All External Redundants', 'All Internal Redundants' and 'Mixed Redundants' or 'General Formulation'. The formulations have been tested using MAIL [4]. This paper has already been extended to 9 pages hence the implementation of these formulations will be presented in a separate paper planned for publication in some next issue of the same journal.

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