RESEARCH ARTICLE OPEN ACCESS

Convection in Anisotropic Porous Medium with Gravity Modulation using Ginzburg-Landau Model

Vanishree, R. K.*, Harshitha, G. E., Gowthami, U. L., Chandrakala, M. N., Chaithra, R.
**(Department of Mathematics, Maharani's Science College for Women, Maharani Cluster University, Bengaluru-01*

ABSTRACT
The effect of gravity modulation, thermo-mechanical anisotropies, inverse Darcy number and Brinkman number is studied on heat transport. The amplitude equation is obtained in the form of Ginzburg-Landau model. It is observed that the gravity modulation , inverse Darcy number and Brinkman number is to reduce the heat transfer whereas the thermo-mechanical anisotropies show opposite effect on heat transfer.

***Keywords* -** Rayleigh-Bènard convection, anisotropy, porous medium, gravity modulation, Ginzburg-Landau model

---

Date of Submission: 07-04-2020 Date of Acceptance: 22-04-2020

I. INTRODUCTION

The study of convection in porous medium has a wide range of applications in geophysics, oil recovery process, in petroleum industry, and in solidification of polymeric liquids. Natural convection in a fluid-saturated horizontal porous medium has received considerable attention owing to its applications in diverse areas. The study has an analogy with the usual Rayleigh-Benard convection problem from a phenomenological view point. The study of natural convection of a Newtonian fluid in a porous medium is now well understood and documented [1-10].

The gravity modulation is one consisting of varying acceleration term in the gravitation Rayleigh number around the gravitational acceleration, i.e, by vertically oscillating a horizontal porous layer. This modulation leads to the variable coefficient in the momentum equation and involves vertical time-periodic vibrations of the system. Many authors ([11-15]) have studied the gravity modulation in porous medium.

Most of the studies on Rayleigh-Benard-Darcy convection focus attention on only the onset of convection. Linear stability analysis is inadequate if one wants to study heat transport. It is with this motive we have made a weakly non-linear analysis of Rayleigh-Bènard-Brinkman convection with gravity modulation . One other motive for the study is to propose time-periodic vertical vibrations (gravity modulation) as an effective external means of controlling convection. Alternately one could make a non-linear study of Rayleigh-Benard-Brinkman convection with gravity modulation using the Lorenz model that would yield a system of three non-autonomous, non-linear ordinary differential equations which are difficult to solve (see[16]). It is on this reason that we adhere to the use of the Ginzburg-Landau equation for our non-linear analysis.

Ginzburg-Landau model is one of the most studied equations in applied mathematics. It describes a vast array of phenomena including non-linear waves, second-order phase transitions, Rayleigh-Bènard convection and superconductivity ([16]). With this motivation a study has been conducted on the heat transport in an anisotropic porous medium with gravity modulation using the amplitude equation (Ginzburg-Landau model).

II. MATHEMATICAL FORMULATION

The physical configuration considered is a horizontal anisotropic, porous layer of infinite extent occupied by Boussinesquian Newtonian fluid confined between two boundaries at $z=0$ and $z=d$ that have a difference in temperature ΔT as a shown in Fig.1. The fluid density is assumed to be a linear function of temperature T. A cartesian co-ordinate system is taken with the origin in the lower boundary and z-axis vertically upwards. An appropriate single-phase heat transport equation is chosen with effective heat capacity ratio and effective thermal diffusivity. Thus, the governing equations for the system with gravity modulation are:

$$\nabla \cdot q = 0 \qquad (1)$$

$$\rho_0 \left[\frac{\partial q}{\partial t} + (\vec{q}\cdot\nabla)\vec{q} \right] = -\nabla p + \rho \vec{g}' + \mu_e \nabla^2 \vec{q} - \frac{\mu_f}{K}\cdot \vec{q} \qquad (2)$$

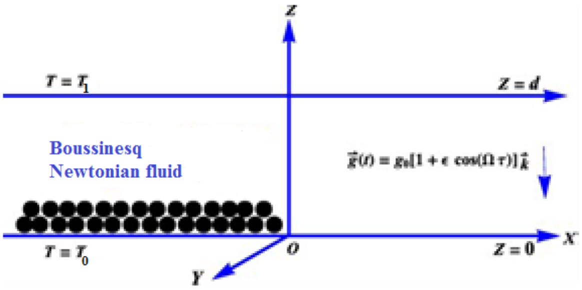

Fig.1: Schematic of the flow of configuration.

$$\gamma \frac{\partial T}{\partial t} + \vec{q}\cdot\nabla T = \chi_v \left[\eta \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right]$$

(3) $\rho = \rho_0 \left[1 - \alpha \left(T - T_0 \right) \right]$

(4)

where $K = \frac{1}{K_h}\hat{i}\hat{i} + \frac{1}{K_v}\hat{k}\hat{k}$ is the permeability tensor, $\vec{g}' = g_0 \left[1 + \delta_1 \varepsilon_1^{\,2} \cos\left(\Omega_0 t\right) \right]$ is the gravity modulation and $\gamma = \frac{(\rho_0 C)_m}{(\rho_0 C)_f}$ is the heat capacity ratio. For simplicity, γ is taken to be unity in the paper. In the case of two- dimensional convection, one can introduce the stream function after eliminating the pressure. Then (2) and (3) take the form:

$$\rho_0 \frac{\partial}{\partial t}(\nabla^2 \psi) = \alpha \rho_0 g_0 \left[1 + \delta_1 \varepsilon_1^{\,2} \cos(\Omega_0 t) \right] \frac{\partial T}{\partial t} - \mu_e \nabla^4 \psi - \frac{\mu_f}{K_v} \left[\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{\varepsilon}\frac{\partial^2 \psi}{\partial z^2} \right] + \rho_0 \frac{\partial(\psi, \nabla^2 \psi)}{\partial (x,z)} \qquad (5)$$

$$\frac{\partial T}{\partial t} = \frac{\partial(\psi, T)}{\partial(x,z)} = \chi_v \left[\eta \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right] \qquad (6)$$

Scaling length by d (depth of the porous channel), ψ by χ_v (thermometric conductivity along z-axis), time by $\frac{d^2}{\chi_v}$ and temperature by ΔT (temperature difference between the horizontal boundaries) in (7) and (8), we get the following non-dimensional equations:

$$\frac{1}{P_r}\frac{\partial}{\partial t}\left(\nabla^2 \psi\right) = R_a \left[1 + \delta_1 \varepsilon_1^{\,2} \cos\left(\Omega_0 t\right) \right] \frac{\partial T}{\partial x} - \Lambda \nabla^4 \psi - Da^{-1} \left[\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{\varepsilon}\frac{\partial^2 \psi}{\partial z^2} \right] + \frac{1}{\Pr}\frac{\partial(\psi, \nabla^2 \psi)}{\partial(x,z)} \qquad (7)$$

$$\frac{\partial T}{\partial t} + \frac{\partial(\psi, T)}{\partial(x,z)} = \eta \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \qquad (8)$$

where ε_1^2 is a small quantity that indicates the weak variation. The non-dimensional parameters in the above equations are: Pr= $\frac{\rho_0}{\chi_v \mu_f}$ (Prandtl number), $Ra = \frac{\alpha \rho_0 \Delta T d^3}{\chi_v \mu_f}$ (Thermal Rayleigh number) and $Da^{-1} = \frac{d^2}{K_v}$ (Inverse Darcy number). The boundary conditions for solving (7) and (8) are:

$$\left. \begin{array}{l} \psi = 0 \text{ and } T = 1 \text{ and } Z = 0 \\ \psi = 0 \text{ and } T = 0 \text{ and } Z = 1 \end{array} \right\} \qquad (9)$$

The conduction profile is given by:

$$T_b(Z) = 1 - Z \quad \text{and} \quad \psi_b = 0 \qquad (10)$$

Now we impose finite amplitude perturbations on the basic quiescent state given by (10) as:

$$T = 1 - Z + \Theta \quad \text{and} \quad \psi = \Psi \qquad (11)$$

Substitution of (11) in (7) and (8) yields the following equations:

$$\frac{1}{\Pr}\left[\frac{\partial}{\partial t}\left(\nabla^2 \Psi\right) + \frac{\partial(\Psi, \nabla^2 \Psi)}{\partial(x,z)} \right] = -\Lambda \nabla^4 \Psi + Ra \left[1 + \delta_1 \varepsilon_1^{\,2} \cos\left(\Omega_0 t\right) \right] \frac{\partial \Theta}{\partial x} - Da^{-1} \nabla_\varepsilon^2 \Psi \qquad (12)$$

$$\frac{\partial \Theta}{\partial t} + \frac{\partial(\Psi, \Theta)}{\partial(x,z)} = \nabla_\eta^2 \Theta \qquad (13)$$

Boundary conditions to solve (12) and (13) are:

$$\left. \begin{array}{l} \Psi = 0 \text{ and } \Theta = 0 \text{ and } Z = 0 \\ \Psi = 0 \text{ and } \Theta = 0 \text{ and } Z = 1 \end{array} \right\}. \qquad (14)$$

we now use the expansion:

$$\left. \begin{array}{l} R = Ra_0 + \varepsilon_1^2 R_2 + \varepsilon_1^3 R_3 + \ldots\ldots\ldots \\ \Psi = \varepsilon_1 \Psi_1 + \varepsilon_1^2 \Psi_2 + \varepsilon_1^3 \Psi_3 + \ldots\ldots \\ \Theta = \varepsilon_1 \Theta_1 + \varepsilon_1^2 \Theta_2 + \varepsilon_1^3 \Theta_3 + \ldots\ldots \end{array} \right\} \qquad (15)$$

We now assume the variation of time only at the slow time scale $\tau = \varepsilon_1^{\,2} t$

At the lowest order, we have:

$$\begin{bmatrix} \Lambda\nabla^4 + Da^{-1}\nabla_\varepsilon^2 & Ra_0 \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} & \nabla_\eta^2 \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Theta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (16)$$

(16)

Now the solutions of lowest order system is given by :

$$\left.\begin{aligned} \Psi_1 &= A(\tau)\sin(k_c x)\sin(\pi z) \\ \Theta_1 &= \frac{k_c A(\tau)}{\delta_\eta^{\,2}}\cos(k_c x)\sin(\pi z) \end{aligned}\right\} \qquad (17)$$

The system (16), together with the solution set (17), gives us Rayleigh number for stationary onset of convection and the expression is given below:

$$Ra_0 = \frac{\left[\Lambda\delta^4 + Da^{-1}\delta_\varepsilon^{\,2}\right]}{k_c^{\,2}}\delta_\eta^{\,2} \qquad (18)$$

This is the classical result of [17], $\varepsilon = \eta = \Lambda = 1$ yields the classical results of [10a] for isotropic porous media. The critical value of Ra_0 is obtained at the critical value of k_c given by the least positive root of $\frac{\partial}{\partial k_c}(Ra_c) = 0.$

We now derive the expression for the Nusselt number and also obtain the Ginzburg-Landau equation for the case of stationary instability.

III. AMPLITUDE EQUATION AND HEAT TRANSPORT FOR STATIONARY INSTABILITY

At the second order, we have:

$$\begin{bmatrix} \Lambda\nabla^4 + Da^{-1}\nabla_\varepsilon^2 & Ra_0 \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} & \nabla_\eta^2 \end{bmatrix} \begin{bmatrix} \Psi_2 \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} R_{21} \\ R_{22} \end{bmatrix} \qquad (19)$$

where:

$$\left.\begin{aligned} R_{21} &= 0 \\ R_{22} &= \frac{1}{2\delta_\eta^{\,2}}k_c^{\,2}\pi A(\tau)\sin(2\pi z) \end{aligned}\right\} \qquad (20)$$

The second order solution is given by:

$$\left.\begin{aligned} \Psi_2 &= 0 \\ \Theta_2 &= -\frac{1}{8\pi\delta_\eta^{\,2}}k_c^{\,2}A^2(\tau)\sin(2\pi z) \end{aligned}\right\} \qquad (21)$$

The horizontally averaged Nusselt number, Nu, for the stationary mode of convection is given by:

$$Nu = \frac{\displaystyle\int_{z=0}^{z=1}\left[\int_0^{2\pi/k_c}(1 - Z + \Theta)_z\,dx\right]_{z=0}}{\displaystyle\int_{z=0}^{z=1}\left[\int_0^{2\pi/k_c}(1 - Z)_z\,dx\right]_{z=0}} \qquad (22)$$

Substituting Θ_2 from (21) into (22) , we get:

$$Nu = 1 + \frac{1}{4\delta_\eta^{\,2}}k_c^{\,2}A^2(\tau) \qquad (23)$$

At the third order, we have:

$$\begin{bmatrix} \Lambda\nabla^4 + Da^{-1}\nabla_\varepsilon^2 & Ra_0 \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} & \nabla_\eta^2 \end{bmatrix} \begin{bmatrix} \Psi_3 \\ \Theta_3 \end{bmatrix} = \begin{bmatrix} R_{31} \\ R_{32} \end{bmatrix} \qquad (24)$$

where,:

$$\left.\begin{aligned} R_{31} &= \frac{1}{\Pr}\frac{\partial}{\partial t}(\nabla^2\Psi_1) + Ra_0\delta_1\cos(\Omega_0^{\,*}t)\frac{\partial\Theta_1}{\partial x} \\ &\quad + R_2\frac{\partial\Theta_1}{\partial x} \\ R_{32} &= \frac{\partial\Theta_1}{\partial\tau} + \frac{\partial\Psi}{\partial x}\frac{\partial\Theta_2}{\partial z} \end{aligned}\right\} \qquad (25)$$

Substituting for Ψ_1 , Θ_1 and Θ_2 from (17) and (21), we get the following expressions for R_{31} and R_{32} :

$$\begin{aligned} R_{31} &= \frac{1}{\Pr}\frac{dA}{d\tau}(k_c^{\,2} + \pi^2)\sin(k_c x)\sin(\pi z) \\ &\quad + \delta_1 Ra_0\cos(\Omega_0^{\,*}t) - \left(\frac{k_c A(\tau)}{\delta_\eta^{\,2}}\sin(k_c x)\sin(\pi z)\right) \\ &\quad + Ra_2\left(-\frac{k_c A(\tau)}{\delta_\eta^{\,2}}\sin(k_c x)\sin(\pi z)\right) \end{aligned} \qquad (26)$$

$$\begin{aligned} R_{32} &= \frac{k_c}{\delta_\eta^2}\frac{dA}{d\tau}\cos(k_c x)\sin(\pi z) \\ &\quad - \frac{k_c^{\,3}A^3(\tau)\pi}{4\delta_\eta^2}\cos(k_c x)\sin(3\pi z) \\ &\quad + \frac{k_c^{\,3}A^3(\tau)\pi}{4\delta_\eta^2}\cos(k_c x)\sin(\pi z) \end{aligned} \qquad (27)$$

Sollvability condition for the existence of the third order solution gives us:

$$\int_{z=0}^{z=1}\int_{x=0}^{x=1}[\Psi_1 R_{31} + Ra_0\Theta_1 R_{32}]\,dx\,dz = 0 \qquad (28)$$

Substituting R_{31} , R_{32} and Θ_1 , Ψ_2 from (26), (27) and (17) in (28) and completing the integration, we get the Ginzburg-Landau equation as below

$$\frac{dA}{d\tau}\left[\delta^2 \frac{\pi}{\Pr k_c} + Ra_0 \frac{k_c \pi}{\delta_\eta^2}\right] = \left[Ra_0 \frac{\delta_1}{\delta_\eta^2}\pi \cos(\Omega_0{}^* t) + \frac{R_2 \pi}{\delta_\eta^2}\right] A(\tau) - \frac{Ra_0}{\delta_\eta^4} k_c A^3(\tau) \tag{29}$$

In what follows we discuss the result based on the computation of the Nusselt number (23) using the solution of (29). The non-autonomous Ginzburg- Landau equation (29) is solved numerically for A(τ) by Runge-Kutta-Fehlberg 45 method.

IV. RESULTS AND DISCUSSIONS

The problem addresses a non-linear realm of Rayleigh-Bènard-Brinkman convection in a Newtonian fluid in a sparsely packed anisotropic porous medium with g-jitter. A weakly non-linear stability analysis is made using a non-autonomous Ginzburg-Landau model. The thermal and mechanical anisotropies arise due to the properties of the porous matrix. The g-jitter or gravity modulation arise due to the Rayleigh-Bènard-Brinkman system being vibrated time periodically in the vertical direction. It is well known that, in general, gravity modulation leads to increase in critical Rayleigh number ([11]). The effect of vertical vibrations on the Rayleigh-Bènard-Brinkman system is assumed to be of $O(\varepsilon_1^2)$. In essence this means that we are considering small amplitude vibrations. Such an assumption facilitates over amplitude of convection in a rather simple and elegant manner and is much easier to obtain than in the case of the Lorenz model. The work of [17] clearly show that the thermal anisotropy and mechanical anisotropy have opposite influence on the Rayleigh number. This can be well understood through the expression of Rayleigh number. The Ginzburg-Landau model in the problem given by (29) is a Bernoulli equation and obtaining the analytical solution is hindered by the non-autonomous nature of the equation. Hence the ODE solver with the type RKF, i.e., Runge-Kutta-Fehlberg method of order 45 in scilab was used to solve (29)

Equation (23) seen in conjunction with (29) clearly reveals that Nu(τ) is a function of the inverse Darcy number, Da^{-1}, mechanical anisotropy parameter, ε , thermal anisotropy parameter, η, Brinkman number, Λ, the critical Rayleigh number Ra_0 and the amplitude of modulation, Ω. Porous medium under consideration is a sparsely packed one and hence we assume Da^{-1} to take the values from 1 to 100. The quantity Λ can assume a range of values that are greater than, equal to or less than one (see [19]). Principle of exchange of stabilities is valid and hence stationary mode is the preferred mode of convection.

From the linear theory, we know that the effect of inverse Darcy number and Brinkman number is to stabilize the system. whereas Ra_{0_c} decreases with increase in ε and increases with increasing η. In the case of non-linear theory, we find from the fig.2 that effect of Da^{-1} on heat transfer is to decrease the same. It is obvious due to the fact that Da^{-1} stabilizes the system and hence the heat transfer decreases.

Figure (3) establishes the fact that the effect of Λ on heat transport is not very dominant. This means that the effective viscosity, μ_e, and fluid viscosity, μ_f, are almost equal for this problem under consideration.

From figs.4-5 it is clear that ε and η have opposing effect on heat transfer. The effect of ε is to increase the heat transport and that of η is to decrease the same. This is in accordance with the linear theory as ε destabilizes and η stabilizes the system.

The effect of frequency of modulation Ω is to diminish the heat transport as can be seen in fig. (6).

From the above discussions it is apparent that the geometry of the porous media and the externally manageable g-jitter can be effectively used in regulating the convection.

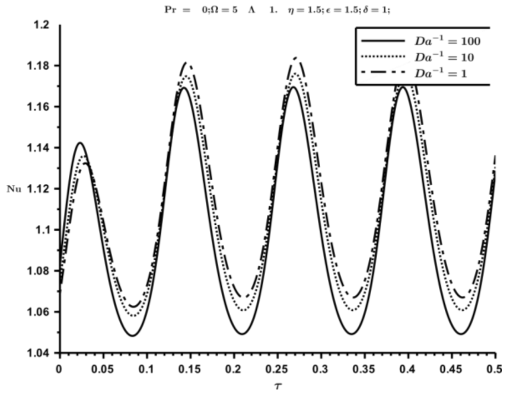

Fig 2: Plot of Nu versus τ for different values of inverse Darcy number, Da^{-1}.

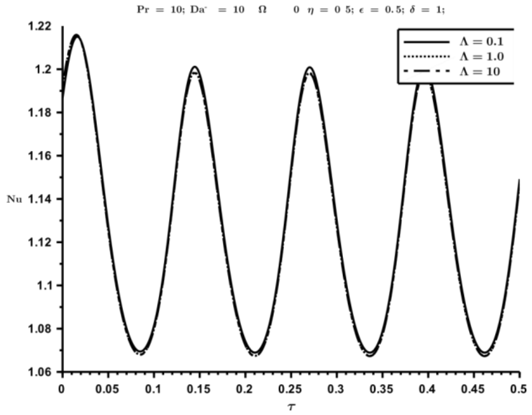

Fig 3: Plot of Nu versus τ for different values of Brinkman number, Λ.

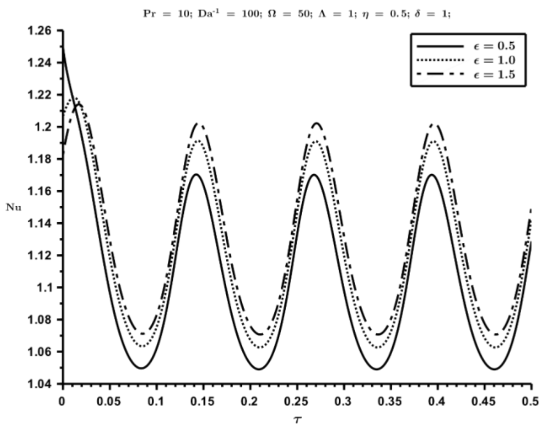

Fig 4: Plot of Nu versus τ for different values of mechanical anisotropy parameter ε.

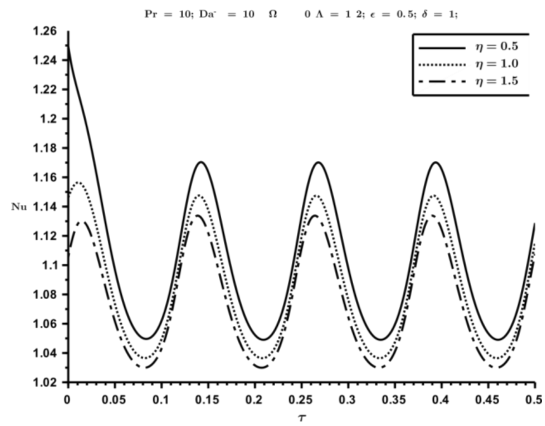

Fig 5:Plot of Nu versus τ for different values of mechanical anisotropy parameter η.

V. CONCLUSION

The following conclusion are drawn from the above study.

i). Convection can be controlled by proper choice of the porosity of the porous medium.

ii). The mechanical and thermal anisotropies have opposite effect on heat transport.

iii). Frequency of modulation can be used as an effective means of controlling convection.

iv). Ginzburg-Landau model can be used effectively to understand heat transport.

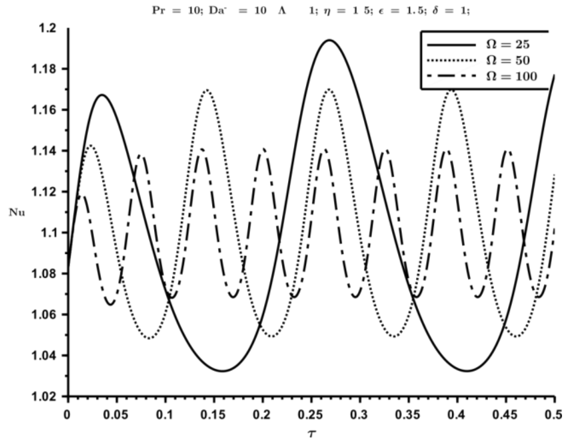

Fig 6: Plot of Nu versus τ for different values of frequency of modulation Ω.

REFERENCES

[1]. J. W. Elder, Numerical experiments with free convection in a vertical slot. J. Fluid Mech., 24, 1966, 823-843.

[2]. S. A. Bories and M. A. Combarnous, Natural convection in a sloping porous layer. J. Fluid Mech., 57, 1973, 63-79.

[3]. P. Cheng, Heat transfer in geothermal systems, Advances in heat transfer (Academic Press), 14. 1978,1-105.

[4]. N. Rudraiah, Non-linear convection in a porous medium with convective acceleration and viscous force. Arab. J. Sci. Engg. 92, 1984, 153-167.

[5]. D.A. Nield, Recent research on convection in a porous medium: Convective flows in porous media, Proc. of a seminar organised by OSIR and CSIRO Institute of Physical Sciences. Wairakei. New Zealand, 1978.

[6]. M. Kaviany, Onset of thermal convection in a saturated porous medium:experiment and analysis. Int. J. Heat Mass Transfer, 27, 1984, 2101-2110.

[7]. D. B. Ingham, I. Pop, Transport phenomena in porous media (Elsevier, 1998).

[8]. K. Vafai, Hand book of porous media, (Marcel Dekker, Inc., New York, 2000).

[9]. N. Rudraiah, P. G. Siddheshwar, T. Masuoka, Nonlinear convection in porous media, J. Porous Media:A Review, 6, 2003.

[10]. D. A. Nield, A. Bejan, Convection in porous media, (3rd edition, Springer-Verlag. Newyork, 2006).
[11]. M. S. Malashetty, V. Padmavathi, Effect of Gravity modulation on the onset of convection in a porous layer. J. Porous Media**,** 1, 1997, 219-226.
[12]. S. Govender, Linear Stability and Convection in a Gravity Modulated Porous Layer Heated from Below: Transition from Synchronous to Subharmonic Solutions, Transp. in Porous Media, 59(2), 2005, 227-238.
[13]. P. G. Siddheshwar, R. K. Vanishree, A. C. Melson, Study of heat transport in Benard-Darcy convection with G-Jitter and thermo mechanical anisotropy in variable viscosity liquids. Transp. Porous Media, 92, 2012, 277-288.
[14]. P. G. Siddheshwar, R. K. Vanishree. Lorenz and Ginzburg-Landau eqations for thermal convection in a high-porosity medium with heat source. Ain Shams Eng J. 9, 2018, 1547-1555.
[15]. Srivastava Alok, B. S. Bhadauria, P. G. Siddheshwar, I. Hashim. 2013. Heat transport in an anisotropic porous medium saturated with variable viscosity liquid under G-jitter and internal heating effects. Transp. Porous Media, 99, 2013, 359-379.
[16]. P. G. Siddheshwar, A series solution for the Ginzburg-Landau equation with a time-periodic coefficient, Appl. Math., 3, 2010, 542–554.
[17]. J. F. Epherre**,** Critere d'apparition de la convection naturelle dans une couche poreuse anisotrope. Rev. Gen Thermique**,** 168, 1975, 949-950.
[18]. E. R. Lapwood, Convection of a fluid in a porous medium, Mathematical Proceedings of the Cambridge Philosophical Society, 44(4), 1948, 508-521
[19]. R. C. Givler, S. A. Altobelli, Determination of the effective viscosity for the Brinkman-Forchheimer flow model. J. Fluid Mech. 258, 1994, 355–370.

Vanishree, R. K. "Convection in Anisotropic Porous Medium with Gravity Modulation using