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# Observer Based Parameter Estimation For Linear Uncertain Discrete-Time Systems

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**ABSTRACT**
Presented is an observer-based parameter estimation solution for a class of linear, discrete-time systems. The proposed formulation embeds the problem of parameter estimation within a parametric uncertain observer formulation where the state and output matrices are expressed as $A = A_0 + \Delta A$ and $C = C_0 + \Delta C$. The methodology is developed by creating general solutions for the uncertainty matrices $\Delta A$ and $\Delta C$. A unique solution for each is recovered by parameterizing the general solution subject to a rank condition. The primary advantage of the proposed method is that individual parameters within the linear state equation matrices can be estimated using input/output data. The methodology is well suited for parameter estimation problems involving multi-energy-domain systems where intermediate measurements between fields are not available. Simulation examples are provided to demonstrate the utility of the proposed parameter estimation method. This result has broad applications to robust feedback solutions and system health monitoring (system diagnostics and prognostics).
**Keywords -** stability, state observer, system identification, uncertain systems

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## I. INTRODUCTION

Quantifying the dynamics of deterministic and stochastic systems via mathematical models serves as a foundation for the engineering sciences. Establishing causalities or discovering mechanistic relationships governing the dynamic response of systems involves theoretical studies validated through empirical observations. State equation systems commonly arise in the framework of multi-physics dynamical systems and can also be found in non-physics based such as macro- and microeconomics [1].

Parametric mathematical models for linear systems generally have two forms, transfer functions and state-equations. While a mapping exists between these two representations, the state-equation form could be considered as the most basic owing to its utility in numerical integration and revealing the interconnection among system states. However, the calibration of state equation models using experimental data has proven challenging. Historically, model fusion with data is achieved using a transfer function representation (i.e., input-output model) where least squares or maximum likelihood solutions accomplish parameter estimation. In these methods, the model parameters are estimated by minimizing an error criterion such as output error or equation errors. The primary limitation associated with parameter estimation using transfer functions is that transfer function coefficients are usually a combination of physical system parameters thus concealing the individual values. This is especially the case when dealing with multi-physics systems.

The goal of this work is to develop a parameter estimation solution that will determine the individual values within the A and C matrices. In this context, the estimation of model parameters for systems represented in state equation form requires estimates of the unknown system state variables. A class of system identification methods that address these challenges employs a prediction-error state observer to improve the state estimates based on the nominal model contained within the observer. These approaches use system input information and measureable system output information to improve state predictions. There is a longstanding knowledge base for state observer design as applied to linear systems with a priori known model structures. The majority of the state observer methodologies converge asymptotically to the system state(s) provided observability and stability conditions are satisfied. Examples include the Luenberger Observer, its derivatives and generalized forms detailed in [2-4]. Other observers are based on Least Squares Method (LSE) to minimize the state prediction error [5-6]. Finite-time state observers have also been discovered [7-8] for both continuous and discrete-time systems. These observers reproduce state estimates using input/output measurements and have convergence within a pre-specified finite-time that is equal to the delay chosen within the observer. It has been shown that it is always possible to construct a state observer with any nonzero time delay. However, this finite time convergence is guaranteed only for system with no uncertainty in the model coefficients.

A limited knowledge base exists for uncertain system identification using observer theory known as adaptive observer [9-10]. Adaptive observers include the uncertain parameters as additional states and are therefore estimated simultaneously with other states of the system [11-17]. The results found using the available methods in the literature have been successfully used for specific applications. One adaptive observer investigation presented in Chen et al. [18] identified an Rôssler hyper-chaotic system with two unstable poles using adaptive observers. Model coefficients as well as the states were estimated using the error between measured and estimated states. A number of states are assumed to be accessible and therefore measurable. Another adaptive observer study given in Dochain [19] employed an adaptive observer to identify the parameters of a chemical reaction. The unknown parameters were assumed to be constant. Therefore, the Dochain investigation cannot track time-varying model parameters. Additional observer-based system identification techniques have also been implemented for model-based fault detection and diagnosis [20-21]. An augmented Luenberger observer is provided in [22] that estimate both internal and external uncertainty of the system. This estimation cancels the effect of uncertainty on the state feedback control design offering better robustness and disturbance rejection. In [23], augmented states are employed to estimate states and unknown parameters simultaneously of a nonlinear invariant system. Conditions on the uncertainty of the system, under which the method is stable, are proposed. In all adaptive observers, augmenting the state vector increases the order of the estimator and therefore they require more computational time.

Presented in this manuscript is a solution to the problem of recovering model parameters directly from observer states. The proposed methodology estimates the individual element system parameters of A and C matrices using input/output data processed through an observer. Estimating these internal parameters permits the isolation of the subsystems for the purposes of health monitoring. In contrast to adaptive observer techniques that estimate system states and parameters simultaneously, the proposed method first estimates the observer states thereby reducing the order of the system. A stability analysis of the observer is carried out ensuring the convergence of the state estimation in the presence of model uncertainty. Using the estimated states, measured data is used to solve for the time-varying parameters of the system treated as system uncertainty. This uncertainty-based estimation approach, applicable to linear discrete-time observable system, is presented in its general formulation. The approach used to solve for the uncertain parameters includes two steps. The first step involves an analytical solution that parameterizes a generic form of the solution. To this end, mathematical arrangements based on pseudo-inverse are used. The second step employs a numerical solution to recover the unique estimates within the general solution via gradient descent optimization algorithm.

## II. PROBLEM STATEMENT

Consider a linear discrete-time system with time varying parametric uncertainty defined as

$$\begin{cases} x(k+1) = A(k)x(k) + Bu(k) \\ y(k) = C(k)x(k) \end{cases} \quad (1)$$

where

$A(k) = A_0 + \Delta A(k) \in R^{nxn}, B \in R^{nxm}, C(k) = C_0 + \Delta C(k) \in R^{pxn}$

and system uncertainty exists in the matrices $\Delta A(k)$ and $\Delta C(k)$. The matrices $A_0$, B and $C_0$ contain the nominal system parameters around which uncertainty of the system is added. The system states are denoted as $x(k) \in R^n$, the system inputs are $u(k) \in R^m$, and the measurable system outputs are $y(k) \in R^p$. The class of systems defined in (1) are limited to those satisfying the observability matrix rank conditions where $\rho[A,C] = Rank \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n, \forall\ \Delta A \ and\ \Delta C$. That is, the uncertain system must be observable for all possible uncertainties.

The objective is to estimate the elements within the uncertain system matrices $\Delta A(k)$ and $\Delta C(k)$ given sampled system inputs $u(k)$, and sampled measurable system outputs $y(k)$ in real-time. The proposed method provides a solution that uniquely identifies the entire uncertain model parameters for a wide range of linear systems in their general form. The validity of the method is subject to conditions on the system matrices and uncertainty to accurately estimate both unknown states and parameters. These conditions will be discussed in the main results section.

## III. MAIN RESULTS

Developed in this section is a model parameter estimation methodology using observer theory. The advantage of an observer-based approach is the ability to specify the estimation convergence rate of the model parameters through the design of the observer gain matrix $K_e$. These model parameters are individual elements of the A and C matrices of the system model thus enabling direct physical meaning in multi-domain systems that would otherwise be difficult if not impossible to isolate and estimate without additional sensors.

The proposed parameter estimation strategy shown in Fig. 1 focuses on the estimation of the uncertainty matrices defined in (1). The estimated observer states are used within a generalized analytical solution for the $\Delta A(k)$ and $\Delta C(k)$ matrices. Estimation of the individual elements within $\Delta A(k)$ and $\Delta C(k)$ are computationally determined such that the estimation error is minimized.

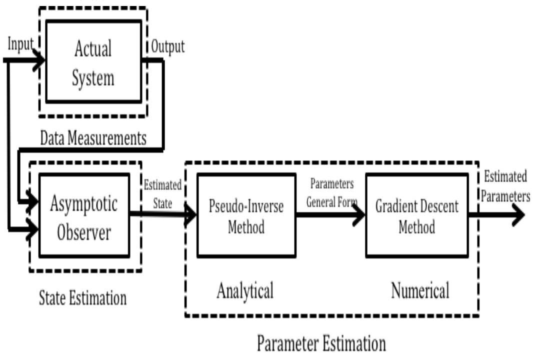


**Figure 1**. Parameter Identification Flow Chart

### 3.1. Estimation of the Observer States

Consider the standard error driven observer that asymptotically estimates the states of a system. The observer, in its traditional form, is

$$\hat{x}(k+1) = A_0\hat{x}(k) + K_e(y(k) - C_0\hat{x}(k)) + Bu(k) \tag{2}$$

where $K_e$ is an observer gain matrix. The design of $K_e$ is such that the observer poles are fast enough to track the system state without unnecessarily extending the observer bandwidth thereby amplifying sensor noise.

**Stability of the Observer**

The stability of the observer is essential to ensure convergence of the states estimates. The required conditions for the stability of the observer depends if the system uncertainty is constant or time varying. For linear time invariant (LTI) systems where the uncertain system matrix $A_0 + \Delta A$ is Schur stable, the observer is asymptotically stable if the observer state matrix $(A_0 - K_e C_0)$ is also Schur stable, that is, eigenvalues of the observer state matrix are inside the unit circle, namely

$$\begin{aligned} &\left|\lambda_j\left(A_0 + \Delta A\right)\right| < 1 \\ &\text{and } \left|\lambda_j(A_0 - K_e C_0)\right| < 1 \qquad j = 1,\ldots,n \end{aligned} \tag{3}$$

Inequalities (3) set conditions on the uncertainty of the system, that is, they define a region in which uncertainty of the system is located. Since condition (3) is sufficient to ensure stability of the observer exclusively in case of linear time invariant systems, it is employed when the unknown parameters of the system are constant. The observer gain is designed by placing observer poles in the desired positions using Ackermann method [24]. Therefore, the observer states as well as the parameters estimated based on pseudo-inverse converge asymptotically.

For linear time variant (LTV) systems, eigenvalues placement method becomes insufficient to guarantee stability of the observer and the application of Lyapunov method is necessary in designing the observer. The observer system is described in the following expression

$$\begin{aligned} \begin{pmatrix} x(k+1) \\ e(k+1) \end{pmatrix} &= \begin{pmatrix} A_0 & 0 \\ 0 & A_0 - K_e C_0 \end{pmatrix} \begin{pmatrix} x(k) \\ e(k) \end{pmatrix} \\ &+ \begin{pmatrix} \Delta A(k) & 0 \\ K_e \Delta C(k) - \Delta A(k) & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ e(k) \end{pmatrix} \\ &\Rightarrow \xi(k+1) = A_\xi \xi(k) + \Delta(k)\xi(k) \end{aligned} \tag{4}$$

where $\xi(k) = \begin{pmatrix} x(k) \\ e(k) \end{pmatrix}$ and $\Delta$(k) is the uncertainty in parameters. Bounds in the uncertainty matrix $\Delta$(k) will be sought that guarantee stability of observer. For this purpose, the norm of the uncertainty is assumed to be bounded; $\left\|\Delta(k)\right\| \le \delta$.

Let Lyapunov function be $V(\xi) = \xi^T \mathrm{P}\xi \ge 0 \Rightarrow P \ge 0$ where P is a symmetric matrix. The Lyapunov function discrete difference is given by

$$\begin{aligned} V(\xi(k+1)) - V(\xi(k)) &= \xi^T(k+1)P\xi(k+1) - \xi^T(k)P\xi(k) \\ &= \left(A_\xi\xi(k) + \phi(k)\right)^T P\left(A_\xi\xi(k) + \phi(k)\right) - \xi^T(k)P\xi(k) \\ &= \xi^T\left(A_\xi^T P A_\xi - P\right)\xi + \xi^T A_\xi^T P\phi + \phi^T P A_\xi \xi + \phi^T P\phi \end{aligned}$$

given $\zeta = \begin{pmatrix} \xi \\ \phi \end{pmatrix}$. Quadratic stability is guaranteed when

$$V(\xi(k+1)) - V(\xi(k)) = \begin{pmatrix} \xi \\ \phi \end{pmatrix}^T \begin{pmatrix} A_\xi^T P A_\xi - P & A_\xi^T P \\ P A_\xi & P \end{pmatrix} \begin{pmatrix} \xi \\ \phi \end{pmatrix} < 0 \tag{5}$$

for $\xi(k) = \begin{pmatrix} x(k) \\ e(k) \end{pmatrix} \neq 0$ that satisfy bounded norm condition

$\|\Delta(k)\| \leq \delta \Rightarrow \phi^T\phi \leq \delta^2\xi^T\xi \Rightarrow \phi^T\phi - \delta^2\xi^T\xi \leq 0.$

$$\Rightarrow \begin{pmatrix} \xi \\ \phi \end{pmatrix}^T \begin{pmatrix} -\delta^2 I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \xi \\ \phi \end{pmatrix} \leq 0 \qquad (6)$$

In order to apply S-procedure to transform the problem into Linear Matrix Inequality (LMI), the set $A = \{(\xi,\phi) \mid \xi \neq 0, (6)\}$ should be equal to the set $B = \{(\xi,\phi) \mid (\xi,\phi) \neq 0, (6)\}$. It suffices to show that $\{(\xi,\phi) \mid \xi = 0, \phi \neq 0, (6)\} = \varnothing$. But this is immediate: If $\phi \neq 0$, then condition (6) cannot hold without having $\xi \neq 0$,

Combining 2 inequalities using the S-Procedure, we find that quadratic stability is equivalent to finding P satisfying

$$P > 0, \begin{pmatrix} A_\xi^T P A_\xi - P + \delta^2 I & A_\xi^T P \\ P A_\xi & P - I \end{pmatrix} \leq 0 \qquad (7)$$

Stability margin is defined as the largest $\delta \geq 0$ for which the system remains stable and computed by solving the LMI problem

$$\text{maximize } \delta$$
$$\text{subject to } \delta \geq 0, P > 0, \begin{pmatrix} A_\xi^T P A_\xi - P + \delta^2 I & A_\xi^T P \\ P A_\xi & P - I \end{pmatrix} \leq 0 \qquad (8)$$

To find the maximum bound $\overline{\delta}$, $\delta$ is maximized subject to the Linear Matrix Inequality constraints (8). In this case, when uncertainty is less in amplitude than the maximum bound, the Lyapunov function is positive and its derivative is negative around the equilibrium and therefore stability of the observer (4) is guaranteed. Therefore, the observer error goes to zero asymptotically.

### 3.2. General Solution for the Uncertainty Matrices

Given that the observer estimates will asymptotically converge subject to condition (3) for LTI systems or condition (8) for LTV systems, the estimated states may be matched to the exact state equation as

$$\begin{cases} \hat{x}(k+1) = A\hat{x}(k) + Bu(k) = (A_0 + \Delta A)\hat{x}(k) + Bu(k) \\ y(k) = C\hat{x}(k) = (C_0 + \Delta C)\hat{x}(k) \end{cases} \qquad (9)$$

where

$$\begin{cases} \Delta A\hat{x}(k) = \hat{x}(k+1) - A_0\hat{x}(k) - Bu(k) \\ \Delta C\hat{x}(k) = y(k) - C_0\hat{x}(k) \end{cases} \qquad (10)$$

There exists a solution to (10) if and only if

$$\begin{cases} \left(\hat{x}(k+1) - A_0\hat{x}(k) - Bu(k)\right)\hat{x}^+(k)\hat{x}(k) = \left(\hat{x}(k+1) - A_0\hat{x}(k) - Bu(k)\right) \\ \left(y(k) - C_0\hat{x}(k)\right)\hat{x}^+(k)\hat{x}(k) = \left(y(k) - C_0\hat{x}(k)\right) \end{cases} \qquad (11)$$

where $\hat{x}(k)$ is the Moore-Penrose pseudo-inverse [25] of the state estimation. Condition (11) is equivalent to saying that there exists a solution to $Ax = b$ in A for known vectors x and b if and only if $bx^+x = b$.

Proof: Solve for A in the equation $Ax = b$. If $bx^+x = b$, then clearly $bx^+$ is a solution to the equation. This proves existence of a solution.

Conversely, if a solution exists, say M, then $Mx = b$ $\Rightarrow Mxx^+x = b$ using $xx^+x = x$ by definition of pseudo-inverse.

$\Rightarrow bx^+x = b$ using $Mx = b$. ■

The equality in (11) is equivalent to the rank conditions

$$\begin{cases} Rank\begin{pmatrix} \hat{x}(k) \\ \hat{x}(k+1) - A_0\hat{x}(k) - Bu(k) \end{pmatrix} = Rank\left(\hat{x}\right) \\ Rank\begin{pmatrix} \hat{x}(k) \\ y(k) - C_0\hat{x}(k) \end{pmatrix} = Rank\left(\hat{x}\right) \end{cases} \qquad (12)$$

where $Rank\left(\hat{x}\right) = 1$ since $\hat{x}$ is a non-zero vector.

Equation (12) is equivalent to $bx^+x = b$ if and only if $Rank\begin{pmatrix} x \\ b \end{pmatrix} = Rank(x)$

Proof: $bx^+x = b \Leftrightarrow b - bx^+x = 0 \Leftrightarrow Rank(b - bx^+x) = 0$

We know from the properties of matrices that

$$Rank\begin{pmatrix} x \\ b \end{pmatrix} = Rank(x) + Rank(b - bx^+x)$$

Consequently, $bx^+x = b \Leftrightarrow Rank\begin{pmatrix} x \\ b \end{pmatrix} = Rank(x)$. ■

Under condition (11) or (12), the uncertainty matrices of the system can be written in the following parameterized form

$$\begin{cases} \Delta A = \left(\hat{x}(k+1) - A_0\hat{x}(k) - Bu(k)\right)\hat{x}^+(k) + Z(I - \hat{x}(k)\hat{x}^+(k)) \\ \Delta C = \left(y(k) - C_0\hat{x}(k)\right)\hat{x}^+(k) + W(I - \hat{x}(k)\hat{x}^+(k)) \end{cases} \qquad (13)$$

where variables Z and W are arbitrary matrices of appropriate dimensions.

**Aside:** Equation (13) is equivalent to saying that a complete set of solutions to $Ax = b$ is given by $M(Z) = bx^+ + Z(I - xx^+)$ as Z, an arbitrary matrix, varies over all possible values [26].

Proof: Consider $M(Z) = bx^+ + Z(I - xx^+)$ $\forall Z$.

Then

$$M(Z)x = \left(bx^{+} + Z(I - xx^{+})\right)x = bx^{+}x + Z(x - xx^{+}x)$$
$$= bx^{+}x + Z(x - x) = bx^{+}x$$

By a condition of existence of solution previously demonstrated; $bx^{+}x = b$

$\Rightarrow$ $M(Z)x = b \Rightarrow$ M(Z) is a solution of equation $Ax = b \quad \forall Z$.

Conversely, now suppose a matrix K is a solution to $Ax = b$. Then $bx^{+} + K(I - xx^{+}) = bx^{+} + K - Kxx^{+}$.

Since K is a solution, then $Kx = b \Rightarrow bx^{+} + K(I - xx^{+}) = bx^{+} + K - bx^{+} = K$

$\Rightarrow$ Every matrix K in the set of solutions may be written in the form $bx^{+} + K(I - xx^{+})$. ■

Note that the general solution of (4) contains an infinite set of possible uncertainty matrices $\Delta A$ and $\Delta C$ since number of unknowns is higher than the number of equations in (4). For each couple of parameters *Z* and *W*, there is a corresponding solution given by (13).

**3.3. Determining Z and W**

First a special case where Z and W are simultaneously zero for the solution to (13) is considered. The uncertainty matrices, in this special case, will be unique and written as

$$\begin{cases} \Delta A = \left(\hat{x}(k+1) - A_0\hat{x}(k) - Bu(k)\right)\hat{x}^{+}(k) \\ \Delta C = \left(y(k) - C_0\hat{x}(k)\right)\hat{x}^{+}(k) \end{cases} \quad (14)$$

The solution (14) is valid under the following conditions

$$\begin{cases} \Delta A\hat{x}(k)\hat{x}^{+}(k) = \Delta A \Leftrightarrow Rank\begin{pmatrix} \hat{x}^{+} \\ \Delta A \end{pmatrix} = Rank\left(\hat{x}^{+}\right) \\ \Delta C\hat{x}(k)\hat{x}^{+}(k) = \Delta C \Leftrightarrow Rank\begin{pmatrix} \hat{x}^{+} \\ \Delta C \end{pmatrix} = Rank\left(\hat{x}^{+}\right) \end{cases} \quad (15)$$

Condition (15) contains both estimated states and system uncertainty matrices. Collectively, both define a set of uncertainty matrices ($\Delta A$ and $\Delta C$) for which this simplification is applicable. In general, the uncertainty of the system does not meet condition (15) and the actual solution of the problem can be located at another nonzero value of parameters *Z* and *W* depending on the uncertainty present in the system. When condition (15) is not true, the actual solution may be determined using a minimization technique.

**General Solution to (13)**

Since condition (15) is stiff, the general form of uncertainty matrices (with parameters *Z* and *W*) will be conserved for the remainder of the work. This imposes no loss in generality and takes into account a wide range of uncertainties in the system. Batching of data including estimated states will be used to find the unique solution corresponding to the problem and is the approach employed here. A minimization procedure is developed to find the solution for wide range of problems without constraining the system and the uncertainty to meet a stiff condition (15).

A gradient descent approach is developed to calculate the Z and W matrices that minimize the fitness functions $E_1(Z)$ and $E_2(W)$ defined as

$$\underset{Z}{Min}\, E_1(Z) = \underset{Z}{Min} \sum_i \left\| (A_0 + \Delta A(Z))\hat{x}(i) + Bu(i) - \hat{x}(i+1) \right\|^2$$
$$\underset{W}{Min}\, E_2(W) = \underset{Z}{Min} \sum_i \left\| (C_0 + \Delta C(W))\hat{x}(i) - y(i) \right\|^2 \quad (16)$$

where Z and W are considered as vectors of dimension that matches the unknown parameters N. For the gradient descent approach, the n-dimensional space is discretized. The objective of the method is to decrease the fitness functions iteratively by moving along the steepest descent direction in Z-space and W-space until convergence. The gradient descent procedure [27] is summarized as

i. Select an initial point $Z_0$ in Z-space and $W_0$ in W-space
ii. The opposite of gradient of $E_1(Z)$ and $E_2(W)$ give the best direction to minimize the functions. Next iteration guess of Z and W: $Z_{i+1} = Z_i - \gamma_1 \nabla E_1(Z_i)$ and $W_{i+1} = W_i - \gamma_2 \nabla E_2(W_i)$
iii. Stop when convergence is reached, $\left\|Z_{i+1} - Z_i\right\| \le dZ_{\min}$ OR $E_1(Z_i) \le Tol$
And $\left\|W_{i+1} - W_i\right\| \le dW_{\min}$ OR $E_2(W_i) \le Tol$

Gradient Descent Step Size Selection

The step size γ can be calculated adaptively for each step using Barzilai and Borwein approach [28], which has been proven for large dimensional problems. Defining $\Delta Z = Z_i - Z_{i-1}$ and $\Delta g(Z) = \nabla E_1(Z_i) - \nabla E_1(Z_{i-1})$, the step size for minimizing $E_1$ is given by

$$\gamma_1 = \frac{\Delta g(Z)^T \Delta Z}{\Delta g(Z)^T \Delta g(Z)}. \quad (17)$$

The same approach to calculate the step size $\gamma_2$ minimizing $E_2$ is used. The step size is calculated directly using (17) that approximates the inverse of Hessian matrix in the Newton method, which is costly to form. Using this variable step size, the method computational cost to convergence is reduced.

In the following section, the method of this section will be demonstrated using a double mass-spring-damper system. Two cases of parameter uncertainty will be studied. The first case considers constant uncertainty while the second case studies step change in the parameters. In both cases, the method will exhibit stability and convergence of the estimates to the exact parameters.

## IV. APPLICATIONS OF OBSERVER BASED PARAMETER ESTIMATION

To demonstrate the utility of the proposed parameter estimation method, simulation examples are performed for double-mass-spring-damper system (see Fig. 2). The objective is to estimate the uncertain parameters of the system about the nominal values given observer estimated states. Input/output data is generated via simulations of the double-mass-spring-damper system. Changes in stiffness and/or damping of the system will be introduced within the simulation environment. The method is verified as estimates of the system parameters change during the simulation.

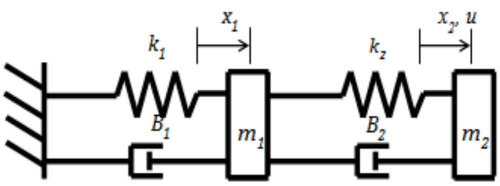


**Figure 2.** Double Mass Spring Damper System

The differential equations governing the motion of this two-degree of freedom system are derived from Newton's second law of motion for each mass $m_1$ and $m_2$

$$\begin{cases} m_1\ddot{x}_1 + B_1\dot{x}_1 + B_2(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_2(x_1 - x_2) = 0 \\ m_2\ddot{x}_2 + B_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = u \end{cases} \quad (18)$$

where $x_1$ and $x_2$ are displacements of mass $m_1$ and $m_2$ respectively. Input u is a force applied to mass $m_2$.

The system comprises two outputs and one input. Hence, the system is described with two transfer functions $G_1(s)$ and $G_2(s)$. The transfer functions $G_1(s)$ and $G_2(s)$ describing the behavior of $m_1$ and $m_2$ respectively due to input u are

$$\begin{cases} G_1(s) = \frac{X_1(s)}{U(s)} \\ = \frac{B_2s + k_2}{m_1m_2s^4 + (m_2(B_1+B_2)+m_1B_2)s^3 + (m_2(k_1+k_2)+m_1k_2+B_1B_2)s^2 + (B_2k_1+B_1k_2)s + k_1k_2} \\ G_2(s) = \frac{X_2(s)}{U(s)} \\ = \frac{m_1s^2 + (B_1+B_2)s + k_1 + k_2}{m_1m_2s^4 + (m_2(B_1+B_2)+m_1B_2)s^3 + (m_2(k_1+k_2)+m_1k_2+B_1B_2)s^2 + (B_2k_1+B_1k_2)s + k_1k_2} \end{cases} \quad (19)$$

Using transfer functions to describe the behavior of the dynamic system, the relationship between coefficients and physical parameters is complicated making the recovery of stiffness and damping coefficients of the system problematic. When the system is discretized and transfer functions are transformed into the z-domain, the relationship becomes more complicated. The physical parameters are combined into the model coefficients. Hence, state space representation is used to gain access to physical parameters of the system directly from the A, B and C matrices. Particularly, when the system to be studied is composed of multiple subsystems of different physical nature including mechanical, thermal, fluid, and electrical, the interior physical parameters are lumped together into transfer functions coefficients. Consequently the recovery of individual parameters is difficult if not impossible. Under these circumstances, the proposed parameter estimation approach is more beneficial as it permits direct access to interior model parameters of every subsystem.

Defining the states of the system as displacement and velocity of the two masses $x = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ x_1 \\ x_2 \end{pmatrix}$ and the measured output comprising displacement of each mass $y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we obtain the state space representation of the system by transforming system of second order equations (18) to first order vector equation

$$\begin{cases} \dot{x}(t) = A_cx(t) + B_cu(t) \\ y(t) = C_cx(t) \end{cases} \quad (20)$$

where the continuous form of state equation representation matrices in (20) are given by

$$A_c = \begin{pmatrix} -\frac{B_1+B_2}{m_1} & \frac{B_2}{m_1} & -\frac{k_1+k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{B_2}{m_2} & -\frac{B_2}{m_2} & \frac{k_2}{m_2} & -\frac{k_2}{m_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (21)$$

$$B_c = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, C_c = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Discrete form of the system after adding uncertainty is given by

$$\begin{cases} x(k+1) = (A_0 + \Delta A)x(k) + B_0u(k) \\ y(k) = (C_0 + \Delta C)x(k) \end{cases} \quad (22)$$

where $x(k)=x(kT_s)$ and discrete system nominal matrices $A_0$, $B_0$ and $C_0$ are approximated using the first order finite derivative as

$$A_0 = e^{A_c T_s} \cong I + T_s A_c$$

$$= \begin{pmatrix} -\frac{B_1+B_2}{m_1}T_s+1 & \frac{B_2}{m_1}T_s & -\frac{k_1+k_2}{m_1}T_s & \frac{k_2}{m_1}T_s \\ \frac{B_2}{m_2}T_s & -\frac{B_2}{m_2}T_s+1 & \frac{k_2}{m_2}T_s & -\frac{k_2}{m_2}T_s \\ T_s & 0 & 1 & 0 \\ 0 & T_s & 0 & 1 \end{pmatrix}, \quad (23)$$

$$B_0 = \int_0^{T_s} e^{A_c\lambda}\, d\lambda B_c \cong \begin{pmatrix} 0 \\ T_s \\ 0 \\ 0 \end{pmatrix}, C_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

As detailed previously, relationships between state equation representation matrices and physical parameters are accessible. Recovery of physical parameters from matrices coefficient is not computationally consuming.

The proposed method is validated through two case of studies applied to the double mass spring damper system described above. The same system arrangement as well as nominal values of physical parameters is used for both cases. In the first case, only spring uncertainties are applied to the system. In the second case, both spring and damper uncertainties are applied. Furthermore, in the latter case the uncertainties are time varying to represent a linear parameter varying dynamic system.

<u>Case 1: Spring Stiffness Uncertainty</u>

Let the double-mass-spring-damper system have the nominal parameters

$m_1$ = 5 kg, $m_2$ = 3 kg, $B_1$ = 5 $Nm^{-1}s$, $B_2$ = 5 $Nm^{-1}s$, $k_1$ = 25 $Nm^{-1}$, $k_2$ = 100 $Nm^{-1}$.

The integration step size h is related to the sampling frequency as $f_s = \frac{1}{T_s} = 1000Hz$. The uncertainties for $k_1$ and $k_2$ are respectively $\Delta k_1 = 25Nm^{-1}, \Delta k_2 = 50Nm^{-1}$. No uncertainty is given to the damping coefficients $B_1$ and $B_2$ for this example. These values of uncertainty are verified to be located in the permitted region meeting the validity conditions of the method (3). In this case, the exact uncertainty matrix is given by

$$\Delta A = \begin{pmatrix} -\frac{\Delta B_1+\Delta B_2}{m_1}T_s & \frac{\Delta B_2}{m_1}T_s & -\frac{\Delta k_1+\Delta k_2}{m_1}T_s & \frac{\Delta k_2}{m_1}T_s \\ \frac{\Delta B_2}{m_2}T_s & -\frac{\Delta B_2}{m_2}T_s & \frac{\Delta k_2}{m_2}T_s & -\frac{\Delta k_2}{m_2}T_s \\ 0 & 0 & 0 & 0 \\ 0 & 0_s & 0 & 0 \end{pmatrix} \quad (24)$$

$$= \begin{pmatrix} 0 & 0 & -0.015 & 0.01 \\ 0 & 0 & 0.01667 & -0.01667 \\ 0 & 0 & 0 & 0 \\ 0 & 0_s & 0 & 0 \end{pmatrix}$$

The method is implemented within the Matlab software environment. Shown in Fig. 3 are the estimated outputs from the observer compared to measured outputs for initial condition problem of double mass spring damper system. The initial condition on state variables is defined as $x(0) = x_{init} = \begin{pmatrix} 0.01 \\ 0.01 \\ 0.01 \\ 0.01 \end{pmatrix}$. The input is set to zero so that the system oscillates freely.

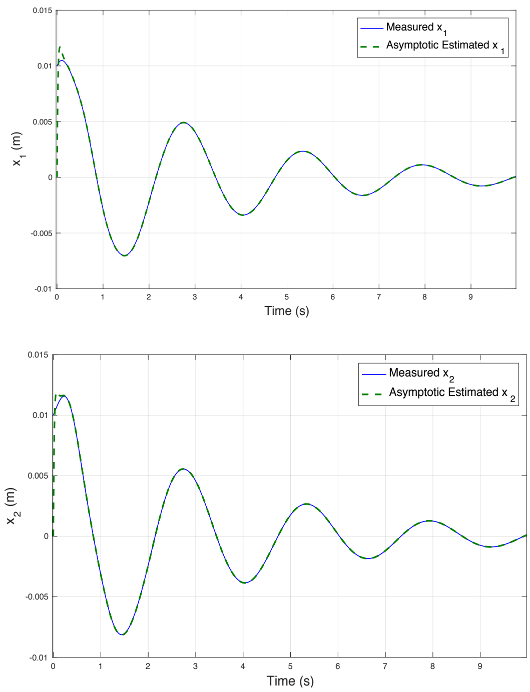


**Figure 3.** Asymptotic Observer Output Estimation

The implemented observer estimates the internal states of the system. The gain of the observer is chosen so that the poles of the observer are placed to the left of the poles of the continuous time nominal system to be fast enough to track the plant dynamics but to the right of noise range of frequencies so as to attenuate it. The continuous time nominal plant poles are [-1.48938 ± 7.28846 i, -0.290431 ± 1.71413 i]. Based on these values, the observer poles are placed at [-45, -40, -35, -30] requiring the gain matrix $K_e = \begin{pmatrix} 1.24028 & -0.192081 \\ -0.192081 & 1.24028 \\ 0.0731667 & -0.00744765 \\ -0.00744765 & 0.0731667 \end{pmatrix}$.

The observer converges asymptotically to the system states.

Once converged, the estimated state variables found by observer can be used to find the uncertain parameters of the system using the approach described in previous section (see (1)-(17)). Gradient descent algorithm (see (16)) is employed to the general form of the solution calculated analytically (see (13)). In this case, 17 iterations are sufficient for the routine to converge and give a value of parameter Z that minimizes the fitness function. Thus, the method proves effective with regard to computation time required in this example.

The cost function at optimum is accurate to the order of 4 $10^{-10}$. The difference between the final two values of parameter Z is $10^{-8}$ quantifying convergence of the results. The tangent plane to the graph of the cost function is locally horizontal at the region of the optimum. These gradient descent properties observed at convergence prove robustness of the result returned by the algorithm. Deviation from nominal matrix $A_0$ is estimated as

$$\Delta A_{est} = \begin{pmatrix} 2.49e-05 & -1.66e-05 & -0.0149 & 0.0099 \\ -2.76e-05 & 2.76e-05 & 0.0166 & -0.0166 \\ -5.26e-07 & -5.62e-07 & -9.27e-08 & -1.01e-07 \\ -2.19e-07 & -2.34e-07 & -3.87e-08 & -4.25e-08 \end{pmatrix}$$

Physical parameters (stiffness of the springs $k_1$ and $k_2$ as well as both damping coefficients $B_1$ and $B_2$) are calculated using estimated uncertainty matrix $\Delta A_{est}$ and equation (24) as shown in Table 1.

**Table 1.** Estimated and actual values of uncertainty for case 1

| | $\Delta k_1$ | $\Delta k_2$ | $\Delta B_1$ | $\Delta B_2$ |
|---|---|---|---|---|
| Actual Value | 25 | 50 | 0 | 0 |
| Estimated Value | 24.86 | 49.74 | -0.04 | -0.08 |
| Percent Error | 0.56% | 0.52% | _ | _ |

**Case 2: Time Varying Uncertainty**

To further validate the method, spring and damper uncertainties are hereafter step changing in time (see Table 2). The method is verified to track change of individual parameters of the system.

**Table 2.** Time varying uncertainty applied for case 2

| | 1-1000 samples | 1001-2000 samples | 2001-3000 samples |
|---|---|---|---|
| Uncertainty Applied | $\Delta k_1 = 25N/m$ | $\Delta k_1 = 0N/m$ | $\Delta k_1 = 50N/m$ |
| | $\Delta k_2 = 50N/m$ | $\Delta k_2 = 0N/m$ | $\Delta k_2 = -25N/m$ |
| | $\Delta B_1 = 0Ns/m$ | $\Delta B_1 = -3Ns/m$ | $\Delta B_1 = 0Ns/m$ |
| | $\Delta B_2 = 0Ns/m$ | $\Delta B_2 = -3Ns/m$ | $\Delta B_2 = 0Ns/m$ |

For this purpose, the observer gain matrix (i.e., its poles) are maintained as in Case 1. An adaptive procedure is implemented that estimates system parameters and thus evaluates their change in time.

In this example, the adaptation is performed in batch mode meaning that estimation of parameters at time t takes into account the measurements from the interval [t-Δt, t]. Windowing the data in batch mode is used to isolate the system state before and after parameter changes so that after Δt, the estimation relies exclusively on measurements after the change in parameters occurs. In this example, a window of 250 back-samples is used to estimate parameters at each time to track changes in the system if any occur and neglect the original system state as the window advances and data is spanned. Shown in Fig. 4 are the estimated physical parameters as function of time.

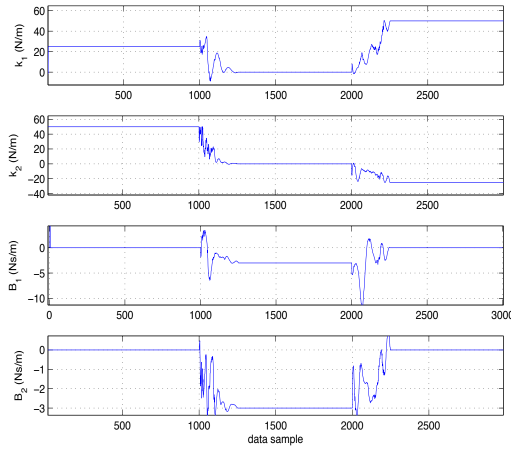


**Figure 4.** Parameters Estimation Adaptation

The procedure estimates the changes in model parameters. When system parameters change, the method asymptotically tracks those variations. The exact change is found when window of data used for estimation completely contains the estimated states of the altered system. Tracking the mechanical characteristics of the system in this manner enables diagnostic and prognostic decisions. Summarized in Table 3 are the actual value, the estimated value at convergence and the corresponding percent error for each combination of uncertainty parameters applied to the system.

**Table 3.** Estimated and actual values of uncertainty for case 2

| | | $\Delta k_1$ | $\Delta k_2$ | $\Delta B_1$ | $\Delta B_2$ |
|---|---|---|---|---|---|
| 1st Combination | Actual Value | 25 | 50 | 0 | 0 |
| | Estimated Value | 24.86 | 49.74 | -0.04 | -0.08 |
| | Percent Error | 0.56% | 0.52% | _ | _ |
| 2nd Combination | Actual Value | 0 | 0 | -3 | -3 |
| | Estimated Value | -0.12 | -0.29 | -2.99 | -2.99 |
| | Percent Error | _ | _ | 0.33% | 0.33% |
| 3rd Combination | Actual Value | 50 | -25 | 0 | 0 |
| | Estimated Value | 49.9 | -24.91 | -0.08 | 0.04 |
| | Percent Value | 0.2% | 0.36% | _ | _ |

## V. CONCLUSIONS

Presented is a parameter estimation methodology formulated as an uncertain linear discrete state space model having measured input/output data. The method utilizes observer theory to estimate system states and upon convergence employs a pseudo-inverse solution computationally solved via gradient-descent to estimate unknown system parameters. This "in-series" approach solves the parameter estimation problem without increasing the observer order thereby reducing computation costs. Validity conditions on model parameters and states of the system have been also been developed. The implementation of this parameter estimation process is suitable for real-time computing and health monitoring applications. Demonstration of the parameter estimation method is applied to a double-mass-spring-damper system where individual parameters are changing in real-time.

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