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Flow-Induced Vibration and Stability Analyses Of Axially Functionally Graded Non-Prismatic Fluid-Conveying PipesResting On Variable Non-Winkler Foundation

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Abstract
In this present study, the analysis of free vibration and stability of axially functionally graded non-uniform fluid-conveying pipes and resting on a two-parameter foundation, such as Pasternak foundation under various sets of boundary conditions namely Clamped-clamped, Clamped-pinned, Pinned-pinned is carried out. An approach known as Variational Iteration Method (VIM) is used to carry out the analysis. The present study assumes that each of the Young modulus, material density, cross sectional area, moment of inertia of the pipe and foundation parameters varies along the pipe axis. Natural frequencies and critical flow velocities are obtained for various classical end conditions. Also, influence of various parameters namely mass ratio, foundation stifnesses, non-uniformity terms and non-homogeneous materials on the natural frequencies and critical flow velocities of the present work is examined and the findings are presented. The obtained solutions are compared with some available results in the literature and very good agreements are observed. These new results will also serve as a benchmark for subsequent researchers. Keywords: Axially Fuctionally Graded Materials (AFGMs), Critical velocity, Free Vibration, Natural Frequency, Variational Iteration Method (VIM), Non-uniform, Non-Winkler foundation

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I INTRODUCTION

A class of materials known as functionally graded materials (FGMs) may be defined as a group of composite materials possessing different material properties from one surface to another. These materials have certain structural performance requirements and they are useful in many areas of application such as aerospace, transportation, biomedical installations, automobile sector, and defense industries. In particular, researchers have shown great interest in this broad range of applications due to the fact that such materials have the benefit of being able to withstand high-temperature, low density and high toughness while maintaining structural integrity [1]. In addition, it is remarked at this juncture that the corresponding Young modulus and density of such materials (FGMs) are not constant. It is reported in [2] that studies involving functionally graded beam (FGB) structures have become a fertile area of research since beam structures have been widely used in various fields which involve significant requirement of material properties. [3] investigated the effect of material distribution, velocity of the moving load and excitation frequency on the dynamic response of beams. It was assumed that the material properties of the

beam vary continuously in the thickness direction according to the power law. Free vibration analysis of functionally graded beams (FGBs') with simply supported edges was presented in [4]. Different higher order shear deformation theories and classical beam theories are used in the analysis. Natural frequencies and some mode shapes are also obtained for different material properties and slenderness ratio.
It is of good note that studies on pipeline conveying fluid have also been continuously carried out in the past decades. Indeed, vibration and stability of pipes conveying fluid have been studied for more than six decades both theoretically and experimentally in the field of fluid structure interaction,[5]. To suppress or avoid the instability of flow-induced vibrations of pipes conveying fluid, several control methods have been proposed. The available control methods for pipes conveying fluid may be grouped into two: the passive methods and active methods [6].

Non-uniform (non-prismatic) pipes are tubular structure or hollow cylinder of metal, wood or other material usually having variable cross-sectional area and moment of inertia, used for the conveyance of either liquid, gaseous substances or their mixtures. Non-uniform pipes transporting fluid have received vast attraction over the past decades due to their applicability of fluid flowing through them from one place to another. Several of these applications could be found in various engineering and medical fields such as oil and gas exploration, hydropower, nuclear reactor, exhaust pipes, heat exchanger, agricultural irrigation, marine risers, blood flowing through human/animal arteries and flow via pulmonary and urinary systems. Furthermore , it is hereby remarked that using non-uniform pipes in industries may be less costly, especially in situation where good stiffness and light-weight structures are taken into consideration [7]. Mathematically speaking and unlike prismatic pipes, the partial differential equation governing the motion of non-uniform pipes is usually made up of variable coefficients. This makes the governing equation amenable to direct integration only in some special cases. Otherwise, it is solved using either an approximate technique or a numerical method. Hence, a dynamical problem involving both functionally graded materials and non-uniform property such as the present case is in general complicated since the number of variable coefficient increases. Examples of previous works in this direction includes the work in [8] where the lateral vibration of uniform pipes conveying fluid made of FGMs using Simpletic method is investigated. It was found that for a clamped-clamped FGM pipe conveying fluid, the dimensionless critical flow velocity for first mode divergence and the critical coupled-mode flutter were obtained. [9] also investigated the dynamic behaviour of Axially Functionally Graded (AFG) clamped-clamped pipes conveying fluid using the Generalized Integral Transform Technique (GITT). The effects of Young's modulus variation, material distribution and flow velocity on natural frequencies and vibration amplitude of pipes conveying fluid were analyzed. However, the pipe considered was a uniform one. [6] studied the dynamics of AFG clamped free uniform pipes conveying fluid using Differential Quadrature Method (DQM). [10] examined the dynamic behaviour of cracked uniform functionally graded (FG) material pipe conveying fluid using Galerkin Method. The associated natural frequencies and stability of the system were determined. [11] studied the vibration of pipes conveying fluid with variable cross section (that is, non-uniform pipes) employing Galerkin technique. The influence of the flow velocity and variable cross-section on the natural frequencies, critical flow velocity and system stability were discussed. It is noted however, that the materials involved in this work were not functionally graded. Recently, [12] studied the dynamic response of axially functionally graded conical pipes (a non-uniform structure) conveying fluid. It was found that the natural frequency and critical velocity increase with increasing volume fraction index.

It is remarked at this juncture, however, that in all the above previous studies, no consideration is given to the effect of elastic subgrades on the natural frequencies or critical velocity of the dynamics of pipes conveying fluid. For practical applications, it is relevant and useful to consider fluid-conveying pipes resting on an elastic foundation. For instance, in real life, long pipelines transporting fluid (petroleum, gas, water etc.) are usually supported on elastic foundation like soil made up of various types of terrain. The governing equation of motion for this type of system must therefore contain a model describing the characteristics of such a soil. The most common and simplest of such a model is known as Winkler foundation having foundation constant K [13]. Some of the works that dealt with dynamics of pipes conveying fluid resting on Winkler foundation, only includes the works in [13] , [14] and [15]. However, since the characteristic feature of this well-known foundation model is the discontinuous behaviour of the axial displacements [16] while, in practice, the axial displacements continue beyond the force axis, a more realistic elastic foundation model called Pasternak foundation (i.e a non-Winkler foundation) model is considered in this present work. Such a model is characterized with two foundation constants namely spring constant K and shear modulus G. [17] examined the critical velocity of a fluid flowing in a pipeline and resting on Pasternak foundation. [18] investigated the effect of Pasternak foundation on vibrations of fluid-conveying pipes. Also, [19] applied Galerkin Finite Element Method to analyze the dynamic behaviour of a fluid-conveying pipe resting on Pasternak foundation. Later, [20] examined the dynamics and stability of multi-span pipe conveying fluid embedded in Pasternak foundation. Notwithstanding, it is hereby remarked that all the above previous work on either Winkler or Pasternak foundation do not take into consideration the functionally graded and non-uniformity properties of the pipe. Several techniques have been adopted for vibration and stability analysis of pipes conveying fluid. Various approximate solutions are available in literature for vibrational characteristics of pipes as mentioned above. Additional instances include the following; Fourth Order Runge-Kutta Method (4th Order RKM) was used in [21] to study the effect of open crack and a moving mass on the dynamic behaviour of a simply supported pipe with moving mass, Generalized Differential Quadrature Method (GDQM) was applied in [22] to determine the critical flow speed of pipes, Generalized Integral Transform Technique (GITT) was also employed in [23] to examine the dynamic behaviour of pipes conveying fluid, Adomian Decomposition Method (ADM) and Differential Transform Method (DTM) were used in [24] to investigate the vibrational behaviour of fluid conveying Timoshenko pipeline, Central Difference Method (CDM) was used in [25] to examine the dynamic analysis of a cantilevered pipe conveying fluid, [26] used Complex Mode Technique to investigate the Frequency analysis of Functionally Graded Curved Pipes Conveying Fluid and Harmonic Differential Quadrature Method was later used by [27] to examine the dynamics of

AFG conical pipes conveying fluid. For pipes, however, where higher modes are required, it is difficult to compute the necessary results employing the above listed methods [28]. Notwithstanding, higher mode results from practical point of view are essential, especially, in the design of AFG pipes [28]. Hence, a reliable and efficient approach such as Variational Iteration Method (VIM) is introduced in this work. According to [28], this method has been shown to handle high mode natural frequency problem efficiently. This method was first proposed by [29] to solve various linear and non-linear differential equations. This method was first used to solve vibrational problem involving conveying fluid pipes in [30].

In the above context, the current article aimed, at determining the dynamic characteristics of functionally graded (FG), non-uniform fluid-conveying pipes resting on variable Pasternak foundation subjected to various classical boundary conditions using Variational Iteration Method (VIM). This problem, to the best of the author's knowledge has not been investigated. Emphatically speaking, this work has appropriately extended the works in [Chellapeilla (2007 and 2008), and Jiya *et.al.* (2018)] by including the effect of functionally graded materials, non-uniformity of both the pipe and foundation on the dynamic system. Interesting results for varying values of functionally graded materials, non-uniformity and foundation parameters are obtained and presented.

II FORMULATION OF THE PROBLEM

According to the Euler-Bernoulli theory which takes no account of shear deformation and rotatory inertia, the governing equation of motion for free vibration of axially functionally graded (AFG), non uniform fluid-conveying pipes resting on variable two parameter elastic foundation is expressed as: [12] and [18]

$$\frac{\partial^2}{\partial x^2}\left[E(x)I(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right] + m_f V^2 \frac{\partial^2 w(x,t)}{\partial x^2} + 2m_f V \frac{\partial^2 w(x,t)}{\partial x \partial t} + [m_f + \rho(x)A(x)]\frac{\partial^2 w(x,t)}{\partial t^2}$$
$$+k_w(x)w(x,t) + \frac{\partial^2}{\partial x^2}\left[k_p(x)\frac{\partial w(x,t)}{\partial x}\right] = 0, \quad (0 \le x \le L) \tag{1}$$

where $w(x,t)$ is the transverse deflection of the pipe at position x along the axial coordinate and time t, $\rho(x)A(x)$ is the variable mass of the pipe per unit length which depends on both the material density $\rho(x)$ and the cross sectional area $A(x)$. m_f is the mass of the fluid, V is the fluid flow velocity, $k_w(x)$ and $k_p(x)$ is defined as the variable Winkler foundation's coefficient and the second Pasternak foundation's coefficient, respectively. $E(x)I(x)$ is the variable flexural rigidity depending on both Young's modulus $E(x)$ and the area moment of inertia $I(x)$ of the pipe while L is the length of the pipe.

The equation of motion may be subjected to various sets of boundary conditions. However, the present study considers the following three various types of boundary conditions namely; Simply-supported pipe, Clamped-clamped pipe and Clamped-pinned pipe.
For Simply-Supported pipe
at x = 0;

$$w(0,t) = 0, \qquad \frac{\partial^2 w(0,t)}{\partial x^2} = 0 \tag{2}$$

at x = L;

$$w(L,t)=0, \qquad \frac{\partial^2 w(L,t)}{\partial x^2}=0 \tag{3}$$

Clamped-Clamped pipe
at x = 0;

$$w(0,t)=0, \qquad \frac{\partial w(0,t)}{\partial x}=0 \tag{4}$$

at x = L;

$$w(L,t)=0, \qquad \frac{\partial w(L,t)}{\partial x}=0 \tag{5}$$

Clamped-Pinned pipe
at x = 0;

$$w(0,t)=0, \qquad \frac{\partial w(0,t)}{\partial x}=0 \tag{6}$$

at x = L;

$$w(L,t)=0, \qquad \frac{\partial^2 w(L,t)}{\partial x^2}=0 \tag{7}$$

Assuming that equation (1) is harmonic, so that

$$w(x,t)=W(x)e^{i\omega t} \tag{8}$$

It follows, therefore, that equation (1) reduces to the following ordinary differential equation:

$$\frac{d^2}{dx^2}\left[E(x)I(x)\frac{d^2W}{dx^2}\right]+m_fV^2\frac{d^2W}{\partial x^2}+2m_f i\omega V\frac{dW}{dx}-\left[m_f+\rho(x)A(x)\right]\omega^2 W + k_w(x)W(x)+\frac{d}{dx}\left[k_p(x)\frac{dW}{dx}\right]=0 \tag{9}$$

where ω is the natural frequency of the vibrating system and $i=\sqrt{-1}$
For simplicity, the following dimensionless parameters are introduced:

$$\begin{gathered}\overline{W}=\frac{W}{L}, \quad \overline{x}=\frac{x}{L}, \quad D(\overline{x})=\frac{E(x)I(x)}{E_0I_0}, \quad \overline{v}=\sqrt{\left[\frac{m_f}{D(\overline{x})E_0I_0}\right]}VL, \\ P(\overline{x})=\frac{m_f+\rho(x)A(x)}{m_f+\rho_0A_0}, \quad \overline{\omega}=\omega\sqrt{\frac{(m_f+\rho_0A_0)L^4}{E_0I_0}}, \\ F(\overline{x})=\frac{K_w(x)}{K_{w_0}}, \quad T(\overline{x})=\frac{K_p(x)}{K_{p_0}}, \quad \beta=\frac{m_f}{m_f+\rho_0A_0}\end{gathered} \tag{10}$$

where β is the mass ratio, $\overline{\omega}$ is the dimensionless frequency parameter and $\overline{v}$ is the dimensionless flow velocity, E_0I_0 is the flexural rigidity at $\overline{x}=0$, ρ_0A_0 is the mass per unit length at $\overline{x}=0$, K_{w_0} is the Winkler foundation stiffness at $\overline{x}=0$ and K_{p_0} is the Pasternak foundation stiffness at $\overline{x}=0$
Substituting equation (10) into equation (9) leads to:

$$\begin{gathered}\frac{d^4\overline{W}}{d\overline{x}^4}+2\frac{D'(\overline{x})}{D(\overline{x})}\frac{d^3\overline{W}}{d\overline{x}^3}+\frac{D''(\overline{x})}{D(\overline{x})}\frac{d^2\overline{W}}{d\overline{x}^2}+\overline{v}^2\frac{d^2\overline{W}}{d\overline{x}^2}+2i\overline{v}\,\overline{\omega}\sqrt{\beta}D(\overline{x})\frac{d\overline{W}}{d\overline{x}} \\ -\frac{P(\overline{x})}{D(\overline{x})}\overline{\omega}^2\overline{W}+\frac{F(\overline{x})}{D(\overline{x})}\overline{K}_w\overline{W}-\overline{K}_p\frac{T(\overline{x})}{D(\overline{x})}\frac{d^2\overline{W}}{d\overline{x}^2}-\overline{K}_p\frac{T'(\overline{x})}{D(\overline{x})}\frac{d\overline{W}}{d\overline{x}}=0\end{gathered} \tag{11}$$

where $\overline{K}_w=\frac{K_{w_0}L^4}{E_0I_0}$; $\quad \overline{K}_p=\frac{K_{p_0}L^2}{E_0I_0}$
For simplicity, it is noteworthy to define the following coefficients:
$Q_1(\overline{x})=2\frac{D'(\overline{x})}{D(\overline{x})}, \quad Q_2(\overline{x})=\frac{D''(\overline{x})}{D(\overline{x})}, \quad Q_3(\overline{x})=\sqrt{\frac{1}{D(\overline{x})}}, \quad Q_4(\overline{x})=\frac{P(\overline{x})}{D(\overline{x})},$
$Q_5(\overline{x})=\frac{F(\overline{x})}{D(\overline{x})}, \quad Q_6(\overline{x})=\frac{T(\overline{x})}{D(\overline{x})}, \quad Q_7(\overline{x})=\frac{T'(\overline{x})}{D(\overline{x})}$
In view of the above seven functions, equation (11) becomes

$$\begin{gathered}\frac{d^4\overline{W}}{d\overline{x}^4}+2Q_1(\overline{x})\frac{d^3\overline{W}}{d\overline{x}^3}+Q_2(\overline{x})\frac{d^2\overline{W}}{d\overline{x}^2}+\overline{v}^2\frac{d^2\overline{W}}{d\overline{x}^2}+2i\overline{v}\overline{\omega}\sqrt{\beta}Q_3(\overline{x})\frac{d\overline{W}}{d\overline{x}} \\ -Q_4(\overline{x})\overline{\omega}^2\overline{W}+Q_5(\overline{x})\overline{K}_w\overline{W}-Q_6(\overline{x})\overline{K}_p\frac{d^2\overline{W}}{d\overline{x}^2}-Q_7(\overline{x})\overline{K}_p\frac{d\overline{W}}{d\overline{x}}=0\end{gathered} \tag{12}$$

The associated set of dimensionless boundary conditions, in view of equations (2)- (8) are as follows:

(i) Simply-Supported: at $\overline{x} = 0$

$$\overline{W}(0) = 0, \quad \frac{d^2\overline{W}(0)}{d\overline{x}^2} = 0 = 0 \tag{13}$$

at $\overline{x} = 1$

$$\overline{W}(1) = 0, \quad \frac{d^2\overline{W}(1)}{d\overline{x}^2} = 0 = 0 \tag{14}$$

(ii) Clamped-Clamped: at $\overline{x} = 0$

$$\overline{W}(0) = 0, \quad \frac{d\overline{W}(0)}{d\overline{x}} = 0 = 0 \tag{15}$$

at $\overline{x} = 1$

$$\overline{W}(1) = 0, \quad \frac{d\overline{W}(1)}{d\overline{x}} = 0 = 0 \tag{16}$$

(iii) Clamped-Pinned: at $\overline{x} = 0$

$$\overline{W}(0) = 0, \quad \frac{d\overline{W}(0)}{d\overline{x}} = 0 = 0 \tag{17}$$

at $\overline{x} = 1$

$$\overline{W}(1) = 0, \quad \frac{d^2\overline{W}(1)}{d\overline{x}} = 0 = 0 \tag{18}$$

Furthermore and following [28], the breadth of the pipe is assumed to vary linearly, so that both the cross-sectional area $A(x)$ and moment of inertia $I(x)$ along the pipe axis can be expressed as:

$$A(\overline{x}) = A_0(1 - \alpha\overline{x}) \tag{19}$$

$$I(\overline{x}) = I_0(1 - \alpha\overline{x}) \tag{20}$$

where A_0, I_0 are the cross-sectional area and moment of inertia at $\overline{x} = 0$ respectively and α is the breadth non-uniformity parameter.
Also, the linear variation of both the Winkler foundation stiffness $K_w(x)$ and Pasternak foundation stiffness $K_p(x)$ can be written as

$$K_w(\overline{x}) = K_{w_0}(1 - \eta\overline{x}) \tag{21}$$

$$K_p(\overline{x}) = K_{p_0}(1 - \eta\overline{x}) \tag{22}$$

where η is the non-uniformity foundation stiffness parameter.
Furthermore, in this present work, the material properties Young modulus $E(x)$ and mass density $\rho(x)$ along the pipe axis were assumed to be constant, varying linearly or parabolically [28]. In particular,
For the constant case:
E_0 and ρ_0 are the Young modulus and material density of the pipe at $\overline{x} = 0$, respectively.
For the linearly varying case:

$$E(\overline{x}) = E_0(1 + \overline{x}) \tag{23}$$

$$\rho(\overline{x}) = \rho_0(1 + \overline{x}) \tag{24}$$

For the parabolic case:

$$E(\overline{x}) = E_0(1 + \overline{x} + \overline{x}^2) \tag{25}$$

$$\rho(\overline{x}) = \rho_0(1 + \overline{x} + \overline{x}^2) \tag{26}$$

The dimensionless equation of motion in (12) is further simplified by inserting appropriate equations out of those in equations (19-26)

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III METHOD OF SOLUTION

A semi-analytical method known as Variational Iteration Method (VIM) is applied in this research work to obtain the natural frequencies and critical flow velocities of AFG tapered pipes resting on variable Pasternak foundation.
To illustrate the basic idea of VIM, a general non-linear differential equation can be considered

$$LU(\overline{x}) + N\overline{U}(\overline{x}) = g(\overline{x}) \tag{27}$$

where L is the linear operator, N is the non-linear operator and $g(x)$ is the continuous function. Thus, the correctional functional can be constructed in the form:

$$U_{m+1}(\overline{x}) = U_m(\overline{x}) + \int_0^{\overline{x}} \lambda[LU_m(\tau) + N\overline{U}_m(\tau) - g(\tau)]d\tau \tag{28}$$

where λ is a Lagrange multiplier which can be determined optimally via the variational theory, $\overline{U}_m$ is a restricted variation which means $\delta\overline{U}_m = 0$ and "m" is the mth approximation
By making use of equation (28), the correctional functional for the dimensionless governing equation (12) can be written as:

$$\begin{aligned}\overline{W}_{m+1}(\overline{x}) = \overline{W}_m(\overline{x}) + \int_0^{\overline{x}} \lambda(\tau)[\overline{W}^{iv}_m(\tau) + 2Q_1(\tau)\overline{W}'''_m(\tau) + Q_2(\tau)\overline{W}''_m(\tau)\\ + \overline{v}^2\overline{W}''_m(\tau) + 2i\overline{v}\sqrt{\beta}Q_3(\tau)\omega\overline{W}'_m(\tau) - Q_4(\tau)\omega^2\overline{W}_m(\tau)\\ + Q_5(\tau)\overline{k_w}\overline{W}_m(\tau) - Q_6(\tau)\overline{k_p}\overline{W}''_m(\tau) - Q_7(\tau)\overline{k_p}\overline{W}'_m(\tau)]\end{aligned} \tag{29}$$

The following condition is obtained when the correction functional is taken into consideration:

$$\begin{aligned}\delta\overline{W}_m : \quad & 1 - \lambda''' + 2\lambda' Q_1\\ & + 2\lambda'' Q_1 + 2\lambda Q''_1 + 2\lambda' Q'_2 - \overline{v}^2\lambda' + \overline{k_p}\lambda' Q_6\\ & + \overline{k_p}\lambda Q'_6 + 2i\overline{v}\sqrt{\beta}\lambda' Q_3 - \overline{k_p}\lambda Q_7|_{\tau=\overline{x}} = 0\end{aligned} \tag{30}$$

$$\delta\overline{W}'_m : \quad \lambda'' - 2\lambda' Q'_1 + \lambda' Q_1 + Q_2\lambda + \lambda\overline{v}^2 - \overline{k_p}\lambda Q_6|_{\tau=\overline{x}} = 0 \tag{31}$$

$$\delta\overline{W}''_m : \quad -\lambda' + 2\lambda Q_1|_{\tau=\overline{x}} = 0 \tag{32}$$

$$\delta\overline{W}'''_m : \quad \lambda|_{\tau=\overline{x}} = 0 \tag{33}$$

$$\begin{aligned}\int \delta\overline{W}_m(\tau) : \quad & \int_0^{\overline{x}} [\lambda^{iv} - 2\lambda''' Q_1 - 2\lambda'' Q'_1 - 2\lambda' Q'_1 - 2\lambda' Q''_1\\ & - 2\lambda Q'''_1 + \lambda Q''_2 - \lambda' Q'_2 + \lambda' Q_1 + \lambda'' Q_2\\ & - \lambda' Q'_2 + \overline{v}^2\lambda'' + Q_4\omega^2\lambda - Q_5\overline{k_w}\lambda - Q_6\overline{k_p}\lambda''\\ & - Q'_6\overline{k_p}\lambda' - Q'_6\overline{k_p}\lambda' - Q''_6\overline{k_p}\lambda - 2\overline{v}i\omega\sqrt{\beta}\lambda Q_3\\ & - 2\overline{v}i\sqrt{\beta}\omega\lambda' Q_3 + Q_7\overline{k_p}\lambda' + \lambda\overline{k_w}Q'_7]d\tau = 0\end{aligned} \tag{34}$$

In order to satisfy the equation (30) - (34), the lagrange multiplier can be obtained as:

$$\lambda = \frac{(\tau - \overline{x})^3}{3!} \tag{35}$$

Substituting equation (35) into (29), an iteration procedure can be achieved:

$$\begin{aligned}\overline{W}_{m+1}(\overline{x}) = \overline{W}_m(\overline{x}) + \int_0^{\overline{x}} \frac{(\tau - \overline{x})^3}{3!}[\overline{W}^{iv}_m(\tau) + 2Q_1(\tau)\overline{W}'''_m(\tau) + Q_2(\tau)\overline{W}''_m(\tau)\\ + \overline{v}^2\overline{W}''_m(\tau) + 2i\overline{v}\sqrt{\beta}Q_3(\tau)\omega\overline{W}'_m(\tau) - Q_4(\tau)\omega^2\overline{W}_m(\tau)\\ + Q_5(\tau)\overline{k_w}\overline{W}_m(\tau) - Q_6(\tau)\overline{k_p}\overline{W}''_m(\tau) - Q_7(\tau)\overline{k_p}\overline{W}'_m(\tau)]d\tau\end{aligned} \tag{36}$$

If $m = 0, 1, 2, 3, ..., k$, one can obtain the following successive iteration formula based on the proposed method:

$$\begin{aligned}
\overline{W}_1(\overline{x}) =& \overline{W}_0(\overline{x}) + \int_0^{\overline{x}} \frac{(\tau-\overline{x})^3}{3!}[\overline{W}_0^{iv}(\tau) + 2Q_1(\tau)\overline{W}_0'''(\tau) \\
&+ Q_2(\tau)\overline{W}_0''(\tau) + \overline{v}^2\overline{W}_0''(\tau) + 2i\overline{v}\sqrt{\beta}Q_3(\tau)\omega\overline{W}_0'(\tau) \\
&- Q_4(\tau)\omega^2\overline{W}_0(\tau) + Q_5(\tau)\overline{k_w}\overline{W}_0(\tau) \\
&- Q_6(\tau)\overline{k_p}\overline{W}_0''(\tau) - Q_7(\tau)\overline{k_p}\overline{W}_0'(\tau)]d\tau \\
\overline{W}_2(\overline{x}) =& \overline{W}_1(\overline{x}) + \int_0^{\overline{x}} \frac{(\tau-\overline{x})^3}{3!}[\overline{W}_1^{iv}(\tau) + 2Q_1\overline{W}_1'''(\tau) \\
&+ Q_2(\tau)\overline{W}_1''(\tau) + \overline{v}^2\overline{W}_1''(\tau) + 2i\overline{v}\sqrt{\beta}Q_3(\tau)\omega\overline{W}_1'(\tau) \\
&- Q_4(\tau)\omega^2\overline{W}_1(\tau) + Q_5(\tau)\overline{k_w}\overline{W}_1(\tau) \\
&- Q_6(\tau)\overline{k_p}\overline{W}_1''(\tau) - Q_7(\tau)\overline{k_p}\overline{W}_1''(\tau)]d\tau \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\overline{W}_k(\overline{x}) =& \overline{W}_{k-1}(\overline{x}) + \int_0^{\overline{x}} \frac{(\tau-\overline{x})^3}{3!}[\overline{W}_{k-1}^{iv}(\tau) + 2Q_1\overline{W}_{k-1}'''(\tau) \\
&+ Q_2(\tau)\overline{W}_{k-1}''(\tau) + \overline{v}^2\overline{W}_{k-1}''(\tau) + 2i\overline{v}\sqrt{\beta}Q_3(\tau)\omega\overline{W}_{k-1}'(\tau) \\
&- Q_4(\tau)\omega^2\overline{W}_{k-1}(\tau) + Q_5(\tau)\overline{k_w}\overline{W}_{k-1}(\tau) \\
&- Q_6(\tau)\overline{k_p}\overline{W}_{k-1}''(\tau) - Q_7(\tau)\overline{k_p}\overline{W}_{k-1}'(\tau)]d\tau
\end{aligned} \tag{37}$$

where $\overline{W}_0(\overline{x})$ is denoted to be an initial approximation to start the iteration process. This can be in the form:

$$\overline{W}_0(\overline{x}) = \overline{W}(0) + \overline{W}'(0)\overline{x} + \frac{\overline{W}''(0)}{2!}\overline{x}^2 + \frac{\overline{W}'''(0)}{3!}\overline{x}^3 \tag{38}$$

where $\overline{W}(0)$, $\overline{W}'0$,$\overline{W}''(0)$ and $\overline{W}'''(0)$ are unknown constants to be determined while applying the above dimensionless boundary condition
After obtaining $\overline{W}_k(\overline{x})$, the solution for the dimensionless equation can be expressed as:

$$\overline{W}(\overline{x}) = \lim_{k\to\infty} \overline{W}_k(\overline{x}) \tag{39}$$

Selecting infinity is impossible for the iteration process, so a large number "m" is chosen and substituted into equation (39) on the basis of accuracy required.

$$\overline{W}(\overline{x}) = \overline{W}_m(\overline{x}) \tag{40}$$

For clarity, equation (40) is then substituted into four boundary conditions which gives four systems of simultaneous equation expressed in a matrix form as follows:

$$\{P_{yz}^k(\overline{\omega}_i^m)\}_{4\times4} \begin{Bmatrix} \overline{W}(0) \\ \overline{W}'(0) \\ \overline{W}''(0) \\ \overline{W}'''(0) \end{Bmatrix} = 0 \quad (y, z = 1, 2, 3, 4) \tag{41}$$

P_{yz}^k are evaluated polynomial of the eigenvalue $\overline{\omega}$ with respect to the superscript k, $(\overline{\omega}_i^m)$ is the ith estimated dimensionless natural frequency corresponding to m and also $\overline{W}(0)$, $\overline{W}'(0)$, $\overline{W}''(0)$ and $\overline{W}'''(0)$ are the column vector of the unknown coefficient. Dimensionless natural frequencies of AFG tapered pipes resting on variable elastic foundation were obtained when considering a non-trivial solution such that determinant of the coefficient matrix are set to zero. Therefore, the characteristics root of the determinant are solutions to the problem. However, if the fluid velocity reaches a certain value $\overline{v}_c$, the natural frequency becomes zero [17]. That is, if we set $\omega_i = 0$ in the dimensionless equation, the critical flow velocity will be obtained.
Testing the convergence provides us a condition that the absolute value of ϵ must be less than or equal to 1×10^{-5}. However, if the convergence criterion is not satisfied, the procedure will be repeated by increasing the number of iteration until the convergence is realized. This can only be obtained when

$$\left|\omega_i^m - \omega_i^{m-1}\right| \le \epsilon \tag{42}$$

IV NUMERICAL RESULT AND DISCUSSION

For the purpose of verifying the accuracy and efficiency of the method used, several numerical examples are considered and results are presented in tabular and graphical form in this section. In particular, the vibration and stability characteristics of axially functionally graded (AFG) non-uniform fluid-conveying pipes resting on variable elastic foundation is numerically investigated. A computer software known as "Mathematica" was used to compute the numerical results using VIM. The modulus of elasticity E, and pipe mass density ρ are assumed to, be constants, vary linearly, or parabolically, respectively, for all given examples except otherwise stated. The first example is devoted to examining the vibration of uniform pipes with (β =0, $\overline{v}$ =0, i.e no fluid, no flow pipes - beam) resting on a uniform Winkler foundation. In this case, the non-uniformity parameter α is set to zero while the foundation stiffness variation parameter $\eta = 0$ and also the effect of axially Functionally Graded Materials (AFGMs) as well as the Pasternak foundation stiffness $\overline{K}_p$ are neglected. The results for the first four natural frequencies when the pipes are subjected to different sets of boundary conditions are presented in Table 1. By comparison, it is found that the numerical results give an excellent agreement with those obtained in [31], [32] and [33] for which methods different from VIM are used. All the partially empty boxes in Table 1 imply that corresponding article does not provide the pertinent results.

Table 2 presents the fundamental frequency of non uniform no fluid, no flow pipes (i.e non-uniform beams) subjected to Pinned-Pinned and Clamped- Pinned boundary conditions. This second example neglects the effect of $\overline{K}_p$, $\overline{K}_w$, effect of material properties, fluid velocity and mass ratio. The computed results were therefore validated by comparing them with those obtained in [35], and a good agreement was observed. It is seen from Table 2, that for the Clamped-pinned case, the fundamental frequencies increase as α increases, while when the Pinned-pinned vibrating configuration is considered, an increase in α leads to a decrease in the fundamental frequencies. In this context, it might be remarked that the natural vibration behaviour of the non-prismatic, non axially functionally graded pipe with no fluid is unpredictable [35]. The third example deals with the results of first four dimensionless natural frequencies of non-uniform pipes with no fluid resting on Winkler Foundation as illustrated in Tables 3-5. Each of the Tables contains the results for the three vibrating configurations (Pinned-pinned, Clamped-pinned and Clamped-clamped, respectively). This analysis ignores the impact of AFGMs, β, $\overline{K}_p$, $\overline{v}$ and η on vibration of pipes resting on an elastic foundation. However, various values of α and a fixed value of $\overline{K}_w$ were used. Present VIM results are compared with those of [31] and [34] and excellent agreement is obtained.

Table 1: Natural frequencies of uniform no fluid, no flow (beams) and non-functionally graded pipes with various boundary conditions resting on Winkler foundation. Note that DQEM denotes Differntial Quadrature Element Method

End conditions	Method	ω_1	ω_2	ω_3	ω_4
Pinned-pinned	Present	9.92014	39.4911	88.8321	157.9168
	ADM [31]	9.92014	39.4911	88.8321	157.9168
	DTM [32]	9.92014	39.4911	88.8321	
	DQEM [33]	9.92014	39.4913	89.4002	
Clamped-Pinned	Present	15.4506	49.9749	104.253	178.273
	ADM [31]	15.4506	49.9749	104.2525	178.2725
	DTM [32]				
	DQEM [33]				
Clamped-Clamped	Present	22.3956	61.6809	12.908	199.862
	ADM [31]	22.3956	61.6809	12.908	199.862
	DTM [32]	22.3733	61.6728	120.903	199.859
	DQEM [33]	22.3956	61.6811	120.910	199.885

Tables 6 and 7 present fundamental frequencies of uniform no fluid pipes resting on Winkler Pasternak foundation subjected to Clamped-Clamped and Simply-Supported boundary conditions for various values of foundation stiffness parameters (i.e $\overline{K}_w$ and $\overline{K}_p$) with $\overline{v} = 0$; $\beta = 0$, $\eta = 0$ and $\alpha = 0$. It is also remarked that this particular example neglects the effects of functionally graded materials on vibration response of the pipes. From the results obtained using VIM, good agreement can be observed with those of [18] and [35].

Table 2: Fundamental frequencies for various values of non-uniform parameter α under Clamped-Pinned and Pinned-Pinned conditions. Note that PSM denotes Power Series Method

Boundary Conditions	α	Method	ω
Clamped-Pinned	0.1	PSM/RRM [35]	15.527
		Present	15.5274
	0.3	PSM [35]	15.768
		Present	15.7686
	0.5	PSM [35]	16.044
		Present	16.0444
Pinned-Pinned	0.1	PSM/RRM [34]	9.868
		Present	9.86853
	0.3	PSM [35]	9.860
		Present	9.85741
	0.5	PSM [35]	9.825
		Present	9.82537

Table 8 presents the values of the fundamental frequency parameter of a prismatic pipe conveying fluid resting on two parameter foundation for different sets of boundary conditions. [18] employed the Fourier Series Techniques to calculate the fundamental frequencies for this example. This special case assumes the pipe's cross section to be uniform but with different values of $\overline{K}_w$, and $\overline{K}_p$, for η and α being set to zero while $\overline{v} = 2$. In order to examine the accuracy of the obtained results, the present work was compared with the results gotten from the reference work as reported in Table 8. It is observed that the present results show good agreement with those of the previous literature. Also as the values of $\overline{K}_p$ increases, for various fixed values of $\overline{K}_w$, the fundamental frequencies increase.

Table 3: Vibration characteristics of non-uniform Pinned-pinned pipes with no fluid resting on Winkler elastic foundation

End Condition	α	Method	ω_1	ω_2	ω_3	ω_4
Pinned-pinned	0.1	Present	9.92173	39.4928	88.834	157.9189
		ADM [31]	9.9217	39.4928	88.834	157.9189
		HPM [34]	9.9217	39.4928	88.834	157.9189
	0.3	Present	9.91701	39.5047	88.8511	157.935
		ADM [31]	9.9170	39.5047	88.8511	157.9389
		HPM [34]	9.9170	39.5047	88.8511	157.9389
	0.5	Present	9.89358	39.5344	88.8986	157.996
		ADM [31]	9.8932	39.5340	88.8986	157.9966
		HPM [34]	9.8932	39.5340	88.8986	157.9966

Table 4: Vibration characteristics of non-uniform Clamped-pinned pipes with no fluid resting on Winkler elastic foundation

End Condition	α	Method	ω_1	ω_2	ω_3	ω_4
Clamped-pinned	0.1	Present	15.5615	50.0781	104.356	178.376
		ADM [31]	15.5615	50.0781	104.3556	178.3756
		HPM [34]	15.5615	50.0781	104.3556	178.3756
	0.3	Present	15.8069	50.3052	104.584	178.604
		ADM [31]	15.8069	50.3052	104.5836	178.6036
		HPM [34]	15.8069	50.3052	104.5836	178.6036
	0.5	Present	16.0882	50.569	104.855	178.878
		ADM [31]	16.0879	50.5685	104.8544	178.8785
		HPM [34]	16.0879	50.5685	104.8544	178.8785

Table 5: Vibration characteristics of non-uniform Clamped-clamped pipes with no resting on Winkler elastic foundation

End Condition	α	Method	ω_1	ω_2	ω_3	ω_4
Clamped-Clamped	0.1	Present	22.3922	61.6751	120.901	199.855
		ADM [31]	22.3922	61.6752	120.9009	199.8549
		HPM [34]	22.3922	61.6752	120.9009	199.8549
	0.3	Present	22.3477	61.6114	120.83	199.78
		ADM [31]	22.3477	61.6114	120.8302	199.7803
		HPM [34]	22.3477	61.6114	120.8302	199.7803
	0.5	Present	22.213	61.419	120.615	199.551
		ADM [31]	22.2120	61.4183	120.6114	199.5512
		HPM [34]	22.2120	61.4183	120.6114	199.5512

Table 6: Fundamental frequencies of uniform Pinned-Pinned non-functionally graded pipes (not conveying fluid) with respect to Pasternak elastic parameters. Note that FST denotes Fourier Series Technique

K_p	Methods	K_w		
		10^2	10^4	10^6
0.5	Present	3.961	10.036	31.589
	FST [18]	3.960	10.036	31.623
	DQM [36]	3.960	10.036	31.623
1.0	Present	4.144	10.048	31.542
	FST [18]	4.143	10.048	31.625
	DQM [36]	4.143	10.048	31.622
2.5	Present	4.582	10.084	31.621
	FST [18]	4.582	10.084	31.623
	DQM [36]	4.582	10.083	31.625

Table 7: Fundamental frequencies of uniform Clamped-Clamped non-functionally graded pipes (not conveying fluid) with respect to Pasternak elastic foundation parameters

$\overline{K_p}$	Methods	$\overline{K_w}$		
		0	10^2	10^4
0.5	Present	4.867	5.071	10.137
	FST [18]	4.916	5.114	10.142
	DQM [36]	4.869	5.071	10.137
1.0	Present	4.993	5.182	10.152
	FST [18]	5.083	5.264	10.162
	DQM [36]	4.994	5.182	10.152
2.5	Present	5.318	5.477	10.194
	FST [18]	5.505	5.649	10.222
	DQM [36]	5.320	5.477	10.194

Next, Tables 9 - 11 illustrates the effect of axially Functionally Graded Materials (FGMs') on the first four dimensionless natural frequencies of nonuniform pipes conveying fluid and resting on Pasternak elastic foundation. Three different boundary conditions (Pinned-pinned, Clamped-clamped and Clamped-pinned) and the following fixed values of pertinent parameters ($\overline{K_w}$ = 10, $\overline{K_p}$ = 2.5, η = 0.2, α = 0.2, $\overline{v}$=2 and β = 0.5) were considered. Following [28], three types of material models were also used in this example. They are the homogeneous type (i.e both E and ρ are assumed constants), homogeneous/nonhomogeous type (i.e only one of E and ρ is, constant while the other varies either linearly or parabolically) and Non-homogeneous type (i.e when each of E and ρ varies linearly or parabolically). It is observed that the values of natural frequencies for the Homogeneous Materials (HMs) are higher than those of nonhomogeneous material for all boundary conditions considered.

Table 8: Fundamental frequency values for uniform pipes conveying fluid versus various values of $\overline{K_w}$ and $\overline{K_p}$, for $\eta = 0$, $\alpha = 0$ and $\beta = 0.5$ for various vibrating configurations

$\overline{K_p}$	$\overline{K_w}$	Pinned-pinned		Clamped-pinned		Clamped-clamped	
		Chellapilla [18]	Present	Chellapilla [18]	Present	Chellapilla [18]	Present
0.01	0.01	7.52	7.52	13.66	13.66	20.99	20.99
	0.50	7.55	7.55	13.68	13.68	21.00	21.00
	2.50	7.68	7.68	13.75	13.75	21.05	21.04
0.5	0.01	10.37	10.17	15.82	15.64	22.70	22.52
	0.50	10.40	10.20	15.84	15.72	22.71	22.60
	2.50	10.49	10.30	15.90	15.80	22.76	22.68
1.0	0.01	12.80	12.40	17.86	17.78	24.40	24.23
	0.50	12.82	12.53	17.87	17.80	24.41	24.31
	2.50	12.90	12.78	17.93	17.86	24.45	25.37
2.5	0.01	18.78	18.70	23.33	23.21	30.22	30.15
	0.50	18.80	18.72	23.34	23.24	30.23	30.19
	2.50	18.86	18.75	23.39	23.28	30.27	30.22
10.0	0.01	40.39	40.21	45.63	45.34	52.21	52.13
	0.50	40.40	40.23	45.64	45.36	52.22	52.14
	2.50	40.45	40.31	45.68	45.43	52.26	52.19

Moreover, tapered fluid-conveying pipes comprising of constant modulus of elasticity and parabolically varying mass per unit volume produce the least natural frequencies for all the boundary conditions considered while non-uniform AFG pipes having parabolically varying modulus of elasticity with constant material mass density yield the highest frequencies (1) for both Clamped-clamped and pinned-pinned vibrating configurations. (2) However, for Clamped-pinned end condition, the highest natural frequency comes from non-uniform AFG pipes, for which E varies linearly while ρ is a constant. Hence, the impact of material properties plays a key role in the designs of tapered pipes conveying fluid and resting on a two-parameter elastic foundation. Furthermore, it is hereby remarked that Table 9 is for the clamped-clamped boundary condition, Table 10 is for the clamped-pinned boundary condition, while Table 11 is for the pinned-pinned boundary condition.

Tables 12 - 14 depict the effect of $\overline{K_w}$ and $\overline{K_p}$ on the natural frequencies of AFG tapered fluid-conveying pipes resting on a variable two-parameter foundation. In this case, the data used for the analysis were η = 0.2, α = 0.2, $\overline{v}$=2 and β = 0.5, while the three boundary conditions were considered. This example assumes that the material properties of the pipe $E(x)$ and $\rho(x)$ varies linearly and parabolically, respectively, as expressed in equations (23) and (26). We observed that the natural frequencies increase with the increase of Winkler foundation stiffness parameter $\overline{K_w}$ for various fixed values of $\overline{K_p}$. This trend holds for each of the three vibrating configurations considered (see Tables 12-14) However, frequencies become minimum for Pinned pinned AFG pipes compared to Clamped clamped and Clamped pinned pipes. Similarly, the free vibration characteristics of AFG tapered pipes carrying fluid are sensitive to Pasternak foundation stiffness parameter $\overline{K_p}$. It is hereby remarked that the first four natural frequencies increase significantly with increasing $\overline{K_p}$. From the observed numerical results, it can however be concluded that $\overline{K_p}$ has a greater effect than $\overline{K_w}$ on vibrational frequencies of AFG nonuniform fluid-conveying pipes.

The first four dimensionless natural frequencies for axially functionally graded (AFG) pipes resting on variable Pasternak elastic foundation with various values of non-uniformity parameter α and nonuniform foundation stiffness parameter η are calculated using VIM, and the results are tabulated in Tables 15 - 17. Each of the three vibrating configurations is used per Table.

Table 9: Effect of non-homogeneous material on natural frequencies of AFG non-uniform clamped-clamped fluid-conveying pipes resting on Pasternak foundation ($\overline{K_w} = 10$, $\overline{K_p} = 2.5$, $\eta = 0.2$, $\alpha = 0.2$, $\overline{v}$=2 and $\beta = 0.5$)

E	ρ	ω_1	ω_2	ω_3	ω_4
Constant	Constant	21.829	61.3076	120.552	199.528
	Linear	17.9	50.175	98.8387	190.144
	Parabolic	16.4733	46.0214	90.5882	172.313
Linear	Constant	25.9189	73.1171	143.234	236.176
	Linear	21.456	60.7885	120.056	199.065
	Parabolic	19.7978	55.6666	109.491	181.101
Parabolic	Constant	28.0528	80.7929	160.233	266.66
	Linear	23.4413	66.0071	129.699	214.299
	Parabolic	21.7706	61.2746	120.684	199.797

Table 10: Effect of non-homogeneous material on natural frequencies of AFG non-uniform clamped-pinned fluid-conveying pipes resting on Pasternak foundation ($\overline{K_w} = 10$, $\overline{K_p} = 2.5$, $\eta = 0.2$, $\alpha = 0.2$, $\overline{v}$=2 and $\beta = 0.5$)

E	ρ	ω_1	ω_2	ω_3	ω_4
Constant	Constant	15.1347	49.8042	104.1190	178.1680
	Linear	12.1610	40.4934	85.1025	145.9670
	Parabolic	11.0124	36.9732	77.8342	133.5690
Linear	Constant	17.1752	58.3409	122.0660	208.7760
	Linear	14.0106	48.6801	102.9980	177.0450
	Parabolic	12.6644	44.1773	93.3703	160.3870
Parabolic	Constant	18.5536	65.4039	138.9560	198.2070
	Linear	14.8302	52.2409	110.3260	189.3650
	Parabolic	13.5914	48.5543	102.9910	177.1030

Table 11: Effect of non-homogeneous material on natural frequencies of AFG non-uniform pinned-pinned fluid-conveying pipes resting on variable Pasternak foundation ($\overline{K_w} = 10$, $\overline{K_p} = 2.5$, $\eta = 0.2$, $\alpha = 0.2$, $\overline{v}$=2 and $\beta = 0.5$)

E	ρ	ω_1	ω_2	ω_3	ω_4
Constant	Constant	9.40833	39.0584	88.4588	157.5830
	Linear	7.74599	32.0043	72.5925	129.4210
	Parabolic	7.1232	29.3221	66.4794	118.5040
Linear	Constant	10.7549	46.1869	104.2450	185.2100
	Linear	8.9437	38.7615	88.2200	157.3710
	Parabolic	8.2137	35.3315	80.1140	142.6920
Parabolic	Constant	11.5477	51.5829	118.3880	212.2480
	Linear	9.4795	41.6783	94.5904	168.4030
	Parabolic	8.7905	38.8291	88.3658	157.5540

Examining the numerical results critically, it can be seen that the natural frequencies decrease slightly with increasing non-uniformity parameter α for Clamped- Clamped pipe (see Table 17). This same report also applies for stiffness variation coefficient parameter η. The vibration frequencies of AFG uniform pipes conveying fluid and lying on two parameter uniform elastic foundation, (i.e for $\alpha = 0$ and $\eta = 0$) have higher values than those of the non-uniform AFG fluid-conveying pipes resting on variable two-parameter elastic foundation. In the case of Pinned-pinned boundary condition, increasing values of α leads to a small rise in the first mode, but the presence of η decelerates the natural frequency when α are 0, 0.2, 0.4 and 0.6, respectively. For Clamped-pinned boundary condition, an increasing trend of variation of natural frequencies with respect to α is observed. We can as well see that the increasing value of η leads to a decline in the values of the computed frequency parameter for all the boundary conditions considered.

Table 12: Effect of $\overline{K_w}$ and $\overline{K_p}$ on natural frequencies of AFG non-uniform clamped-pinned fluid-conveying pipes resting on variable Pasternak foundation (η = 0.2, α = 0.2, $\overline{v}$=2 and $\beta = 0.5$)

$\overline{K_w}$	$\overline{K_p}$	ω_1	ω_2	ω_3	ω_4
0.5	0.1	12.0034	43.5566	92.7312	168.5250
	10	14.3765	46.3202	95.7050	173.9220
	50	21.4084	56.1160	106.8700	174.738
5	0.1	12.1072	43.5885	92.7465	168.5250
	10	14.4637	46.3501	95.7199	173.9225
	50	21.4678	56.1362	106.8830	174.7460
10	0.1	12.2216	43.6240	92.7635	168.5260
	10	14.5599	46.3834	95.7363	173.9230
	50	21.5335	56.1635	106.8980	174.7550
100	0.1	14.1207	44.2576	93.0693	168.5350
	10	16.1939	46.9783	96.0323	173.9330
	50	22.6837	56.6529	107.1620	174.9180

Based on the computed results in Tables 15-17, it is remarked that the non-uniformity parameter α and variation foundation stiffness parameter η have remarkable influence on vibration of AFG non-uniform fluid-conveying pipes resting on variable elastic foundation. Hence, it ascertains that the present work is efficient and reliable for structural design.

The influence of mass ratio β and velocity $\overline{v}$ on transverse vibration of AFG tapered pipes transporting fluid and resting on variable Pasternak elastic foundation is presented in Tables 18 - 20, where ($\overline{K_w}$ = 10, $\overline{K_p}$ = 2.5, α = 0.2 and η = 0.2). Specifically, the computed results of the natural frequencies with respect to various values of β and $\overline{v}$ for (i) AFG tapered clamped-clamped pipes resting on a variable Pasternak foundation, (ii) AFG tapered clamped-pinned pipes resting on a variable Pasternak foundation, and (iii) AFG tapered pinned-pinned pipes resting on a variable Pasternak foundation are presented respectively in each of the three Tables 18, 19, and 20. It can be seen that the effect

Table 13: Effect of $\overline{K_w}$ and $\overline{K_p}$ on natural frequencies of AFG non-uniform pinned-pinned fluid-conveying pipes resting on variable Pasternak foundation (η = 0.2, α = 0.2, $\overline{v}$=2 and $\beta = 0.5$)

$\overline{K_w}$	$\overline{K_p}$	ω_1	ω_2	ω_3	ω_4
0.5	0.1	7.0855	34.3489	79.1266	141.6850
	10	10.4405	37.7169	82.5388	145.1190
	50	18.4705	48.9735	95.1377	158.345
5	0.1	7.1924	34.4879	79.2906	141.8660
	10	10.4667	37.8214	82.6854	145.2900
	50	18.5422	49.0025	95.1528	158.3540
10	0.1	7.3944	34.5340	79.3108	141.8770
	10	10.6066	37.8634	82.7048	145.3010
	50	18.6216	49.0348	95.1696	158.3640
100	0.1	10.3743	35.3550	79.6737	142.0810
	10	12.8638	38.6121	83.0527	147.8560
	50	19.9952	49.6115	95.4719	158.5470

of mass ratio β decreases the first frequency parameter, while each of the second, third and fourth frequencies increases as the value of the mass ratio increases for various fixed values of $\overline{v}$. It is also noteworthy to point out the dependency of the first four frequencies on the fluid flow velocity of the AFG non-uniform fluid-conveying pipes. It is observed that the natural frequencies decrease with increasing velocity, and the reduction tends to be more noticeable with higher velocity for the three Tables. Based on this outcome, it is hereby remarked that the fluid flow velocity plays a vital role on the natural frequencies of the AFG tapered pipes.

The stability of the present system was also studied by computing the relevant critical flow velocities and then examining the effect of various relevant parameters on critical flow velocities of the AFG tapered fluid-conveying pipes resting on variable Pasternak elastic foundations under different boundary conditions. To this end, it is noted that there exists a critical fluid velocity near which the natural frequency of a system tends to zero [17]. It is worthy to also note that when the fluid velocity is smaller than the critical fluid velocity $\overline{v_c}$ (i.e $\overline{v} < \overline{v_c}$), then the system is stable, while instability sets in at $\overline{v} = \overline{v_c}$. Thus, it is important to predict the critical flow velocities of pipes transporting fluid systems. Therefore, the following examples were considered. Table 21 presents the values of critical velocities of non-AFG uniform fluid-conveying pipe not resting on Pasternak foundation. It can be found that the present results agree with the numerical results obtained by [5].

As disclosed in [5], two types of instability can take place in dynamics of fluid-conveying piping system, namely; Instability by divergence/buckling, and Instability by flutter. These type of instability is wholly dependent on higher flow velocity involved in the piping system. Based on this, Table 22 illustrates a form of instability that occurred, while, examining Critical flow velocities of non-uniform AFG fluid-conveying pipes resting on variable Pasternak foundation. The values of the critical velocity for the clamped clamped condition are the highest whil the lowest are those of Pinned pinned boundary condition. Computed results for critical flow velocity of uniform pipe with various values of Winkler foundation parameter used in [13] and [18] are tabulated in Tables 23-24 for the three vibrating configurations. For this numerical example, we set $\beta = 0.1$, $\eta = 0$, $\alpha = 0$, $\overline{K_p} = 0$, and assumed that the effect of functionally graded materials is also ignored. Comparative studies show that satisfactory agreement between the results in [13], [17] and the present ones were found. Although, there exist slight differences in the results most especially for higher values of Winkler parameter $\overline{K_w}$ obtained with those of [17] using Galerkin approach.

Table 14: Effect of $\overline{K_w}$ and $\overline{K_p}$ on natural frequencies of AFG non-uniform clamped-clamped fluid-conveying pipes resting on variable Pasternak foundation ($\eta = 0.2$, $\alpha = 0.2$, $\overline{v} = 2$ and $\beta = 0.5$)

$\overline{K_w}$	$\overline{K_p}$	ω_1	ω_2	ω_3	ω_4
0.5	0.1	19.1726	54.9834	108.7680	180.3500
	10	20.9634	57.4140	111.4720	183.2150
	50	26.9124	66.2755	121.7650	194.352
5	0.1	19.2424	55.0089	108.7810	180.3580
	10	21.0272	57.4384	111.4840	183.2230
	50	26.9621	66.2966	121.7770	194.3590
10	0.1	19.3196	55.0372	108.7950	180.3660
	10	21.0979	57.4655	111.4990	183.2320
	50	27.0173	66.3200	121.7900	194.3680
100	0.1	20.6594	55.5447	109.0560	180.5250
	10	22.3315	57.9514	111.7530	183.3880
	50	27.9915	66.7405	122.0230	194.5140

Table 15: Effect of η and α on natural frequencies of AFG non-uniform clamped-pinned fluid-conveying pipes resting on variable Pasternak foundation ($\overline{K_w} = 10$, $\overline{K_p} = 2.5$, $\overline{v} = 2$ and $\beta = 0.5$)

η	α	ω_1	ω_2	ω_3	ω_4
0	0	12.4693	44.0167	93.2194	160.2430
	0.2	12.7679	44.2573	93.4455	160.4600
	0.4	13.1268	44.5508	93.7206	160.7240
	0.6	13.5686	44.9316	94.0858	161.0790
0.1	0	12.4235	43.9824	93.1876	160.2120
	0.2	12.7162	44.2173	93.4079	160.4240
	0.4	13.0677	44.5030	93.6754	160.6800
	0.6	13.4990	44.8723	94.0829	161.0230
0.3	0	12.3314	43.9139	93.1238	160.1500
	0.2	12.6123	44.1372	93.3327	160.3510
	0.4	12.9484	44.4074	93.5848	160.5920
	0.6	13.3585	44.7535	93.9150	160.9110
0.5	0	12.2385	43.8452	93.0601	160.0880
	0.2	12.5075	44.0570	93.2575	160.2770
	0.4	12.8279	44.3116	93.4941	160.5030
	0.6	13.2162	44.6344	93.8010	160.7990

Table 16: Effect of η and α on natural frequencies of AFG non-uniform pinned-pinned pipes resting on variable Pasternak foundation ($\overline{K_w} = 10$, $\overline{K_p} = 2.5$, $\overline{v} = 2$ and $\beta = 0.5$)

η	α	ω_1	ω_2	ω_3	ω_4
0	0	8.1954	35.3495	80.1498	142.7320
	0.2	8.3387	35.4159	80.1953	142.7660
	0.4	8.5087	35.5208	80.2800	142.8390
	0.6	8.7125	35.6950	80.4429	142.9950
0.1	0	8.1393	35.3131	80.1171	142.7010
	0.2	8.2764	35.3737	80.1568	142.7290
	0.4	8.4387	35.4707	80.2338	142.7950
	0.6	8.6323	35.6333	80.3851	142.9380
0.3	0	8.0259	35.2402	80.0516	142.6380
	0.2	8.1505	35.2892	80.0798	142.6550
	0.4	8.2968	35.3704	80.1413	142.7050
	0.6	8.4692	35.5097	80.2692	142.8550
0.5	0	7.9109	35.1672	79.9861	142.5750
	0.2	8.0224	35.2024	80.0028	142.5810
	0.4	8.1522	35.2698	80.0488	142.6150
	0.6	8.3025	35.3855	80.1532	142.7120

Table 17: Effect of η and α on natural frequencies of AFG non-uniform clamped-clamped pipes resting on variable Pasternak foundation ($\overline{K_w} = 10$, $\overline{K_p} = 2.5$, $\overline{v} = 2$ and $\beta = 0.5$)

η	α	ω_1	ω_2	ω_3	ω_4
0	0	19.8778	55.7371	109.5640	181.7700
	0.2	19.8614	55.7333	109.5580	181.1700
	0.4	19.7885	55.6570	109.4730	181.0790
	0.6	19.5964	55.4174	109.2030	180.7910
0.1	0	19.8494	55.7076	109.5340	181.1470
	0.2	19.8296	55.6999	109.5250	181.1360
	0.4	19.7521	55.6184	109.4340	181.0390
	0.6	19.5535	55.3710	109.1550	180.7420
0.3	0	19.7924	55.6485	109.4750	181.0880
	0.2	19.7659	55.6332	109.4570	181.0670
	0.4	19.6793	55.5410	109.3540	180.9580
	0.6	19.4673	55.2780	109.0580	180.6430
0.5	0	19.7352	55.5894	109.4160	181.0290
	0.2	19.7020	55.5663	109.3890	180.9990
	0.4	19.6061	55.4636	109.2750	180.8780
	0.6	19.3807	55.1847	108.9620	180.5440

Table 18: Effect of β and $\overline{v}$ on natural frequencies of AFG tapered clamped-clamped pipes resting on variable Pasternak foundation ($\overline{K_w} = 10$, $\overline{K_p} = 2.5$, $\alpha = 0.2$ and $\eta = 0.2$)

β	$\overline{v}$	ω_1	ω_2	ω_3	ω_4
0.1	0	20.9731	56.9731	110.8130	182.4530
	1	20.9714	56.5912	110.4480	182.0720
	2	19.9157	55.5716	109.3460	180.9230
	3	18.5080	53.8323	107.4850	178.9940
0.3	0	20.9731	56.9731	110.8130	182.4530
	1	20.6950	56.5986	110.4620	182.0900
	2	19.8408	55.0646	109.4020	180.9950
	3	18.3433	53.9204	107.6180	179.1580
0.5	0	20.9731	56.9731	110.8130	182.4530
	1	20.6761	56.6059	110.4750	182.1070
	2	19.7668	55.5363	109.4570	181.0650
	3	18.1831	54.0017	107.7480	179.3220

Table 19: Effect of β and $\overline{v}$ on natural frequencies of AFG tapered clamped-pinned pipes resting on variable Pasternak foundation ($\overline{K_w} = 10$, $\overline{K_p} = 2.5$, $\alpha = 0.2$ and $\eta = 0.2$)

β	$\overline{v}$	ω_1	ω_2	ω_3	ω_4
0.1	0	14.3462	45.7837	94.9755	161.9900
	1	14.0070	45.4036	94.5780	161.5830
	2	12.9369	44.2456	93.3760	160.3570
	3	10.9258	42.2511	91.3402	157.2920
0.3	0	14.3462	45.7837	94.9755	161.9900
	1	13.9927	45.4107	94.5921	161.6010
	2	12.8822	44.2778	93.4348	160.4310
	3	10.8147	42.3389	91.4810	158.4660
0.5	0	14.3462	45.7837	94.9755	161.9900
	1	13.9785	45.4177	94.6062	161.6990
	2	12.8281	44.3088	93.4930	160.5050
	3	10.7070	42.4203	91.6191	158.6360

Table 20: Effect of β and $\overline{v}$ on natural frequencies of AFG tapered pinned-pinned pipes resting on variable Pasternak foundation ($\overline{K_w} = 10$, $\overline{K_p} = 2.5$, $\alpha = 0.2$ and $\eta = 0.2$)

β	$\overline{v}$	ω_1	ω_2	ω_3	ω_4
0.1	0	10.1846	37.0686	81.7810	144.3160
	1	9.7630	36.6294	81.3405	143.8750
	2	8.3784	35.2811	80.0054	142.5460
	3	5.3642	32.9193	77.7325	140.3050
0.3	0	10.1846	37.0686	81.7810	144.3160
	1	9.7504	36.6394	81.3576	143.8960
	2	8.3326	35.3266	80.0766	142.6310
	3	5.2913	33.0439	77.9039	140.5020
0.5	0	10.1846	37.0686	81.7810	144.3160
	1	9.7378	36.6493	81.3746	143.9170
	2	8.2876	35.3705	80.1471	142.7150
	3	5.2213	33.1598	78.0717	140.6970

It is found that the values for critical velocity increase as the Winkler parameter increases. It simply implies that higher values of $\overline{K_w}$ leads to higher critical velocity $\overline{v_c}$. Tables 25 -26 present variation in critical velocity of uniform fluid-conveying pipes resting on two-parameter elastic foundation for Pinned-pinned and Clamped-clamped cases. It is observed that higher rise exists in the critical velocity when $\overline{K_p} \geq 100$, whereas the effect of $\overline{K_w}$ increases the critical velocity. Nevertheless, satisfactory agreement is found between the present results and those of [17]. According to this observation, it can be said that the values of the critical velocity for $\overline{K_p}$ are greater than those of $\overline{K_w}$.

Table 21: Dimensionless Critical velocity of uniform fluid-conveying pipes without resting on Pasternak foundation ($\alpha = 0$, η =0, $\overline{K_w}$ =0, $\overline{K_p} = 0$)

Boundary conditions	β	DTM [5]	Present	Form of Instability
Pinned-pinned	0.1	3.1416	3.1416	Divergence
Clamped-pinned	0.5	4.4934	4.4934	Divergence
Clamped-clamped	0.5	6.2832	6.2832	Divergence

Table 22: Critical velocities of AFG non-uniform fluid-conveying pipes resting on variable Pasternak foundation ($\overline{K_w} = 2.5$, $\overline{K_p} = 1$, $\eta = 0.1$, and $\beta = 0.1$)

Boundary Conditions	α	$\overline{v}_c$	Form of Instability
Clamped-Clamped	0.2	6.3138	Divergence
	0.4	6.2632	Divergence
	0.6	6.1966	Divergence
Clamped-Pinned	0.2	4.4453	Divergence
	0.4	4.5335	Divergence
	0.6	4.6559	Divergence
Pinned-Pinned	0.2	3.3004	Divergence
	0.4	3.3165	Divergence
	0.6	3.3380	Divergence

Table 23: Dimensionless Critical velocity for various values of Winkler foundation parameters for the uniform Pinned-Pinned case ($\beta = 0.1$, $\alpha = 0$, $K_p = 0$)

K_w	Doare and de Langre [13]	Chellapilla [17]	Present
1	3.1577	3.15768	3.15768
10	3.2989	3.29891	3.29891
100	4.47233	4.47233	4.47233
800	7.7293	7.72934	7.72934
1700	9.0851	9.08515	9.08536
3500	11.3196	11.31965	11.3196
4500	11.8105	12.38809	11.8105
6000	12.505	13.83691	12.5049
7000	12.9473	14.72381	12.9472

Table 24: Dimensionless Critical velocity for various values of Winkler foundation parameters for the uniform Clamped-Clamped pipe ($\beta = 0.1$, $\alpha = 0$, $\overline{K_p} = 0$)

$\overline{K_w}$	Doare and de Langre [13]	Chellapilla [17]	Present
1	6.2892	6.38505	6.28923
10	6.3434	6.44208	6.34331
100	6.8613	6.98684	6.85614
800	8.9447	9.9984	9.856
1700	10.7415	10.93215	10.7338
3500	12.323	12.59364	12.2449
4500	13.1194	13.42815	13.3804
6000	13.9649	14.5907	13.9646
7000	14.3231	15.31678	14.3197

Table 25: Critical velocity for various values of foundation stiffness parameters ($\overline{K_w}$ and $\overline{K_p}$) with $\alpha = 0$ and $\beta = 0.1$ for the uniform Pinned-Pinned end condition

$\overline{K_p}$	$\overline{K_w} = 10$		$\overline{K_w} = 10^4$	
	Present	Galerkin [17]	Present	Galerkin [17]
10^{-6}	3.29891	3.2989	17.1109	17.1108
10^{-4}	3.29893	3.2989	17.1109	17.1108
1	3.44715		17.1401	
10	4.4569	4.4586	17.4006	17.4006
100	10.5301	10.4823	19.8187	19.8187
1000	31.6528	31.7786	39.0646	39.9552
5000	71.9676	70.7875	73.006	72.751
10000	100.967	100.054	102.782	101.4533

Table 26: critical velocity for various values of foundation stiffness parameters ($\overline{K_w}$ and $\overline{K_p}$) with $\alpha = 0$ and $\beta = 0.1$ for the uniform Clamped-Clamped end condition

$\overline{K_p}$	$\overline{K_w} = 10$		$\overline{K_w} = 10^4$	
	Present	Galerkin [17]	Present	Galerkin [17]
10^{-6}	6.34331	6.4420	17.33	17.3133
10^{-4}	6.34332	6.4420	17.33	17.3133
1	6.42165		17.41	
10	7.0875	7.1763	17.5812	17.5997
100	11.8422	11.895	19.8236	19.9937
1000	32.2955	32.2722	36.0519	36.0520
5000	71.694	71.0035	72.79	72.7993
10000	100.831	100.207	101.492	101.487

Tables 27-29 give numerical results obtained for critical flow velocities of AFG tapered pipe conveying fluid subjected to the three boundary conditions under consideration, and lying on two parameter variable elastic foundation. Based on those results with $\beta = 0.1$ and $\eta = 0.1$, the effect of $\overline{K_w}$ and $\overline{K_p}$ on critical flow velocity of the pipe was investigated. It can be observed that for a fixed value of $\overline{K_p}$, there is, in general, a sharp increase in the values of the critical flow velocities when $\overline{K_w} \geq 10$ for Pinned-Pinned, Clamped-Pinned, and Clamped-clamped cases. For the three boundary conditions considered, the critical flow velocity increases when $\overline{K_w}$ and $\overline{K_p}$ increases. Thus, it can be concluded that each of the foundation stiffnesses has significant effect on the critical velocity of the system considered.

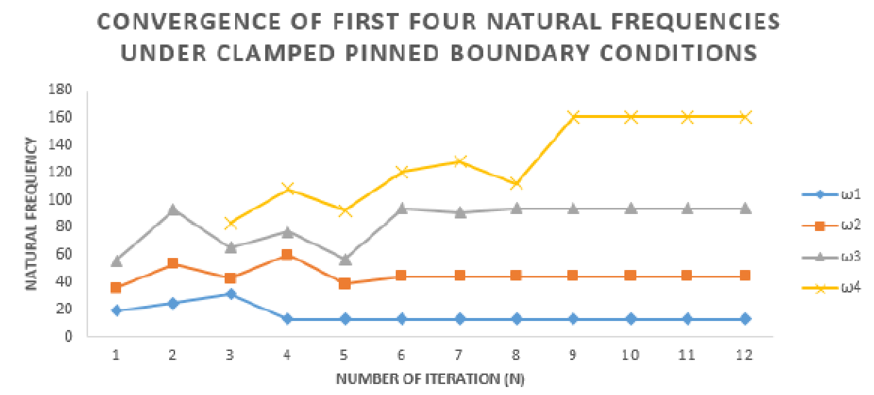

Figure 1: Convergence of the first dimensionless natural frequency for Clamped-Pinned pipe

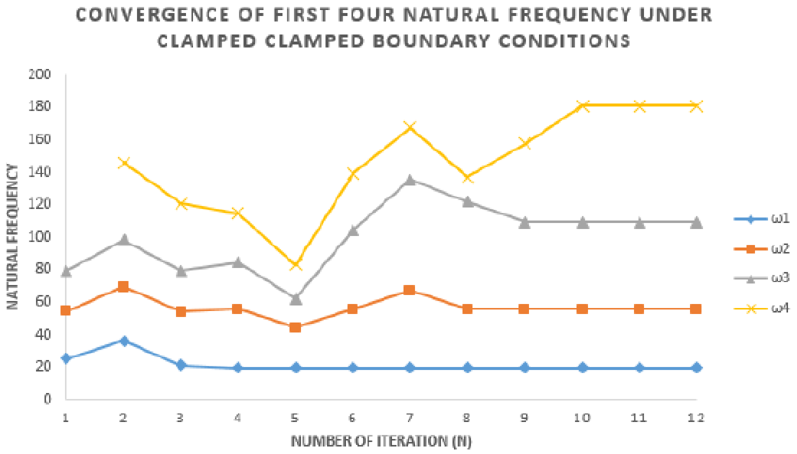

Figure 2: Convergence of the first dimensionless natural frequency for Clamped-Clamped pipe

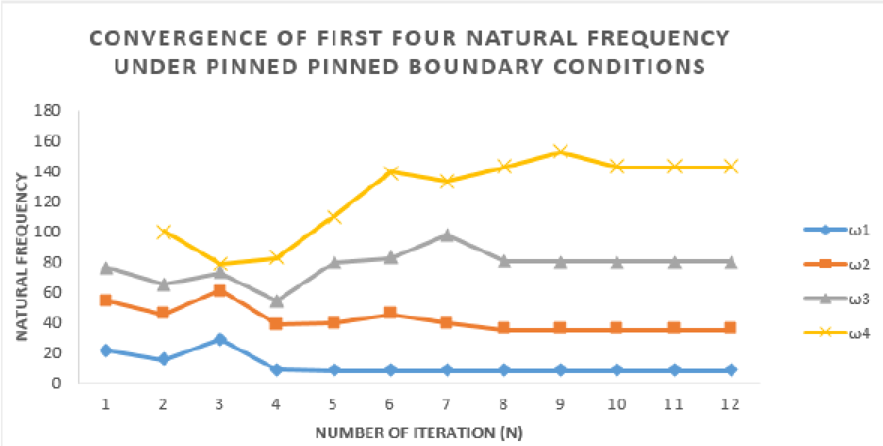

Figure 3: Convergence of the first dimensionless natural frequency for Pinned-Pinned pipe

Next, the impact of non-uniformity parameter α on the critical flow velocities of the non-uniform AFG fluid-conveying pipes resting on variable Pasternak foundation is investigated. The corresponding results for Clamped-Pinned, Pinned-pinned, and Clamped-Clamped conditions are presented in Tables 27-29, respectively. It is vividly noticed that the critical flow velocities increase with increasing α for fixed values of $\overline{K_w}$ and $\overline{K_p}$ in the case of the first and second boundary condition (See Tables 27 and 28). On the other hand for Clamped-Clamped end condition, it is found that the values of the critical flow velocities decrease as α increases when when $\overline{K_p} \leq 1$. Though, there exists a little discrepancy in the values of the critical flow velocities as α increases, especially, when $\overline{K_p} \geq 10$. (See Table 29).

Table 27: Effect of various values of α, $\overline{K_w}$ and $\overline{K_p}$ on critical flow velocity of AFG Clamped-pinned pipes

		$\overline{K_p}$				
α	$\overline{K_w}$	10^{-6}	10^{-1}	1	10	100
0.2	1	4.3590	4.3667	4.4356	5.0729	9.2854
	10	4.4179	4.4255	4.4934	5.1234	9.3122
	50	4.6690	4.6762	4.7404	5.3406	9.4295
	100	4.9614	4.9691	5.0286	5.5970	9.5722
0.4	1	4.4352	4.4440	4.5225	5.2437	9.8955
	10	4.5016	4.5103	4.5877	5.2999	9.9251
	50	4.7838	4.7920	4.8649	5.5414	10.0548
	100	5.1107	5.1183	5.1866	5.8256	11.9474
0.6	1	4.5398	4.5503	4.6431	5.4856	10.7321
	10	4.6173	4.6276	4.7190	5.5503	10.7665
	50	4.9455	4.9551	5.0407	5.8275	10.9180
	100	5.3235	5.3324	5.4122	6.1533	11.1040

Furthermore, the present numerical analysis is also carried out to include an examination of the convergence of the natural frequencies of AFG non-uniform fluid-conveying pipes resting on variable Pasternak elastic foundation. To this end, the convergence of the first four dimensionless natural frequencies of the said system for the case of $\beta = 0.1$, $\alpha = 0.3$, $\overline{K_w} = 2.5$, $\overline{K_p} = 10$ and $\overline{v}$ =2 were plotted in Fig. 1 - 3 for the three end conditions. It is found that, for the three boundary conditions, the first four natural frequencies converge after n = 10 where n is the iteration steps used to analyze the convergence study. The dimensionless ω_m converges faster as the number of iteration steps decreases. Higher mode frequencies might be obtained if the iteration steps increases. The convergence of lower order frequencies can be guaranteed if small terms are chosen.

Results are also presented in Fig. 4-6 to illustrate the effect of functionally graded materials on the frequency of AFG tapered fluid-conveying pipes resting on variable Pasternak elastic foundation with various values of $\overline{K_w}$ and $\overline{v}$ for Clamped-Pinned, Pinned-Pinned, and Clamped-Clamped end conditions. It can be seen that the vibration frequency and velocity increase as the Winkler foundation parameter increases for various material models considered in this work (ie constant, linear and parabolic type). However, the presence of functionally graded materials decreases significantly the frequency and flow velocity values, with $\overline{K_w}$ =100, $\overline{K_w}$=500 and $\overline{K_w}$ =1000. These results also holds for $\overline{K_p}$ = 10, β =0.5, η =0.5, and α =0.4. Similar behaviour can also be remarked for all the boundary conditions.

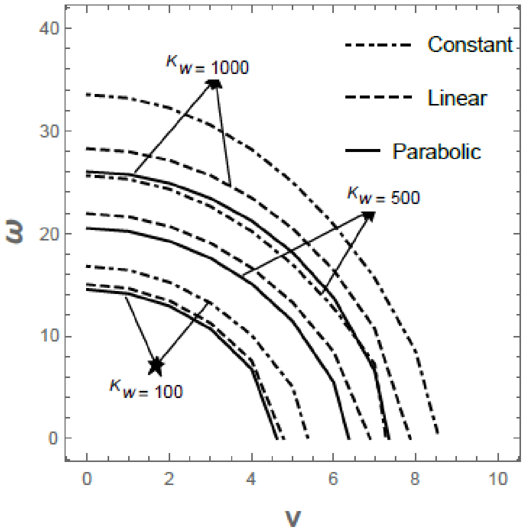

Figure 4: Frequency $\overline{\omega}$ variation against $\overline{v}$ with various values of $\overline{K_w}$ corresponding to different material distribution for Pinned-pinned pipes conveying fluid, ($\overline{K_p}$ = 10, η =0.5, α =0.4, β =0.5)

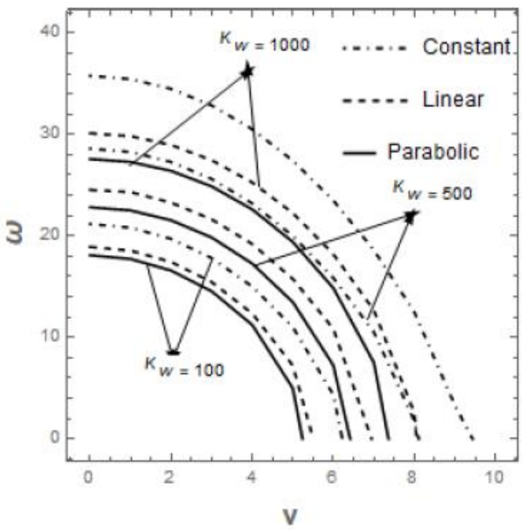

Figure 5: Frequency $\overline{\omega}$ variation against $\overline{v}$ with various values of $\overline{K_w}$ corresponding to different material distribution for Clamped-pinned pipes conveying fluid, ($\overline{K_p}$ = 10, η =0.5, α =0.4, β =0.5)

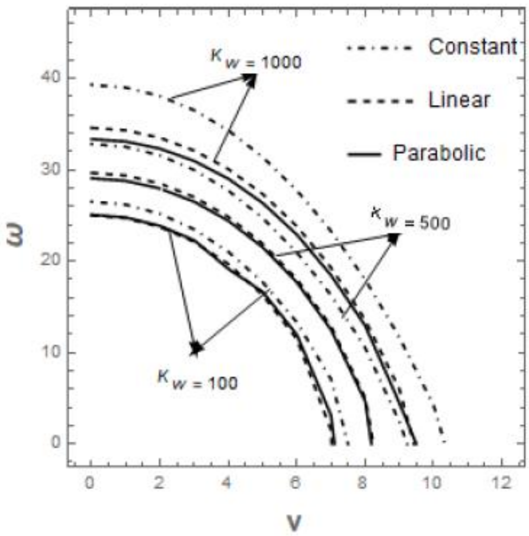

Figure 6: Frequency $\overline{\omega}$ variation against $\overline{v}$ with various values of $\overline{K_w}$ corresponding to different material distribution for Clamped-clamped pipes conveying fluid, ($\overline{K_p} = 10$, η =0.5, α =0.4, β =0.5)

Finally, the critical flow velocities of pipes conveying fluid and resting on Pasternak elastic foundation were calculated as a function of functionally graded materials property. The corresponding results generated can be seen in Table 30. For Pinned-pinned and Clamped-pinned pipes, it can be observed that the parabolic variation of elastic modulus corresponding to the constant, linear and parabolic variations of mass density gives the smallest critical flow velocity but highest critical velocity is obtained when the elastic modulus is constant and mass density varies in a constant, linearly and parabolic manner. However, maximum critical flow velocity was obtained when $E(x)$ varies parabolically with constant, linear and parabolic variation of mass density $\rho(x)$, for clamped-clamped boundary conditions. Based on this observation, the material properties may be said to have significant effect on the stability of the AFG tapered pipes conveying fluid.

Table 28: Effect of various values of α, $\overline{K_w}$ and $\overline{K_p}$ on critical flow velocity of AFG Pinned-pinned pipes

		$\overline{K_p}$				
α	$\overline{K_w}$	10^{-6}	10^{-1}	1	10	100
0.2	1	3.1722	3.1835	3.2838	4.1532	8.9873
	10	3.2742	3.2852	3.3824	4.2319	9.0220
	50	3.6937	3.7034	3.7879	4.5631	9.1741
	100	4.1587	4.1673	4.2442	4.9457	9.3591
0.4	1	3.1730	3.1860	3.2979	4.2597	9.5100
	10	3.2871	3.2994	3.4078	4.3453	9.5480
	50	3.7525	3.7632	3.8586	4.7069	9.7152
	100	4.2633	4.2728	4.3570	5.1230	9.9195
0.6	1	3.1737	3.1883	3.3168	4.3996	10.1407
	10	3.3185	3.3044	3.4421	4.4947	10.1823
	50	3.8318	3.8439	3.9511	4.8954	10.3651
	100	4.4029	4.4135	4.5071	5.3542	10.5892

Table 29: Effect of various values of α, $\overline{K_w}$ and $\overline{K_p}$ on critical flow velocity of AFG Clamped-clamped pipes

		$\overline{K_p}$				
α	$\overline{K_w}$	10^{-6}	10^{-1}	1	10	100
0.2	1	6.2509	6.2566	6.3073	6.7936	10.4671
	10	6.2903	6.2959	6.3463	6.8298	10.4908
	50	6.4618	6.4673	6.5164	6.9883	10.5952
	100	6.6690	6.6744	6.7220	7.1805	10.7239
0.4	1	6.1926	6.1990	6.2559	6.7992	10.8183
	10	6.2367	6.2430	6.2995	6.8393	10.8435
	50	6.4282	6.4343	6.4892	7.0144	10.9544
	100	6.6583	6.6642	6.7172	7.2258	11.0906
0.6	1	6.1149	6.1222	6.1881	6.8126	11.3049
	10	6.1661	6.1734	6.2388	6.8586	11.3329
	50	6.3879	6.3949	6.4581	7.0587	11.4558
	100	6.6527	6.6594	6.7021	7.2993	11.6067

Table 30: Effect of non-homogeneous material on critical velocity of AFG tapered fluid-conveying pipes resting on variable Pasternak foundation ($\overline{K_w} = 10$, $\overline{K_p} = 2.5$, $\eta = 0.2$, $\alpha = 0.2$ and $\beta = 0.5$)

E	ρ	pinned-pinned	clamped-clamped	clamped-pinned
		$\overline{v}_c$	$\overline{v}_c$	$\overline{v}_c$
Constant	Constant	3.6587	6.5380	4.9226
	Linear	3.6587	6.5380	4.9226
	Parabolic	3.6587	6.5380	4.9226
Linear	Constant	3.5202	6.4200	4.5882
	Linear	3.5202	6.4200	4.5882
	Parabolic	3.5202	6.4200	4.5882
Parabolic	Constant	3.4861	6.5694	4.4790
	Linear	3.4861	6.5694	4.4790
	Parabolic	3.4861	6.5694	4.4790

V CONCLUSION

This article, the free vibration and stability analysis of axially functionally graded (AFG) tapered fluid-conveying pipes resting on variable two-parameter elastic foundation under various end conditions has been investigated. In particular, three typical end conditions, namely, Clamped-Clamped, Clamped-Pinned and Pinned-Pinned were considered. Natural frequencies and critical flow velocities of the fluid-conveying pipes under consideration for the said three end conditions were computed. The semi-analytical method known as Variational Iteration Method (VIM) is used to obtain the desired solutions. The efficiency and accuracy of this method are well demonstrated.

The main findings are as follows:

(i) For the purpose of validating the present formulation, results obtained for the present analysis are compared with those of the available published articles for the cases considered under various end conditions, and good agreement is found.

(ii) The influence of non-uniformity parameter α on the critical flow velocities of the non-uniform AFG fluid-conveying pipes resting on variable Pasternak foundation is investigated. Results for Clamped-Pinned, Pinned-pinned and Clamped-Clamped end conditions are obtained. It is noted that the critical flow velocities increase with increasing α for fixed values of $\overline{K_w}$ and $\overline{K_p}$ in the case of the first and second boundary conditions. On the other hand for Clamped-Clamped end condition, it is found that the critical flow velocities decrease as α increases.

(iii) Results are obtained for the impact of both the mass ratio β, and velocity $\overline{v}$ on the frequency of the system under consideration for the three vibrating configuration. It is found that the influence of β decreases the fundamental frequency $\overline{\omega}_1$, while, each of $\overline{\omega}_2$, $\overline{\omega}_3$ and $\overline{\omega}_4$ increases as the values of β increases for fixed values of the fluid velocity $\overline{v}$.

Moreover,the frequency decreases with increasing velocity $\overline{v}$, and the reduction is more noticeable with higher values of $\overline{v}$ for each of the vibrating configurations.

(iv) The effect of material properties on the frequency of the AFG tapered fluid-conveying pipes resting on variable two-parameter foundation under the three vibrating configurations is investigated. It is found that the pipes made up of constant modulus of elasticity and parabolically varying mass per unit volume produces the least frequencies considering all the three boundary conditions. On the other hand, pipes having parabolically varying modulus of elasticity with constant material mass density yields the highest frequency. Thus, it can be concluded that the effect of material properties plays a vital role in the design of the pipes.

(v) New results for AFG non-prismatic fluid-transporting pipes resting on a two-parameter elastic foundation are presented. The influence of the foundation parameters ($\overline{K_w}$, $\overline{K_p}$) on both natural frequency and critical flow velocity is studied. It is observed that the natural frequency and critical flow velocity of the system increase, with the increasing $\overline{K_w}$ for fixed values of $\overline{K_p}$ and this observation holds for the three vibrating configurations considered. However the frequency becomes minimum for Pinned-Pinned AFG non-prismatic fluid-conveying pipes.

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