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Kolmogorov Potential Well

José Roberto Mercado-Escalante*, Pedro Antonio Guido-Aldana**
**(Retired researcher, Mexican Institute of Water Technology, Mexico)*
*** (Researcher, Mexican Institute of Water Technology, Coordination of Water Systems, Assistant Coordinator's Office of Physical Experimentation and Technological Innovation, Mexico)*

ABSTRACT
The method applied lies in the study of the symmetries or groups of Lie of the differential equations. We observe the form of the drag coefficient as a gradient of a logarithmic potential as the transcendental manifestation of the symmetries of the Kolmogorov-f equation.
Said logarithmic potential is omni-present in the very diverse natural phenomena, an affirmation that is verified through the method of Pearson distributions. In particular, we highlight its presence in what we have called the Kolmogorov potential well. The drag coefficient also participates in the potentials, and in the "quanta" of the eigenvalues, of the stationary Schrödinger equation. The solution of the inverse problem for the drag coefficient allows to introduce the law of supply and demand in the Black-Scholes equation for derivatives.
Keywords- Drag coefficient, Lie Symmetries Groups, Kolmogorov Equations, Schrödinger Equation, Fokker-Planck Equation, Black-Scholes Equation.

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I. INTRODUCTION

This article is guided by the *Principle of Duality*, which we place somewhere in the middle, although far away, between today and about 3000 years ago. Principle that can be observed in historical monuments and in the myths of the native peoples of present-day Mesoamerica, such as Tlaloc as God of the East, symbolizing the water that falls as rain and in red color, having as a counterpart the Goddess Coatlicue, in the West, like running water and with the color white as its chromatic. Principle, which we receive again from Europe in Hegel's Dialectic of the 19th century. Principle, which reappears in the Helmholtz Theorem when observing the field lines as open or closed, and which allows us to characterize the drag coefficient of "potential" type in the case of the Langevin field. Principle, which also leads us to random or deterministic variables. When considering the diffusion processes, we again find the duality in a complementary way between the contribution of the Probability Theory and that of the partial differential equations of parabolic type.

We remember that the inverse problems have their main origin in the study of the Schrödinger equation, in dispersion problems. This equation depends on the interaction potential, and in principle, if the functional form of the potential is known, it's possible to solve the problem of finding the eigenvalues and the corresponding eigenfunctions, which is known as the direct spectral problem (the quantization as an eigenvalues problem, [1]). If waves are made to impact on a neighborhood where this potential operates, scattered and reflected waves are produced; and if, in addition, the reflection coefficient is defined as the quotient between the amplitude of the reflected wave by the incident one, it is found that this is determined by the phase shifts; and from which one wants to find the functional form of the potential or the properties that lead to its determination. This is the inverse scattering problem.

On the other hand, in the mathematical field, one wants to determine the potential of the spectral function, which has been known as the inverse Sturm-Liouville problem or inverse spectral problem. The inverse scattering problem in physics turns out to be related to the inverse Sturm-Liouville problem in mathematics. In both one and the other, the potential appears as a coefficient in the differential equation, so it becomes generic to call the inverse problem, obtaining the coefficients of the various differential equations.

Kinetic processes are a natural consequence of the presence of imbalances in a medium and tend to restore equilibrium. In a diffusion process, the flow density is considered and we represent it by the vector field of components proportional and opposite to the transversal gradient, to which we can add the drag components, then we consider the conservation of the probability mass in its differential form as Euler's equation. Thus arises the Kolmogorov-f (forward) equation for the probability density that combines two coefficients: drag (convection, advection, transfer or drift) and diffusion. These coefficients, in turn, are the memory coefficients relative to the same probability density and represent the average values of the displacements per unit of time and of the quadratic displacements per unit of time, the first linked to macroscopic phenomena and faster effects, and the second, to the microscopic and slower effects. Diffusion processes can be imagined as a mirror image of the Kolmogorov-f equation. In 2001 we published about the solution to the inverse problem of the aforementioned equation, [4], [2], [3].

On the other hand, many empirical phenomena can be modeled through homogeneous stochastic Markov processes, with the property of being unigenerational and governed by the evolution equation known as Chapman-Kolmogorov, which presents the notable property of the semigroup. Through this semigroup, a partial differential equation known in mathematical circles as Kolmogorov-f (or Fokker-Planck in physicists) can be formed. And it contains three main ingredients, which we call: the diffusive, the drag and the reactive. In this way it is observed that mathematical models frequently lead to the description through differential equations and in particular, of partial differential equations.

The applied method is based on the study of the symmetries present in nature and reflected in its laws and therefore, in its differential equations. The method is known as Lie groups. We will make a summary exposition of this mentioned method that can be consulted, in extenso, in the references, [5], [6].

The symmetric structure together with a local differentiable structure, allow to describe in a general way and with the characteristic of universality, the various objects, and their evolution, as they participate in mathematical modeling. The most important characteristic is found in the correspondence between vector fields and symmetries, called uniparametric transformation groups, each group has a vector field associated as its infinitesimal generator; and although the elements of the group behave in a non-linear way, their generators do so in a linear way. Differential equations can be seen as surfaces in higher dimensional spaces and infinitesimal generators, and other objects, flow on them describing their evolution. A differential equation rises to these hyperspaces and produces a transformed equation, which leads to a system of differential equations that carry the components of the infinitesimal generator and the coefficients of the partial differential equation under study; by solving this system, we find the functional relationships for the coefficients and the relationships that classify them.

In particular, when considering the coefficient of the displacements per unit of time, it is observed that it is a vector field, so its determination refers us to its divergence and its rotational, as expressed by the Helmholtz theorem, [7].

The study of symmetries for the Kolmogorov-f (or Fokker-Planck) produced, as one of the possible results for the drag coefficient, its logarithmic form, with two possible signs; or as a gradient of a logarithm; and we call this potential Kolmogorov.

If the drag coefficient, in its vector field form, has an irrotational characteristic, so it's a field derivable from a potential; and if it is also of the logarithmic type, then the drag coefficients are linked to the Pearson distributions and their determination from 4 parameters: mean, variance, asymmetry and kurtosis, with the additional advantage that they can be arise in field.

According to Born's statistical interpretation, the probability density is represented as a product of two solutions of the Schrödinger equation, one is a solution of the equation and the other is a solution of the adjoint one, but these solutions depend on the potential of the same name. But through a Ricatti equation there is an important connection between the aforementioned potential for the Schrödinger equation and the drag coefficient, which in turn is linked to the Kolmogorov potential, so the latter will manifest its presence in the probability density.

The density of the Beta distribution serves as a reference, which, as a Pearson distribution, depends on the Kolmogorov potential and for this type of drag coefficient, the probability density is represented as the exponential of a potential. Said Beta density can be factored into two functions with antagonistic tendencies linked to success or failure. But then also the density factors into a pair of exponentials of two potentials. Using the "rhombus geometry" we can associate one diagonal to the sum of two magnitudes and the other diagonal to the difference of the same magnitudes, with the characteristic of orthogonality representing independence, antagonism and non-correlation.

From the Kolmogorov-f (or forward) equations and its adjoint Fokker-Planck, two alternative and parallel diffusion processes arise, one in an evolutionary sense, the other in an involutive sense.

In summary, we approach a summary exposition of the symmetry theory, and in particular, under the constant diffusion hypothesis, to obtain a drag coefficient described by a logarithm. Then we recall the elements of the Helmholtz theorem. Next, we recall the elements of stochastic modeling. We see the relationship between the Kolmogorov potential and the Pearson distributions. We expose the Kolmogorov potential well. Then the presence of the Kolmogorov potential in the Schrödinger potential is discussed. Finally, as another example we present our alternative proposal of the Black-Scholes equation that contains the effects of the law of supply and demand of the subjectivist theory.

II. ASPECTS OF SYMMETRY GROUP THEORY

In a group there is an associative binary operation (semigroup), with a unit element (monoid) and all the elements are invertible.

Groups can operate on sets, transferring their systematization to them, through group actions, where the unit acts as an identity operator and the product of two elements acts as a semigroup property. They can also operate on the same group, for example through translations and conjugations.

In the groups, the topological properties of their elements can also be observed and thus the so-called topological groups are obtained. When their topological structure is locally analogous to that of finite-dimensional Banach spaces, they are said to have the variety structure, in addition to their algebraic structure, and are called Lie groups, [5], [6].

The method that we have applied lies in the study of the symmetries of a differential equation or method of Lie groups. We will make a summary exposition of the method, which can be consulted in extenso in the references, [5], [6]. The groups reveal the symmetries of nature, expressed in their shapes or in their equations.

The following theorem shows the conversion of the differential equation into the transformed equation, [6]:

Theorem 1: On Invariance

A local group of transformations G is a symmetry group of the equation Δ, with $\Delta\big(J^n\mathbf{v}(g)\big) = 0$, and of maximum rank, $\Leftrightarrow$ for all $J^n\mathrm{v}'(g') \in S_\Delta$, in which the equation is locally soluble, one has,

$$L_{J^n\mathbf{v}}\Delta|_{\Delta=0} = J^n\mathbf{v}\big(\Delta|_{\Delta=0}\big) = 0 \qquad (1)$$

for every infinitesimal generator $\mathbf{v}$ of G and where $L_{J^n\mathrm{v}}$ denotes the Lie derivative in the direction of the vector field $J^n\mathrm{v}$.

The differential equation is presented as the function Δ:

$$\Delta = u_{xx} + A(u)u_x + f(u) - B(u)u_t \qquad (2)$$

the transformed equation arises from the application of the invariance criterion, (1), on this equation, and is calculated as the Lie derivative of the differential equation along the extended generator, (4).

The generator is,

$$\mathbf{v} = \xi\partial_x + \tau\partial_t + \emptyset\partial_u \qquad (3)$$

and the extended generator is the vector field

$$j^{(2)}\mathbf{v} = \mathbf{v} + \phi^x\partial_{u_x} + \phi^t\partial_{u_t} + \phi^{xx}\partial_{u_{xx}} + \phi^{xt}\partial_{u_{xt}} + \phi^{tt}\partial_{u_{tt}} \qquad (4)$$

when this is applied to the Δfunction it produces:

$$j^{(2)}\mathbf{v} \mapsto \begin{array}{lcl} \xi\partial_x & \mapsto & 0 \\ \tau\partial_t & \mapsto & 0 \\ \phi\partial_u & \mapsto & \phi(A'u_x + f' - B'u_t) \\ \phi^x\partial_{u_x} & \mapsto & \phi^x A \\ \phi^t\partial_{u_t} & \mapsto & -B\phi^t \\ \phi^{xx}\partial_{u_{xx}} & \mapsto & \phi^{xx} \\ \phi^{xt}\partial_{u_{xt}} & \mapsto & 0 \\ \phi^{tt}\partial_{u_{tt}} & \mapsto & 0, \end{array} \qquad (5)$$

then the following equation results:

$$\phi A'u_x + \phi f' - \phi B'u_t + \phi^x A - B\phi^t + \phi^{xx}|_{\Delta=0} = 0$$

which evaluated at $\Delta = 0$ produces:

$$\phi\left(f' - \frac{B'}{B}f\right) + \phi A\left(\frac{A'}{A} - \frac{B'}{B}\right)u_x - \phi\frac{B'}{B}u_{xx} + A\phi^x - B\phi^t + \phi^{xx} = 0 \qquad (6)$$

this is the transformed equation and especially, it depends on the infinitesimals of the extended generator.

2.1 Constant diffusion

To study the behavior of the equations system under the premise of constant diffusion, [8], [9], we assume $B = cte$in the mentioned equations system, which will assume the form:

$$\begin{array}{ll} \tau=\tau(t) & \tau_t-2\xi_x=0 \\ \xi_u=0 & A'\phi-A\xi_x+A\tau_t+B\xi_t+2\phi_{xu}-\xi_{xx}=0 \\ \phi_{uu}=0 & f'\phi+A\phi_x-B\phi_t-f(\phi_u-\tau_t)+\phi_{xx}=0 \end{array} \quad (7)$$

Now from the fourth and first in (7), we get:

$$\xi_{xx}=0 \quad (8)$$

so, the system for $B=cte$, is expressed by:

$$\begin{array}{ll} \tau=\tau(t) & \tau_t-2\xi_x=0 \\ \xi_u=0 & A'\phi+\left(-\frac{1}{2}\right)A\tau_t+B\xi_t+2\phi_{xu}=0 \\ \phi_{uu}=0 & f'\phi+A\phi_x-B\phi_t-f(\phi_u-\tau_t)+\phi_{xx}=0 \end{array} \quad (9)$$

Now we differentiate with respect to $x$to the fifth in (9), then we do it with respect to u, and since $\emptyset_{uu}=0, \xi_u=0$, and $\tau_u=0$, then,

$$(A'\phi_x)_u=0 \quad (10)$$

Or,

$$A''\phi_x+A'\phi_{xu}=0 \quad (11)$$

This equation admits multiple possibilities. According to the considered group it could be $\phi_x=0$, then $\phi_{xu}=0$, and then $A''$and A'can be arbitrary. On the other hand, if $A'=0$and $A''=0$, that is, if $A=\text{constant}$, we could have arbitrary $\phi_x$and ϕ_{xu}, and in that case the group is arbitrary; like the Galilean, where the two derivatives of the infinitesimal are different from zero, or the first is non-zero and the second is null as in the linear differential equation, or both are null as in the translations group. And in mixed form, as a Galilean group with non-linear drag, where $A^{''}\neq 0$ and $\emptyset_{xu}\neq 0$,

$$\frac{A'}{A''}=-\frac{\phi_x}{\phi_{xu}}\ , \quad (12)$$

then since ϕ_{xu}does not depend on u, by the third equation in (9), we derive with respect to u and it is obtained that if $B=cte$, $A''\neq 0$ and $\emptyset_{xu}\neq 0$, then the drag coefficient is a solution of the ordinary equation

$$\left(\frac{A'}{A''}\right)_u=-1 \quad (13)$$

Integrating (13), with $A'\neq 0$, $cte=a$, two possibilities emerge

$$A=\begin{cases}-b\ln(-u+a)+c & \text{if } -u+a>0, \\ b\ln(u-a)+c & \text{if } \quad u-a>0\end{cases} \quad (14)$$

We can consider another analogous system but with time denoted $\hat{t}$ and the coefficient $\hat{A}$. We carry out analogous calculations and the system for $B=cte$ results in the following alternative for $\hat{t}$ and $\hat{A}$,

$$\begin{array}{ll} \tau=\tau(\hat{t}) & \tau_{\hat{t}}-2\xi_x=0 \\ \xi_u=0 & \left(\hat{A}\right)'\phi+\left(-\frac{1}{2}\right)\left(\hat{A}\right)\tau_{\hat{t}}+B\xi_{\hat{t}}+2\phi_{xu}=0 \\ \phi_{uu}=0 & f'\phi+\left(\hat{A}\right)\phi_x-B\phi_{\hat{t}}-f(\phi_u-\tau_{\hat{t}})+\phi_{xx}=0 \end{array} \quad (15)$$

$$\frac{(\hat{A})'}{(\hat{A})''}=-\frac{\phi_x}{\phi_{xu}}$$

if we now make $\hat{t}=-t$ and $\hat{A}=-A$, the result is $\frac{(-A)'}{(-A)''}=-\frac{\phi_x}{\phi_{xu}}$ and the equation remains invariant:

$$\frac{A'}{A''}=-\frac{\phi_x}{\phi_{xu}} \quad (16)$$

From which results the "symmetry" commented by Kolmogorov about what Schrödinger proposed: "The following considerations, despite their simplicity, seem to me new and not without interest for certain physical applications, in particular for the analysis of the reversibility of the statistical laws of nature, which Mr. Schrödinger has carried out in the case of a specific example.", [1].

The drag coefficient could also be in the form of a logarithmic gradient. Because if $A(u)=\frac{c}{u}$, then $A'=-\frac{c}{u^2}$, but $c$cannot be null because it would cancel out the drag coefficient, then we can consider $a(u)=\frac{A(u)}{c}$and prove that $a$satisfies equation:

$$\left(\frac{A'}{A''}\right)_u\Bigg|_{A=a}=-1$$

The equation $(A'\phi_x)_u=0$produces the representation ofϕ_x, as $\phi_x=\frac{1}{2}u\phi_{xu}$; thus, the cited equation produces$(A'u\phi_{xu})_u=0$, then$A'=-uA''$, then $\left(\frac{A'}{A''}\right)_u=-1$.

In the space of symmetries $c$is a parameter with a value marked in one of the real multi-axes $(\phi^x\partial_{u_x})$ and cannot vary with x, however in the physical space of movement we can think of the parameter as$c(x)=u_x$. Therefore, the form $A(u)=\frac{u_x}{u}$is admitted, thus $A(u)=\frac{u_x}{u}=\frac{\partial}{\partial x}lnu$. In a later section we will consider the particular and diverse case where $A(u)$is a quotient of two polynomials: the numerator, of order 1 over another of order 2, being their coefficients real, (meanwhile, the case $u=cte$ will be included within the Beta density).

2.2 About the Helmholtz theorem

On the other hand, if we fix the attention in the coefficient of the displacements per unit of time, in general it is a vector field and this refers us to its determination from its divergence and its rotational, as affirmed by the Helmholtz theorem.

Within a certain region it's necessary to know the divergence of the field as a scalar potential φ and the rotational of the field as a vector potential, in addition to the value of the field in its normal component at the boundary of the region$F_n|_{\partial\Omega}$, [7], [10]. The field is broken down into two parts: one, with zero divergence, image of the solenoidal projector, (part free of divergence, solenoidal, with vector potential and analogous to a magnetic field) and the second, with zero rotational (irrotational or with scalar potential(φ), analogous to an electric field).

$$\mathbb{F} = P\mathbb{F} \oplus \text{grad}\varphi \tag{17}$$

If the drag field is in the image of the Solenoid Projector, as a solenoidal field it is divergence free, which means the condition gauge $div\ b = 0$. With the analogy of a solenoidal field with a magnetic field, it is observed that if time sign changes, it is equivalent to changing the sign of the moving charge, that is, changing the direction of the stream, which in turn reverses the direction of the magnetic field. So, the symmetry is to change the time sign and simultaneously change the sign of the solenoidal field. Schrödinger noted the importance of this symmetry, as outlined by Kolmogorov in accordance with the already mentioned citation from Nagasawa, [1].

III. STOCHASTIC MODELING

A certain object phenomenon can receive the influence of a multiplicity of other so many phenomena, which under the principle of duality we group into deterministic and random, although the latter are frequently much more abundant. The subsequent classification into main and secondary, allows us to place them in the existence of at least two antagonists, one deterministic, the other random. In a graph we illustrate the evolution of the trajectory of the phenomenon under consideration X, the change is symbolized by $X_{t+\Delta t} - X_t$, which results from the superposition of a deterministic change $b\Delta t$, and a random change $\sigma(t, X_t)(W_{t+\Delta t} - W_t)$. As examples of phenomena with a deterministic predominance we can mention:

The relative loss of the velocity of a particle due to friction in a movement within a fluid at rest, with the proportionality linked to the viscosity of the fluid $\frac{dv}{v} = -\gamma dt$, while the second change depends on a Langevin force, process random with a very narrow standard deviation, [2].

The model of the growth of the number of rabbits, known as Fibonacci numbers, where the relative rate of growth, analogous to a rate of return, is close to the *"golden number"*: $\frac{\Delta F}{F} = \frac{F_{n+1} - F_n}{F_n} \approx \varphi, \varphi = 0.618 = \frac{\sqrt{5}-1}{2}$, [11].

The Pythagorean theorem seen as doubling squares, just as it originated. In descending scales, if the length of the initial side is l_0, the diagonal 1 is $(1/\sqrt{2})l_0$, diagonal 2 is $(1/\sqrt{2})^2 l_0$, diagonal $n$is $(1/\sqrt{2})^n l_0$. The relative decrease rate is $-(1 - 1/\sqrt{2})$, of the order of 30%. If now the circles in which the successive squares are inscribed have diameters $(1/\sqrt{2})^2 l_0$, the relative decreasing rate of the diameters is also $-(1 - 1/\sqrt{2})$. If these circles represent the size of the vortices in a fluid or "Descartes eddies", the decrease will continue until reaching the Kolmogorov length, which depends on the viscosity and is proportional to the ¾ power of the kinematic viscosity of the fluid (meanwhile, the constant of proportionality is $(1/\varepsilon)^{\frac{1}{4}}$, with ε being the rate of energy transfer).

On the other hand, when the random contributions are important, the so-called Itô diffusions emerge, where the random changes are modeled by the Brownian movements, [12]. The differential form of the stochastic differential equation is:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{18}$$

There are two coefficients: drag(b)and diffusion proportional to σ, the standard deviation. An important property is its unigenerational or Markovian character. A second one leads to the construction of a second-order partial differential operator, called the process generator, formed of its two coefficients, and it results $L^+ f(x) = +b(x)\frac{\partial}{\partial x} f + D(x)\frac{\partial^2}{\partial x^2} f$, or more generally,

$$L^+ f(x) = \mathbf{b} \cdot \nabla f + D^{ij} \frac{\partial^2}{\partial x^i \partial x^j} f \tag{19}$$

In particular, assuming $b = 0$and $D^{ij} = \delta^{ij}$, the solution remains $(x, y \in \mathbb{R}^n)$,

$$u(t, x, y) = \frac{1}{\sqrt{(2\pi t)^n}} e^{-\frac{\|x-y\|^2}{2 \cdot t}} \tag{20}$$

which has the form of a Gaussian or "normal" distribution, characteristic of a Brownian motion.

Kinetic processes are the natural consequence of the presence of imbalances in a medium, and tend to restore equilibrium. The density of the flow plays an essential role and if the flow is also disturbed by a drag or drift, we must discount this drag. In particular, consider a probability distribution. According to the principle of conservation of probability mass, the change in density is measured by the divergence of the flow. Thus, the evolution of a probability density is given by the anti-gradient or negative gradient of the probability density, then a diffusion equation arises.

The density of the flow is represented by the vector field $J \mapsto J^i = -\frac{\partial}{\partial x^j} D^{ij}$, then we take into account the drag and add $(J-b)^i = -\frac{\partial}{\partial x^j} D^{ij}$. So, the current probability remains: $J^i = b^i - \frac{\partial}{\partial x^j} D^{ij}$. Now, we consider the principle of mass conservation expressed as continuity equation: $\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}Ju = 0$; or with a source f: $\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}Ju = f$. And finally, the Kolmogorov-f (or adjoint Fokker-Planck) equation arises,

$$Ju = \left(b - \frac{\partial}{\partial t}D\right)u \text{ and } \frac{\partial}{\partial t}u = -\frac{\partial}{\partial x}Ju + f \quad (21)$$

where $u$is the probability density of the random variable and J the probability current of the same variable, then

$$\frac{\partial}{\partial t}u = L_K u \quad (22)$$

Now, we consider another Kolmogorov-f equation,

$$J\hat{u} = \left(\hat{a} - \frac{\partial}{\partial x}D\right)\hat{u} \text{ and } \frac{\partial}{\partial \hat{t}}\hat{u} = -\frac{\partial}{\partial x}J\hat{u} + f \quad (23)$$

where $\hat{u}$ is the probability density of the random variable and J the probability current of the same variable, then:

$$\frac{\partial}{\partial \hat{t}}\hat{u} = L\hat{u} \quad (24)$$

where later we will take the case: $\hat{t} = -t$ and $\hat{a} = -b$.

Next, we look for a representation for an operator adjoint the Kolmogorov-f. For this we consider the stationary solution, which leads us to:

$$\begin{gathered}(\partial_x b(x,t) - \partial_{xx} D(x,t))u = 0\,, \\ \frac{b(x)}{D(x)}(D(x)u) - \partial_x (D(x)u) = 0\,,\end{gathered} \quad (25)$$

This (25) is a linear equation of the type: $\frac{\partial}{\partial x}y + p(x)y = 0$, with $y \leftrightarrow D(x)u$, and has a solution $D(x)u = Ce^{\int \frac{b}{D}d\bar{x}}$,$u = C\left(\frac{1}{D}e^{\int \frac{b}{D}d\bar{x}}\right) = Ce^{-\Phi}$. The probability stream is now $J(x,t) = -De^{-\Phi}\frac{\partial}{\partial x}e^{+\Phi}u$and the Kolmogorov operator is:$L_K = \frac{\partial}{\partial x}De^{-\Phi}\frac{\partial}{\partial x}e^{+\Phi}$. We define the operator $L_{FP} = e^{\Phi}L_K$which turns out to be its adjoint in the direction $\langle u, (e^{\Phi}L_K)g\rangle = \langle L_K u, g\rangle$ under appropriate boundary conditions [2], [12] and by applying integration by parts.

In summary, by integration by parts the derivative operator and the memory coefficient are transposed, the second derivative does it twice and the sign changes two times and remains the same as the original, while the first derivative does so only once and therefore its sign changes. In various dimensions the two operators: the Kolmogorov-f and the adjoint, Fokker-Planck, are

$$\begin{gathered}L_K(x,t) = -\partial_{x^i} b^i(x,t) + \partial_{x^i x^j} D^{ij}(x,t) \\ L_{FP}(x,t) = +b^i(x,t)\partial_{x^i} + D^{ij}(x,t)\partial_{x^i x^j}\end{gathered} \quad (26)$$

The equations produce the fundamental solutions, with the initial condition of acute or determined value in y, s: $\phi_i(x,t\,|y,s)|_{s=t} = \partial(x-y)$

$$\begin{bmatrix}\frac{\partial}{\partial t} \\ (-)\frac{\partial}{\partial s}\end{bmatrix}\phi_i(x,t|y,s) = \begin{bmatrix}L_K(x,t) \\ L_{FP}(x,t)\end{bmatrix}\phi_i(x,t|y,s) \quad (27)$$

If from another part and under suitable conditions, we find the solution of the evolution equation that produces a probability density

$$\frac{\partial}{\partial t}u = L_K u \quad (28)$$

We can consider the semigroup generator, defined by its action on the measurable and bounded functions with domain in the phasic space and ranges in the real ones, which allows to find its average value or mathematical expectation,

$$T_{t+\tau}f(x_0) = \int_X u(t+\tau, x_0, dx)\, f(x) \quad (29)$$

which leads to defining the process *memory coefficients* as averages:

$$b(x,t) = \lim_{\tau \to 0} \int_X u(t+\tau, \bar{x}, dx) \frac{(x-\bar{x})}{\tau},$$
$$D(x,t) = \lim_{\tau \to 0} \int_X u(t+\tau, \bar{x}, dx) \frac{1}{2} \frac{(x-\bar{x})^2}{\tau}, \quad (30)$$

and the generator of the semigroup is obtained:
and the generator of the semigroup is obtained,

$$L_K = \partial_x((-b)u) + \partial_{xx}((+D)u) + f \quad (31)$$

IV. DRAG AND POTENTIAL

In the context of Helmholtz's theorem, we consider the case of the coefficient in its form as a drag vector field, with an irrotational characteristic and, consequently, as a field derivable from a potential, but in particular, being of logarithmic type, as it has turned out from the symmetries study. If, in addition, the drag is represented by a quotient of polynomials: the numerator, of order 1 over the other of order 2, being their coefficients real, then the drag coefficient is linked to the Pearson distributions. In general, since the second polynomial, the one of order 2, has 2 roots, a diversity of distributions of the characteristics of those roots will emerge.

We now focus on the cases of two real and different roots, to later focus on the case of equal roots, which will provide an axis of symmetry and the mode or anti-mode context. The two polynomials contain 5 real coefficients, since the numerator must be of order 1, the coefficient of the power 1 must be different from zero, so we can set its value to 1 or what is equivalent, the remaining 4 are determined relative to the coefficient of the power 1. Thus, we deal with a space of 4 real dimensions, [13].

On the other hand, knowledge of a density of probability distributions allows us to determine the different central moments, but reciprocally, knowledge of the different moments makes it possible, in general, to determine the distribution. In particular, the survey of 4 sample moments accesses to 4 real numbers, which we will put in correspondence with 4 real coefficients, so this is a point in the four-dimensional space that provides us a representation for dragging in the form of a vector field.

Let the drag coefficient be: $\frac{b}{D} = -\frac{d}{dx}\ln\rho(x) = -\frac{P}{Q}$, with $\ln\rho(x)$ determining the Kolmogorov potential. For two different real roots, $x_1 \leq x_2$, the quotient is decomposed into partial fractions, $\left(\frac{\alpha-1}{x+x_1} + \frac{\beta-1}{-x+x_2}\right)$ with $x + x_1 > 0$and $x_2 - x > 0$, the potential remains $\ln\rho(x) = \int_{x_0}^{x} \frac{P}{Q} d\bar{x} = \int_{x_0}^{x} \frac{(\alpha-1)}{+x_1} d\bar{x} + \int_{x_0}^{x} \frac{(\beta-1)}{x_2-\bar{x}} d\bar{x}$, and integrand, $\ln\rho(x) = \ln C(x+x_1)^{\alpha-1}(-x+x_2)^{\beta-1}$, so the probability density is $\rho(x) = C(x+x_1)^{\alpha-1}(-x+x_2)^{\beta-1}$, with change of variable $\frac{x+x_1}{x_2+x_1} = p$ y $\frac{-x+x_2}{x_2+x_1} = 1-p$, it results: $\rho(p(x)) = \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1}$, which corresponds to the Beta distribution, normalized with the Euler Beta function $B(\alpha,\beta)$ and we remember that this distribution has already been cited since the time of Thomas Bayes (1763). At the end, the current distributions represented by:

$$\rho(p(x)) = \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1} \quad (32)$$

From the coefficients of the polynomial Q, both the classification vector $V = (d, \lambda)$, and its roots. But in addition, the powers of the so-called "success" and "failure" are calculated from the coefficients of the polynomial $P$and the roots of the polynomial Q. Case I corresponds to a discriminant $d < 0$, with $\lambda > 1$,

$$d = ac - b^2 = ac(1-\lambda)\lambda = \frac{b^2}{ac}$$
$$x_1 = -\frac{b}{a} + \frac{1}{a}\sqrt{-d} x_2 = +\frac{b}{a} + \frac{1}{a}\sqrt{-d}$$
$$\alpha - 1 = \frac{-a_0 + a_1(x_1)}{a(x_1+x_2)} = \frac{-a_0 + a_1(x_1)}{2\sqrt{-d}} \quad (33)$$
$$\beta - 1 = \frac{a_0 + a_1(x_2)}{a(x_1+x_2)} = \frac{a_0 + a_1(x_2)}{2\sqrt{-d}}$$

The mode position criterion is $X_M = \frac{\propto-1}{\propto+\beta-2} = \frac{a}{a_1}\frac{-a_0+a_1(x_1)}{a(x_1+x_2)} = \frac{a}{a_1}(\alpha-1)$, if $\propto > 1$ and $\beta > 1$, or if $\propto + \beta > 2$; it is a modal type distribution, or otherwise, an anti-modal one. The reference is the logistic mode and is located at:$x_{M,L} = \frac{1}{2}$, being $\propto = \beta = 2$.

With the 4 sample central moments μ_i , $i \in \{1,4\}$, the skewness and kurtosis parameters $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$, $\beta_2 = \frac{\mu_4}{\mu_2^2}$ are constructed, along with a convenient combination of the two:$B = 10\beta_2 - 12\beta_1 - 18$. The 3 coefficients of the quadratic polynomial Q and a linear coefficient P, with $a_1 = 1$,

$$a = -\frac{1}{B}(2\beta_2 - 3\beta_1 - 6)$$
$$b = -\frac{1}{2}a_0$$
$$c = -\frac{\mu_2}{B}(4\beta_2 - 3\beta_1) \quad (34)$$
$$a_1 = 1$$

$$a_0 = \frac{\sqrt{\mu_2 \beta_1}}{B}(\beta_2 + 3)$$

It is advisable to specify the units or physical dimensions used, the drag coefficient is $\left[\frac{b^1}{D}\right] \sim \frac{cm}{s}$, then it must be $e^{-U_K(p(x))} = \rho^D(p(x))$or,$-U_K(p(x)) = D\ln\rho$. And below we cite some examples of construction of the Kolmogorov potential or drag, of vector field type.

Example 1 Corn

The first is vital because it has to do with the energy of humans, that is, their food. In particular we highlight the corn grown in the so-called Mezquital Valley, Mexico. The density is platykurtic because it has kurtosis less than 3, $K_u < 3$, the asymmetry is positive which shows thicker tails on the left than on the right. The discriminant is negative, with $\lambda < 0$, so the density is type I or $Beta$. The "logistics" serves as a reference, for which the shape exponents are equal to 2 and its mode is located at $1/2$, in the middle of the interval $[0,1]$. In our case, the shape exponents are greater than 1, so the density is unimodal, bell-type, with the location of the mode to the left of $1/2$, [14], [13].

Example 2 Conchos River

Before being the vital water that runs through channels to arrive in time to irrigate crops, its accumulation in *"dams"* is required. For the annual data on runoff volumes at the *"Boquilla dam"* in Conchos River, the statistical analysis of the ungrouped data produces the first 4 moments and their subsequent parameters. It is obtained for the asymmetry parameter$\left(\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 1.1593\right)$, thus the distribution is asymmetric with thicker tails on the right. For kurtosis, $\left(\beta_2 = \frac{\mu_4}{\mu_2^2} = 4.0259\right)$ this exceeds the reference kurtosis, the "normal" one, thus it is leptokurtic. The classification parameters$(-8.2823 \times 10^5, -0.49421)$ place the distribution as Beta density, then observing the shape parameters $(-0.0002747, 2.7058)$, it turns out to be a decreasing density, with shifting to the left with respect to the central line of the "logistics", [15].

4.1 Kolmogorov Potential Well

As a third example we consider the particular and important case of the Kolmogorov potential emerging from an anti-modal form of a Pearson Beta distribution. We move the vertical axis to overlap it with the symmetry axis, with the exponents of the shape parameters coinciding and with a value linked to the other memory coefficient that antagonizes the drag.

We begin with an antimodal form of a Beta distribution: $\frac{1}{B(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1}$, with $\alpha < 1, \beta < 1$ or $\propto +\beta < 2$ and Pearson's Type II form $\propto = \beta$, with $\propto -1 = -D$, wich takes the form $\frac{1}{B(\alpha,\beta)}\left(y(1-y)\right)^{\alpha-1}$. We reinterpret the probability of success as an angle, then see it as small enough to allow a series expansion $\left(y(1-y)\right)^D$. We relocate the vertical axis $\frac{1}{2} + k\bar{x} = y$, $(1-\bar{y}^2)^D$, we put $k = \frac{\pi}{4\sqrt{2a}}$ and it remains$\left(\frac{1}{2}\right)^2\left(1 - \frac{1}{2}\left(\frac{\pi}{2a}\bar{x}\right)^2\right)$.

Or, the density is proportional to: $\left(1 - \frac{1}{2}\left(\frac{\pi}{2a}x\right)^2 + \cdots\right)^D$, which corresponds to an expansion of $cos\left(\frac{\pi}{2}\frac{1}{a}x\right)$. For $x = \pm a, cos\left(\frac{\pi}{2a}x\right)|_{x=\pm a} = 0$, and $U_K(x) \to +\infty$, so the width of the well is $2a$ and its depth is unlimited. In the middle $x = 0, cos\left(\frac{\pi}{2a}x\right)|_{x=0} = 1$ and $U_K(x) \to 0$. Now the width $2a$ is dimensionalized by $a \sim \sqrt{\frac{D}{\gamma}}$, therefore, if γ is too high with respect to D, the well approaches to a δ of Dirac type. Then, $\frac{\pi}{2a}x \sim \frac{\pi}{2}\sqrt{\frac{\gamma}{D}}x$, and it turns out: $\frac{1}{B(\alpha,\alpha)}\left(\frac{1}{2}\right)^{-2D}\left(1 - \frac{1}{2}\left(\frac{\pi}{2}\sqrt{\frac{\gamma}{D}}x\right)^2 + \cdots\right)^{-D}$. Definitely, the Kolmogorov potential becomes: $e^{-U_K(x)} = \left(cos\left(\frac{\pi}{2}\sqrt{\frac{\gamma}{D}}x\right)\right)^D$, or equivalently, [2]:

$$U_K(x) = -Dln\left(cos\left(\frac{\pi}{2}\sqrt{\frac{\gamma}{D}}x\right)\right) \qquad (35)$$

V. SCHRÖDINGER POTENTIAL

We highlight the important connection between the drag coefficient and the inverse problem for the potential in the Schrödinger equation. In a previous work, which can be consulted in [16], [4], [2], we saw that there is an interrelation between the drag coefficient $(b(x))$and a Schrödinger potential. Indeed, it is possible to follow the main path of knowing the Schrödinger potential $V_s(x)$, to then find the drag coefficient as a solution to a Ricatti equation, and complete the vital information of the two "memory coefficients". But reciprocally, the subsidiary path is to determine the drag coefficient in other ways and then find the potential $V_s(x)$. The aforementioned connection is shown in the following relationship, where $b(x) = U_K$,

$$V_S(x) = \frac{1}{4D}(U_K(x))^2 - \frac{1}{2}(U_K'(x)) \qquad (36)$$

Relationship that we illustrate with two simple and important examples. In the case of the Smoluchowski force [2], the drag is an irrotational field, therefore it is derivable from a potential. If drag is proportional to position: $b(x) = \gamma x$, then$b'(x) = \gamma$, so $V_S(x) = \frac{\gamma^2}{4D}x^2 - \frac{1}{2}\gamma = \gamma\left(\frac{1}{4}\left(\sqrt{\frac{\gamma}{D}}x\right)^2 - \frac{1}{2}\right)$. Therefore, a drag proportional to the position produces a parabolic Schrödinger potential and, in addition, $V_S(x) - \frac{1}{2}\gamma = \frac{\gamma^2}{4D}x^2$ results in the potential of a harmonic oscillator with a quadratic frequency proportional to γ^2. Conversely, if we assume the potential to be parabolic, we will find in the solution of the Ricatti equation the drag coefficient proportional to the position. As a precision about the physical units, it is observed that $V_S(x)\sim \frac{E_p(x)}{h}$, (potential energy over Planck's constant) and that $\left[\sqrt{\frac{D}{\gamma}}\right]$ has units of length.

In the second example, we have the alternative route of the Kolmogorov potential (as an irrotational field, it is determined from a scalar potential) $\frac{\gamma}{D}x = -\frac{d}{dx}\ln\mu(x)$, then $\mu(x) = Ce^{-\frac{1}{2}\frac{x^2}{D/\gamma}}$, a Pearson distribution of normal type $\left(0, \sqrt{\frac{D}{\gamma}}\right)$, or type XI, $\sigma^2 = \frac{D}{\gamma}$ and $C = \sqrt{\frac{\gamma}{2\pi D}}$. And it is observed that if $\frac{D}{\gamma} \ll 1$, $\sigma \to 0$ and the distribution approaches a Diracδ, which means that the variable loses its random character or "collapses" as they say in Physics. Note the quicker path in this second example that leads immediately to the probability density.

The solutions of the Schrödinger equation of course generally depend on the potential of the same name $(V_S(x))$. The probability density is represented as a product of two solutions, one is a solution of the equation and the other is a solution of the adjoint, according to Born's statistical interpretation. But this potential $(V_S(x))$ is related to the drag coefficient $(b(x) = U_K)$ and this, in turn, is related to the Kolmogorov potential, so this latter potential will be present in the probability density and its "quantum" consequences.

Especially in the Beta distribution, as a Pearson distribution, it can be observed that the density of the distribution can be factored into two functions with antagonistic tendencies, the first linked to the success of the result and the second to its failure, neither of them in themselves is a density, they are functions of type C^2. But then the density is also factored into a pair of exponentials of two potentials, $\mu = e^{V_1} \cdot e^{V_2}$. If we resort to the "rhombus geometry" that is constructed in a plane using the complex representation that combines the semisum of a complex and its conjugate, with semidifference of the two. We can associate one diagonal to the sum of two magnitudes and the other diagonal to the difference of the same magnitudes and presents the characteristic of orthogonality as a manifestation of independence, antagonism and non-correlation. In turn, we associate the first potential with the sum and the second, with the difference: $\mu = \phi_1\phi_2 = e^{V_1}e^{V_2} = e^{\frac{1}{2}(R+S)}e^{\frac{1}{2}(R-S)} = e^R$, with $\phi_1 = e^{\frac{1}{2}(R+S)}$ and $\phi_2 = e^{\frac{1}{2}(R-S)}$. Due to the Ricatti equation that relates the drag coefficient to the potential $(V_S(x))$, linearity is not possible, but the sum and difference of drag coefficients can be related to the sides in the aforementioned geometry of the rhombus. The drag coefficients will be$-b_1 = \nabla\ln\phi_1$, $-b_1 = \nabla\frac{1}{2}(R+S)$ and$-b_2 = \nabla\ln\phi_2$, $-b_2 = \nabla\frac{1}{2}(R-S)$. Or $b_1 + b_2 = -\nabla R$ *and* $b_1 - b_2 = -\nabla S$. In the case of Born's interpretation, we make the real axis coincide with the first diagonal of the rhombus, while we align the second, the orthogonal, parallel to the imaginary axis, thus ϕ_1 corresponds to the wave functionψ and ϕ_2 to the adjoint that reduces to the complex conjugate of the wave functionψ, $(\phi_1, \phi_2) \leftrightarrow (\psi, \overline{\psi})$, $\psi = e^{R+iS}$, $\overline{\psi} = e^{R-iS}$. The ϕ_i are determined from the drag coefficients b_i as gradients of the Kolmogorov potentials $b_i = -\nabla\ln\phi_i$. Furthermore, with the (addition, subtraction) of the drag coefficients the potentials (R, S) are determined, with $R = \ln\mu$, and $S = \ln(\phi_1/\phi_2)$, [1].

Now what is the meaning of the pair ϕ_1 and ϕ_2? We have already highlighted its analogy with the wave functions of the Schrödinger equation and its complex conjugate.

There are two diffusion processes "in parallel" or overlapping: one evolutionary and the second, involutive. And they produce the pair of lateral fundamental solutions:ϕ_i. Summarizing, we reiterate that the $\phi_i$are mutually determined with the drag coefficients $b_i$as gradients of the Kolmogorov potentials $b_i = -\nabla\ln\phi_i$. But in addition, with the (addition, subtraction) of the drag coefficients the potentials (R, S), are determined, being $R = \ln\mu$, and $S = ln(\phi_1/\phi_2)$. Therefore, the sum of the two drag coefficients determines the probability density. In the "rhombus geometry" and during a chosen time interval, the evolution takes place above the main diagonal, opening the transverse diagonal, while the involution takes place below said diagonal, also opening the transverse diagonal but in the opposite direction to the first; or, the triangle lower than the main diagonal is the mirror image of the upper triangle.

We remember that we represent the flow density by the vector field $J \mapsto J^i = -\frac{\partial}{\partial x^j} D^{ij}$,after having disaggregated the drag: $(J-b)^i = -\frac{\partial}{\partial x^j} D^{ij}$. Its combination with the law of conservation of probability mass produces the Kolmogorov-f (forward) equation. Next, we look for the adjoint of this operator and the Fokker-Planck equation results. Finally, the two operators, the Kolmogorov operator and the Fokker-Planck operator in several dimensions, are formulated as: $L_K(x,t) = -\partial_{x^i} b^i(x,t) + \partial_{x^i x^j} D^{ij}(x,t)$ and $L_{FP}(y,s) = +b^i(y,s)\partial_{y^i} + D^{ij}(y,s)\partial_{y^i y^j}$, which act on functions and produce the probability densities. However, when the initial condition indicates a determined value or with suppression of the random character of the variable, in y,s: $\phi_i(x,t|\mathrm{y},\mathrm{s})|_{s=t} = \delta(x-y)$, the equations produce the fundamental solutions and also for $s \le t$, with initial condition$\phi_2(x,t|\mathrm{y},\mathrm{s})|_{s=t} = \delta(x-y)$, the representation of the evolution to the past or involution is obtained $L_{FP} \leftrightarrow (-)\frac{\partial}{\partial s}$, $L_K \leftrightarrow (+)\frac{\partial}{\partial t}$,

$$\frac{\partial}{\partial t}\phi_1(x,t|y,s) = L_K(x,t)\phi_1(x,t|y,s)$$
$$(-)\frac{\partial}{\partial s}\phi_2(x,t|y,s) = L_{FP}(y,s)\phi_2(x,t|y,s) \quad (37)$$

With the fundamental solutions in turn, we obtain the transition probability densities: by integration over characteristic functions, $u_i(x,t\,|\,\mathrm{B},\mathrm{s}) = \int \phi_i(x,t\,|\,\mathrm{y},\mathrm{s})\chi_B(y)dy$. The $u_i(x,t\,|\,\mathrm{B},\mathrm{s})$ are transition probability densities that obey the Chapmann-Kolmogorov equation: $u_1(x,t\,|\,\mathrm{B},\mathrm{s}) = \int u_1(x,t\,|x_\tau,\tau) \cdot u_1(x_\tau,\tau\,|\mathrm{B},\mathrm{s})dx_\tau$. Then the probability $u_1(x,t\,|\,\mathrm{B},\mathrm{s})$arises from a forward diffusion process in the temporal sense, while $u_2(x,t\,|\,\mathrm{B},\mathrm{s})$ arises from an involutive or backward diffusion process. From the Kolmogorov-f equations and its adjoint, Fokker-Planck, two alternative and parallel diffusion processes arise, one in an evolutionary sense, and the other, in an involutive sense, [2].

VI. ALTERNATIVE BLACK-SCHOLES

We remember some of the important elements in financial phenomena. It is about the evolution of the price of assets and their derivatives. The return rate of the price of an asset, or relative variation thereof, is the superposition of the deterministic price variation, driven by drift, with the random variation, proportional to the variation of the Brownian trajectory, with the standard deviation as proportionality coefficient. If the standard deviation is too small, few volatility, the deterministic type of change predominates driven by the drift parameter, or average value of the changes. According to the Itô Formula, the change in the value of the derivative has its deterministic component with drift decreased by half of the variance and its random component as a change in the trajectory of the Brownian movement, with the same standard deviation, [17], [18].

The variation in the trajectory of a stochastic differential equation is split into a deterministic component $(a_1 dt)$ and its random component $(a_2 dW)$,

$$dX = a_1 dt + a_2 dW \quad (38)$$

According to Itô, the rate of return $\left(\frac{dS}{S}\right)$ is divided into the deterministic component (μdt) and the random component (σdW), where $\mu$is the mean value, $\sigma$is the standard deviation and $W$is a Brownian movement,

$$\frac{dS}{S} = \mu dt + \sigma dW \quad (39)$$

and for the variation in the value of the derivative $(V(S,t))$, $\left(\frac{dS}{S} = dV\right)$,

$$dV = \left(\mu - \tfrac{1}{2}\sigma^2\right)dt + \sigma dW \quad (40)$$

From which it results, under the conditions $V(0,t) = 0$ and $V(S,t) \sim S$when$S \to \infty$, the evolution equation for the value of the derivative

$$\frac{\partial}{\partial t}V + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}V + rg(S)\frac{\partial}{\partial S} - rV = 0 \quad (41)$$

It is a parabolic partial differential equation in the variables (S,t). But it is also a Kolmogorov-f, or adjoint Fokker-Planck, equation. So now we reinterpret this evolution as such an equation, with its 2 memory coefficients.

We imagine the coefficient $g(S)$ as a Langevin field $b = D^{(1)}(S) = 1 - \frac{(a/2)^2}{S^2}$, which is interpreted as $g(S) = \frac{(a/2)^2}{S} + S$, and that satisfies $g(S) \approx S$ if $S \to \infty$. Therefore, this interpretation of the drag coefficient allows us to incorporate supply and demand variations, with their effects on the price of derivatives

$$\frac{\partial}{\partial t}V + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}V + r\hat{g}(S)\frac{\partial}{\partial S}V - rV = 0$$

$$\hat{g}(S) = \frac{a^2}{S} \vee S \quad (42)$$

$$a = S_e$$

The price - elasticity of demand (a^2/S), is $e_{Pd} = -\frac{dQ_d/Q_d}{dS/S} = 1$, or of unitary character, where the relative change in demand equals the relative change in the price of the derivative, [19]. Similarly, the price - elasticity of supply (S), is $e_{Po} = -\frac{dQ_o/Q_o}{dS/S} = 1$, and also has a unitary character, which means that the relative change in supply is equal to the relative change in the price of the derivative.

On the other hand, the two curves intersect at the equilibrium value $S_e = +a$. The supply and demand curves are illustrated in the graph 1, where the price (S) is marked on the horizontal axis and the quantity supplied and/or demanded (Q_i) on the vertical axis, being: $Q_1 \sim S$ (dotted line); $Q_2 \sim S^{-1/2}$(continuous line).

6.1 Supply vs. Demand

Quantities supplied and demanded are plotted against prices of the derivatives: $Q_1 \sim S$ (dotted line); $Q_2 \sim S^{-1/2}$(continuous line) (see Figure 1).

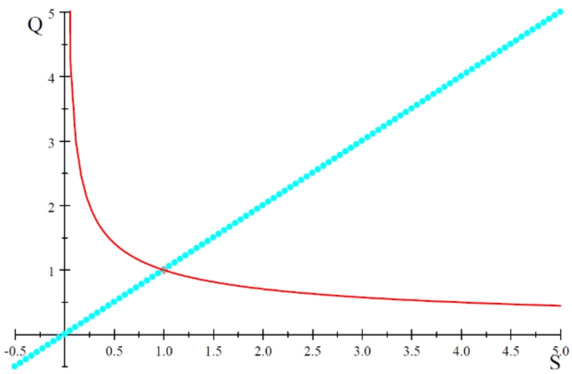

Figure 1. Supply vs demand curves, $Q_1 \sim S$ (dotted line); $Q_2 \sim S^{-1/2}$(continuous line).

VII. CONCLUSIONS

The study of the symmetries contained in the Kolmogorov-f equation produces a drag coefficient of logarithmic type or logarithmic gradient and in particular a Kolmogorov potential well. On the other hand, a Kolmogorov potential is present in the Schrödinger potential through a Ricatti equation that establishes the connection, so it manifests itself on the "quanta" that result from the stationary solution for the eigenvalues of the Schrödinger equation, as in the case of the aforementioned harmonic oscillator.

The probability density arises from the joint action of two logarithmic Kolmogorov potentials, one emerges from the Kolmogorov operator and the other from its adjoint. We imagine them as two vectors in a plane and with a difference in the phase, so their sum is represented by the gradient of one potential and their difference by the gradient of the 2nd potential. Or, the sum determining the potential along one of the diagonals of the rhombus and the difference, along the second diagonal, transversal to the first and determining the second potential. Or, the lower part of the rhombus as a mirror image of the upper part, being the main diagonal the place of the "collapse" of the random variable.

From the above conclusions it can be inferred that the true differential equation for the evolution of a "quantum" system is the Kolmogorov-f equation and its Fokker-Planck adjoint, because they produce the probability density of the phenomenon and its consequences; meanwhile, Schrödinger's equation provides marginal solutions, known by the lyrical expression of "wave functions", which rediscover their path and objective through Born's interpretation. We think that the term "diffusion functions" would have been more appropriate because they are more linked to diffusion processes than to a wave equation.

The Kolmogorov potential of logarithmic type is present in innumerable phenomena of reality with an important random component and can be reconstructed through the Pearson distributions, which are obtained from the first 4 sample moments or by the 4 parameters: mean, variance, skewness and kurtosis.

We observe the principle of Duality in the two possible solutions produced by the symmetry or Lie groups for the drag coefficient, reducing it to the simplest form of orientation opposition, or as a change of orientation in the Langevin field by change in the Kolmogorov potential, either as the existence of a pair of operators: the Kolmogorov operator and its Fokker-Planck adjoint, or as the pair of Kolmogorov evolutions: forward or backward, or as a pair of diffusion processes in parallel, one towards the future, and the second towards the past.

The inverse problem for the drag coefficient in the Black-Scholes evolution equation for derivative securities can be formulated containing the law of supply and demand. Meanwhile, the two coefficients of elasticity result with the particularity of being unitary.

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AUTHORS BIOGRAPHY

José Roberto Mercado-Escalante. Retired researcher from the Mexican Institute of Water Technology –IMTA, Mexico. Dr. of Sciences (Mathematics) by the National Autonomous University of Mexico-UNAM, Master Degree in Mathematics by the Autonomous University of Puebla-BUAP, Mexico; graduates in Physics by the National University of Colombia. Research areas: inverse problems, fractals and fractional derivatives, mathematical aspects of hydraulics.

Pedro Antonio Guido-Aldana. Researcher. Mexican Institute of Water Technology – IMTA, Mexico. Dr. and Master in Hydraulics by the National Autonomous University of Mexico-UNAM, Civil Engineer, Coast University, Colombia. Research interest: hydraulics, fluid mechanics, river mechanics, potamology, experimental research with PIV and LDA.