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Vertex Coloring of Graph Using Adjacency Matrix

T.Ramachandran[1], N.Deepika[2]
[1]Head and Assistant Professor, [2] Research scholar
[1, 2] Department of Mathematics, M. V. Muthiah Government Arts College for Women, Dindigul, Tamil Nadu, India.

ABSTRACT
Graph coloring problem is one of the most popular areas in the field of graph theory and has a long and illustrious history. In a graph coloring, each vertex of the graph is colored in such a manner that no two adjacent vertices have the same color. So far there are several techniques are presented for vertex coloring. In this paper, we propose an algorithm based on the adjacency matrix, to color all the vertices of the given graph with the minimum number of colors and we provide the numerical examples for the proposed algorithm. This algorithm helps us to determine the chromatic number of any graph.
Keywords - Adjacency matrix, Vertex coloring, Chromatic number, Matrix algorithm.

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I. INTRODUCTION

Many real – world situations can conveniently be described by means of a diagram consisting of a set of points together with lines joining certain pairs of these points. For example, the points could represent people with lines joining pair of friends. Notice that in such diagrams one is mainly interested in whether or not two given points are joined by a line; the manner in which they are joined is immaterial. A mathematical abstraction of situations of this type gives rise to the concept of graph [1]. A graph is a set of vertices and edges, the vertices being denoted by set V and edges by set E [2]. Graph coloring has been studied as an algorithmic problem since the early 1970s. The first result about graph coloring deals almost exclusively with planar graphs in the form of the coloring of maps. Graph coloring problem belongs to the class of combinatorial optimization problem and studied due to its lot of application in the area of data science, networking, register allocation and many more. There are many types of coloring such as vertex coloring, edge coloring, total coloring, fractional coloring etc.

Vertex coloring problem can be defined as to assign the color to every vertex of the graph by keeping the constraints that no two adjacent vertices receives the same color such that the number of colors assigned to the vertices should be minimum. The minimum number of colors that will be used to color the vertices of the given graph G is called the chromatic number of the graph and it is denoted by χ (G) [3]. A graph is said to be k –colorable if it can be colored by using k – colors and its chromatic number is k and the graph is called k – chromatic graph [2]. An edge coloring of a graph is a proper coloring of the edges, which means an assignment of colors to edges so that no vertex is incident to edges of the same color. An edge coloring of a graph with k colors is called a k – edge coloring. The smallest number of colors needed for an edge coloring of a graph G is the edge chromatic number and it is denoted by $\chi'(G)$. Total coloring is a type of coloring of both the vertices and edges of a graph. Total coloring is always assumed to be proper in the sense that no adjacent vertices, no adjacent edges and no edge and its end vertices are assigned the same color. The total chromatic number of a graph G is the fewest colors needed in any total coloring of G and is denoted by $\chi''(G)$.

On the greedy algorithms which mostly uses the techniques of deciding the color of vertices sequentially in the coloring process [2]. Greedy algorithm gives the minimum number of colors for vertex coloring but it need not to be a chromatic number. Tabu search techniques provide the optimal coloring of a graph [4]. David S. Johnson et al presented the simulated annealing schemes for graph coloring [5]. Daniel Brelaz presented the new methods to color the vertices of a graph [6]. One of the algorithms uses the machine based learning for graph coloring problem and used 78 identified features for that problem [7]. Amit Mittal et al described a method for graph coloring with minimum number of colors and it takes less time as compared to other techniques [8]. K A Santosa et al, presented the vertex coloring using adjacency matrix [10]. In this paper we propose an algorithm to find the proper coloring of graph using adjacency matrix, which is different from the algorithm proposed by [10], also to suit all types of graph.

II. PRELIMINARIES

Graph coloring is one of the well known parameter in graph theory and many researchers introduced different types of coloring of which vertex coloring is one among them. Although a graph is the pictorial representation of a real – world problem, a matrix is the convenient and useful way of representing a graph.

Some basic definitions and their remarks are presented here under for clear understanding of the algorithm proposed in this paper.

2.1 Definition

Painting all the vertices of a graph with colors such that no two adjacent vertices have the same color is called the *proper coloring* or simply *coloring* of a graph.

A graph in which every vertex has been assigned a color according to a proper coloring is called a *proper colored graph*. A graph G that requires k different colors for its proper coloring, and no less, is called a *k – chromatic graph*, and the number k is called the *chromatic number* of G.

Remarks:

- ✓ A graph consisting of only isolated vertices is 1 – chromatic.
- ✓ A graph with one or more edges (not a self – loop) is at least 2 – chromatic.
- ✓ A complete graph of n vertices is n – chromatic.
- ✓ A graph consisting of simply one circuit with $n \geq 3$ vertices is 2 – chromatic if n is even and 3 – chromatic if n is odd.

2.2 Definition

Let G be a graph with n vertices, e edges, and no self – loops. *Adjacency Matrix* of G is defined by an n by n matrix denoted by $A = [a_{ij}]$, whose n rows and n columns are corresponding to the n vertices. The matrix elements are

$a_{ij} = 1$, if i^{th} vertex is adjacent to j^{th} vertex,
$= 0$, otherwise,

Remarks:

- ✓ Adjacency matrix (or 0,1 matrix) is symmetric, so $a_{ij} = a_{ji}$.
- ✓ Simple graph doesn't have a loop, so the diagonal elements of the adjacency matrix are always zero.
- ✓ The number of 1's in each row equals the degree of the corresponding vertex.
- ✓ For a directed graph, the adjacency matrix is not necessarily symmetric.

III. MATRIX ALGORITHM FOR GRAPH COLORING

For proper coloring of graph using adjacency matrix, a step by step procedure is given below, which help us to color the vertices such that no two adjacent vertices receive the same color.

Algorithm:

Step 1: Construct an adjacency matrix for the given graph.

Step 2: Find the sum of the elements in each row of the matrix. Select the row that has the maximum value.

Case a): If the maximum value is unique, then find the maximal null matrix formed by the zeros in the selected row and go to step 3.

Case b): If there is a tie in the maximum value of the rows, select all those rows and find all maximal null matrices formed by the zeros in the corresponding selected rows then select the largest null matrix among all maximal null matrices and then go to step 3.

Step 3: Check the uniqueness of the null matrix.

Case a): If the maximal null matrix selected in step 2 is unique then go to step 4.

Case b): If there is a tie in the maximal null matrix selected in step 2, find the sum of degrees of all the rows of each maximal null matrix, then choose the null matrix corresponding to the maximum degree.

- If it is unique, then go to step 4.
- If there is tie then select any one null matrix among the tie and go to step 4.

Step 4: Assign a color to the vertices corresponding to the rows of the identified maximal null matrix obtained in step 3 and go to step 5.

Step 5: Remove all the rows and columns associated with the colored vertices, then go to step 2 and repeat the process until all the vertices have been colored.

IV. ILLUSTRATIONS

To understand the above proposed algorithm for proper coloring of vertices of any graph, some of the illustrations are discussed hereunder.

4.1 Illustration

Consider the graph with 11 vertices and 19 edges as shown in the figure 1. Find the proper coloring of a given graph using the proposed matrix algorithm.

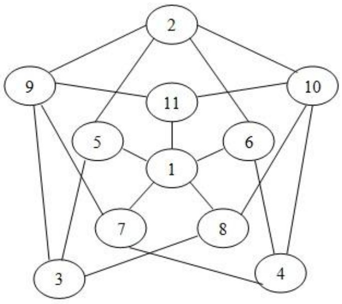

Figure 1

Solution:

As per the algorithm, by the first and second steps, construct the adjacency matrix and compute the sum of the elements in each row of the corresponding matrix that is nothing but the degree of a vertex associated with each row of the matrix and is shown in the table 1.

Table 1

	1	2	3	4	5	6	7	8	9	10	11	degree
1	0	0	0	0	1	1	1	1	0	0	1	5
2	0	0	0	0	1	1	0	0	1	1	0	4
3	0	0	0	0	1	0	0	1	1	0	0	3
4	0	0	0	0	0	1	1	0	0	1	0	3
5	1	1	1	0	0	0	0	0	0	0	0	3
6	1	1	0	1	0	0	0	0	0	0	0	3
7	1	0	0	1	0	0	0	0	1	0	0	3
8	1	0	1	0	0	0	0	0	0	1	0	3
9	0	1	1	0	0	0	1	0	0	0	1	4
10	0	1	0	1	0	0	0	1	0	0	1	4
11	1	0	0	0	0	0	0	0	1	1	0	3

From the table 1, first row has the maximum value 5. By case a) of step 2, it is unique, it's corresponding vertex is 1 and the matrix formed by the zeros in the first row is given in the table 2.

Table 2

	1	2	3	4	9	10
1	0	0	0	0	0	0
2	0	0	0	0	1	1
3	0	0	0	0	1	0
4	0	0	0	0	0	1
9	0	1	1	0	0	0
10	0	1	0	1	0	0

Table 2 shows that, vertices 1, 2, 3 and 4 forms the maximal null matrix. By case a) of step 3, it is unique. By the step four, assign a first color (say pink) to the vertices 1, 2, 3 and 4 associated with the rows of the identified maximal null matrix.

By the step five, remove all the rows and columns associated with the colored vertices of a given graph.

Again by the first and second steps the adjacency matrix and sum of the elements in each row of the matrix for the uncolored vertices of a given graph is given in the table 3.

Table 3

	5	6	7	8	9	10	11	degree
5	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0
7	0	0	0	0	1	0	0	1
8	0	0	0	0	0	1	0	1
9	0	0	1	0	0	0	1	2
10	0	0	0	1	0	0	1	2
11	0	0	0	0	1	1	0	2

Table 3 shows that, the maximum value is 2, and by case b) of step 2 there is tie, select the fifth, sixth, seventh rows and its corresponding vertices are 9, 10 and 11 respectively.

From the fifth row, we find that the vertices 5, 6, 8, 9 and 5, 6, 9, 10 form the maximal null matrices of order 4. From the sixth row, vertices 5, 6, 7, 10 and 5, 6, 9, 10 form the maximal null matrices of order 4. From the seventh row, vertices 5, 6, 7, 8, 11 form the maximal null matrix of order 5.

Select the maximal null matrix among the null matrices formed by the zeros of the selected rows. Vertices 5, 6, 7, 8 and 11 form the largest null matrix among all maximal null matrices. And it is unique by the case a) of step3.

By the step four, assign a second color (say yellow) to the vertices associated with the rows of the identified maximal null matrix. By the step five remove all the rows and columns associated with the colored vertices of given graph.

The remaining vertices 9 and 10 form the null matrix**.** Assign a third color (say blue) to those vertices. The resulting colored graph is depicted in the figure 2.

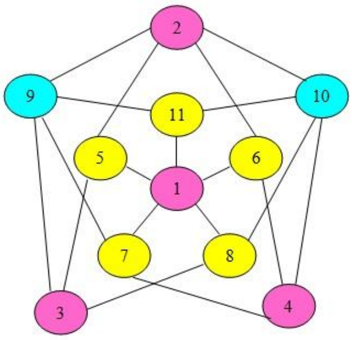

Figure 2

By the proposed matrix algorithm, the vertices of a given graph are colored with the minimum three colors and its chromatic number is 3.

4.2 Illustration

Consider the graph with 8 vertices and 15 edges as shown in the figure 3. Find the proper coloring of a given graph using the proposed matrix algorithm.

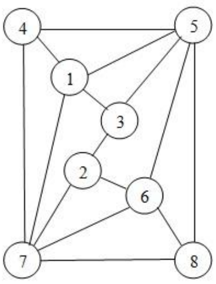

Figure 3

Solution:

As per the algorithm, by the first and second steps, construct the adjacency matrix and compute the sum of the elements in each row of the corresponding matrix that is the degree of a vertex associated with each row of the matrix and is given in table 4.

Table 4

	1	2	3	4	5	6	7	8	Degree
1	0	0	1	1	1	0	1	0	4
2	0	0	1	0	0	1	1	0	3
3	1	1	0	0	1	0	0	0	3
4	1	0	0	0	1	0	1	0	3
5	1	0	1	1	0	1	0	1	5
6	0	1	0	0	1	0	1	1	4
7	1	1	0	1	0	1	0	1	5
8	0	0	0	0	1	1	1	0	3

Table 4 shows that, the maximum value is 5. By case b) of step 2, there is tie in the fifth and seventh rows and the associated vertices are 5 and 7 respectively. Zeros in the fifth row form the maximal null matrices of order 2 with vertices 5, 7 and 5, 2. From the seventh row vertices 7, 5 and 7, 3 form the maximal null matrices of order 2.

By case b) of step 3, there is a tie in the maximal null matrices. Sum of the degrees of all rows of each maximal null matrices formed by the vertices 5, 7 and 5, 2 and 7, 3 are 10, 8 and 8 respectively. Here select the maximal null matrix formed by the vertices 5 and 7, as it has the maximum degree 10 and it is unique. By the step four, assign a first color (say pink) to the vertices 5 and 7 associated with the rows of the identified maximal null matrix.

By step five, remove all the rows and columns associated with the colored vertices of a given graph.

The reduced adjacency matrix and sum of the elements in each row of the matrix for the uncolored vertices of a given graph is given in the table 5.

Table 5

	1	2	3	4	6	8	Degree
1	0	0	1	1	0	0	2
2	0	0	1	0	1	0	2
3	1	1	0	0	0	0	2
4	1	0	0	0	0	0	1
6	0	1	0	0	0	1	2
8	0	0	0	0	1	0	1

Table 5 shows that, the maximum value is 2. By case b) of step 2, there is tie in the first, second, third and fifth row and its associated vertices are 1, 2, 3 and 6 respectively.

From the first row, vertices 1, 2 and 8 form the maximal null matrix of order 3. From the second row, vertices 1, 2, 8 and 2, 4, 8 form the maximal null matrices of order 3. From the third row, vertices 3, 4, 6 and 3, 4, 8 form the maximal null matrices of order 3. From the fifth row, vertices 3, 4 and 6 formed the maximal null matrix of order 3.

By case b) of step 3, there is a tie in the maximal null matrices. Sum of the degrees of all rows of each maximal null matrices formed by the vertices 1, 2, 8 and 2, 4, 8 and 3, 4, 6 and 3, 4, 8 are 5, 4, 5 and 4 respectively. And there is tie in the maximal null matrices formed by vertices 1, 2, 8 and 3, 4, 6 because both have the maximum degree 5. Select any one arbitrarily (here let us choose the maximal null matrix formed by the vertices 1, 2 and 8).

And by the step four, assign a second color (say yellow) to the vertices 1, 2 and 8 of the identified maximal null matrix.

By the step five remove all the rows and columns associated with the colored vertices of given graph.

The remaining vertices 3, 4 and 6 form the null matrix**.** Assign a third color (say blue) to those vertices. The resulting colored graph is depicted in the figure 4.

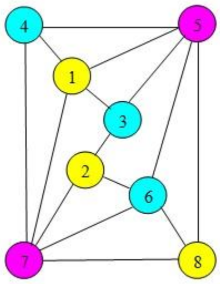

Figure 4

By the proposed matrix algorithm, all the vertices of a given graph are colored with the minimum three colors and its chromatic number is 3.

V. CONCLUSION

In this paper a new matrix algorithm is presented to find the proper coloring of any given graph using adjacency matrix and which suits all types of graphs. The illustrations discussed in the previous section clearly indicate the perfection of the proposed matrix algorithm for proper coloring of any given graph. Further a computer based algorithm can be developed in future by using any computer languages, which will make easy, the coloring of any larger size graph. This will reduce the complexity of finding the chromatic number of any graph.

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