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RESEARCH ARTICLE OPEN ACCESS

Analysis of Convergence of Jacobi and Gauss Siedel Method and Error Minimization.

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ABSTRACT
In this research, we will show that neither of the iterative methods always converges. Implying that the Jacobi and Gauss Seidel Methods do not converge often when applied to a system of linear equations yielding a divergent sequence of approximations. In such cases, the method is termed divergent. Therefore for a system of equations to converge, the Diagonal Dominance of the matrix is necessary before applying any iterative method. The error reduction factor will also be discussed in each Iteration in Jacobi and Gauss Seidel method.

Keywords: Iterative Methods, Convergence, Divergence, Diagonal Dominance, Jacobi Method Gauss, Seidel Method.

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I. INTRODUCTION

An Iterative technique used to solve the system of Ax = b where A is an nxn dimension matrix with initial approximation x_0 to the solution x^*, generating a sequence of vectors $\{x^k\}_{k=0}^{\infty}$ which converges to x^k. Iterative techniques involve methods that convert the system Ax = b into an equivalent system of the form
x = px + α most of the time for some n x n matrix p and vector α. After the selection of the initial condition x_0 , the sequence of approximate solution. Vectors are generated by computing $x^{(k+1)} = p^k x + \alpha$, k = 1,2,3,------. The Jacobian method operates on solving for every variable once. It is not difficult to understand and implement, but convergence is slow. The Gauss Seidel and Jacobi methods are similar except that the updated values are used as soon as they are discovered. Generally, it converges faster than the Jacobi method although still relatively slowly.

1.1 Jacobi Method: The Jacobi Method involves solving an n x n matrix that has no zeros along its diagonal. Each diagonal element is solved, and an approximate value is put in. The process is iterated until it converges. Each of the equations in the linear system are examined easily in the linear system of equations Ax = b. Suppose in the i^{th}equation $\sum_{j=1}^{\infty} a_{ij}x_j = b_i$ the value of x_i
Is solved while the other entries of x are assumed to remain fixed. This gives

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[b_i - \sum_{j \neq i} a_{ij}x^{(k)}\right],\ i = 1, 2, ...n \qquad (1)$$

Which is the Jacobi method. Note that computing $x_i^{(k+1)}$ requires each component in $x^{(k)}$ except $x_i^{(k)}$. The order in which the equations are examined is irrelevant in this method, since the Jacobi method treats them independently. Equation Ax = b, which is decomposed into (L+U+D) x = b, which can be written as
D x = - (L + U) x + b which reduces to
$X_{i+1} = D^{-1}(L+U)\, x_i + D^{-1}\, b$.

$$\begin{bmatrix} m_{1,1} & & & & & \\ & m_{2,2} & & & & \\ & & m_{3,3} & & & \\ & & & m_{4,4} & & \\ & & & & m_{5,5} & \\ & & & & & m_{6,6} \end{bmatrix} x^{k+1} + \begin{bmatrix} & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} & m_{1,6} \\ m_{2,1} & & m_{2,3} & m_{2,4} & m_{2,5} & m_{2,6} \\ m_{3,1} & m_{3,2} & & m_{3,4} & m_{3,5} & m_{3,6} \\ m_{4,1} & m_{4,2} & m_{4,3} & & m_{4,5} & m_{4,6} \\ m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & & m_{5,6} \\ m_{6,1} & m_{6,2} & m_{6,3} & m_{6,4} & m_{6,5} & \end{bmatrix} x^{k} = b$$

Fig 1: Splitting of matrix for Jacobi Method.

Where $P_j = - D^{-1}(L+U)$ and $\alpha_j = D^{-1}b$.
Note: If A is a strictly diagonal dominant matrix, then
$\|P_j\| = \|- D^{-1}(L + U)\|_\infty < \|- D^{-1}\|_\infty \, \|(L+U)\|_\infty = \|(L+U)\|_\infty / \max a_{ii} < 1$

Hence for any $x_0 \in IR_n$, the sequence $\{x^k\}_{k=0}^{\infty}$ defined by
$x_k = p_j x_{k-1} + \alpha_j$: (k= 1,2,-----) $\alpha_j \neq 0$ converges to the unique solution of $x = p_j x + \alpha_j$. Therefore Jacobi's iterative technique will always guarantee a solution to the equation Ax = b whenever A is strictly diagonally dominant.

1.2 Gauss – Seidel Method: The Gauss Seidel method is a modified form of the Jacobi's iteration method. It is defined on matrices with non – zero diagonals, but if the method is diagonally dominant and symmetric positive definite, then convergence is guaranteed. The equation Ax = b which is (L+D+U) x = b, can be written as (D+L) x = - U x + b. This reduces to
$x = - (D + L)^{-1} U x + (D+L)^{-1} b$

$$\begin{bmatrix} m_{1,1} & & & & & \\ m_{2,1} & m_{2,2} & & & & \\ m_{3,1} & m_{3,2} & m_{3,3} & & & \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} & & \\ m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & m_{5,5} & \\ m_{6,1} & m_{6,2} & m_{6,3} & m_{6,4} & m_{6,5} & m_{6,6} \end{bmatrix} x^{(k+1)} + \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} & m_{1,6} \\ & m_{2,2} & m_{2,3} & m_{2,4} & m_{2,5} & m_{2,6} \\ & & m_{3,3} & m_{3,4} & m_{3,5} & m_{3,6} \\ & & & m_{4,4} & m_{4,5} & m_{4,6} \\ & & & & m_{5,5} & m_{5,6} \\ & & & & & m_{6,6} \end{bmatrix} x^{(k)} = b$$

Fig 2: Splitting of Matrix for Gauss Seidel Method.

Therefore,

$x^{(k+1)} = - (D + L)^{-1} U x^{(k)} + (D + L)^{-1}$ Where $p_g = - (D + L)^{-1}U$ and $\alpha_g = (D + L)^{-1} b$.
The iterative Gauss Seidel method for solving the system of equations is given by

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left[b_i - \sum_{j<i} a_{ij} x_i^{(k+1)} - \sum_{j \geq i} a_{ij} x_i^{(k)} \right] \quad (2)$$

Note: The combination of $x_i^{(k+1)}$ have already been computed ($x_i^{(k+1)}$ for j <i). This means that no additional storage is required and the combination in place is possible ($x_j^{(k)}$ for j ≥I).Meaning that no additional storage is required and the computation can be done in place ($x_i^{(k+1)}$ replaces $x^{(k)}$). While this might seem like a rather minor concern, it is unlikely that large systems can be stored for every iteration. If A is a strictly diagonally dominant matrix, then as we have done above, in Jacobi Iteration method, we have
$\|P_j\|_\infty = \|-(D + L)^{-1} U\|_\infty < \|- (D + L)^{-1}\|_\infty \, \|(U)\|_\infty < 1$

Thus for any $x_0 \in IR_n$, the sequence $\{x^k\}_{k=0}^{\infty}$ defined by $x_k = p_g x_{k-1} + \alpha_g$: (k = 1,2,-----) , $\alpha_g \neq 0$ converges to the unique solution of $x = p_g x + \alpha_k$ i.e the Gauss – Seidel iterative technique solves the equation Ax = b. Unlike in the Gauss Seidel method, in the Jacobi method, we can't overwrite $x_i^{(k+1)}$ with $x_i^{(k)}$ as that value will be needed by the rest of the computation. This is the most meaningful difference between the Jacobi and Gauss Seidel methods.

II. IMPORTANCE OF DIAGONALLY DOMINANCE

A square matrix A is diagonally dominant if each $\left|a_{ii}\right| \geq \sum_{j=1}^{n}\left|a_{ij}\right|$ i.e diagonal elements of each row exceeds its absolute value from sum of absolute values of all other entries in that row. A matrix is said to be diagonally dominant if the inequality is strict and if the inequality is greater than or equal to, then we say that the matrix is weakly diagonally dominant. However the combination of these methods with more efficient methods e.g in [3, 4] it can be seen that the associated Jacobi and Gauss Seidel method also converges for any x_0 [1]. If A is symmetric positive dominant then, $s_j = -D^{-1}(L + U)$ is convergent and the Jacobi iteration will converge, otherwise, the method will converge frequently. We must check

$\rho(s_j)$ if A is not diagonally dominant to see if the method can be applied and $\rho(s_j) \leq \| s_j \|$. If $\| s_j \| < 1$, then $S_j$is convergent so we apply Jacobi's Iteration method. Also if $\rho(s_j)$ is nearer to unity, then convergence will be slow. The Jacobi and Gauss Seidel method will diverge if the matrix is not diagonally dominant.

Example 2.1 Solve the linear system of equations.
6x+5y+7z=3; 4x+2y-2z=1, 3x-6y+2z= 9
Using initial approximation (x, y, z) = (0, 0, 0) and show that both methods diverge.
Solution: First, we rewrite the system of equations in the form
X=1/6(3-5y-7z); Y=-1/2(1-4x+2z); Z =1/2(9-3x+6y)
now by using the Jacobi iteration formula(i) with initial approximation (0, 0, 0) we get
$X^{(1)}$ = 1/6(3-15×0-7×0) =0.5
$Y^{(1)}$ = 1/2(1-4×0+2×0) = 0.5
$Z^{(1)}$ = 1/2(9-3×0+6×0) =4.5

As first approximation. By repeating this, we get the sequence of approximation as shown in the table

N	0	1	2	3	4
X	0	0.5	-4.5	-7.625	-17.125
Y	0	0.5	4.0000	17.75	24.25
Z	0	4.5	5.2500	-7.5	81.1875

From the table above, we can conclude that the Jacobi method becomes worse as computations progress. Thus, we can conclude that the method diverges. Next we will see in Gauss Seidel method, the given system of equations diverge faster than the Jacobi method. By using the formula (2) with initial approximation (x,y,z)=(0,0,0) in (iii) we get
$X^{(1)}$ = 1/6(3-15×0-7×0) =0.5
$Y^{(1)}$ = 1/2(1-4×(0.5)+2×0) = -0.5
$Z^{(1)}$ = 1/2(9-3×(0.5)+6 u7×(0.5)) =2.25
shown in the table below

N	0	1	2	3	4	5
X	0	0.5	-3.375	-26.8511	-30.1670	-186.057
Y	0	-0.5	4.5834	25.8962	-55.9591	300.3425
Z	0	2.25	21.3128	15.1877	135.9237	535.2102

Therefore, neither the Jacobi nor Gauss Seidel convergesto the system of linear equations.

2.2. Convergence of Jacobi and Gauss Seidel method by Diagonal Dominance

Interchanging the rows of the given system of equations in example 2.1, we get
4x+2y-2z=3
3x-6y+2z=9
2x+3y+7z= 3
to be strictly diagonally dominant i.e
|4| ≥ |2| +|-2|
|6| ≥ |3| +|2|
|7| ≥ |2| +|3|

We will apply the Jacobi and Gauss Seidel method to get the values below:

Jacobi Iterations:

N	0	1	2	3	4	5
X	0	0.75	0.6429	0.4507	0.5709	1.3488
Y	0	1.4335	1.4643	0.7858	-2.3809	2.995
Z	0	1.2857	0.8571	0.8724	1.1833	0.6406

Gauss Seidel iterations:

N	0	1	2	3	4	5
X	0	0.75	-0.5607	0.5185	0.6199	0.5667
Y	0	1.125	1.8125	1.5742	1.4015	0.8702
Z	0	-0.6964	0.9023	0.5657	1.0349	0.6396

Thus we can conclude that the exact solution is (x, y, z) = (1, 1, 1) because the error reduces in each proceeding iteration. Showing that strict diagonal dominance is a necessary condition for convergence of the Jacobi and Gauss Seidel methods. Evidently Gauss Seidel method converges faster than the Jacobi method, seeing that it achieves more convergence in lesser number of iterations.

III. MEASUREMENT OF REDUCTION OF ERROR

We consider the solution of linear system Ax = b, by the fixed point iteration. Such iteration schemes can all be based on approximate inverse. Implying that any matrix norm [2]. Corresponding to an approximate inverse C for A, we consider the iteration function

$G(x) = Cb + (1 - CA)\, x$
$\quad = x + C\,(b - A\,x)$

i.e,

$g(x) = x - c\, f(x)$ where

$f(x) = Ax - b$. Also $g(x)- g(y)= C\,b + (1 - CA)\,x - [Cb + (1 - CA)y] = (1 - CA)\,(x - y)$.

Consequently, $\| g(x) - g(y) \| \le \| (1 - CA) \| \, \| x - y \|$ with $k = \| (1 - CA) \| < 1$

Therefore fixed point iteration $x^{(m+1)} = x^m + C\,(b - A\,x)$; m = 0, 1, 2, ……..

Starting from any $x^{(0)}$, will converge to a unique solution with the error at each step reduces by at least a factor of $k = \| (1 - CA) \|$

3.1 Article: Suppose the matrix A is strictly row diagonally dominant

I .e $|a_{ii}| > \sum_{j=1}^{n} |a_{ij}|$, $j \ne I$; I = 1, 2, ….n.

Let $D = \mathrm{Diag}\,(a_{11}, a_{22}, a_{33}, \ldots\ldots a_{nn})$ be the diagonal of A.

Then $\| 1 - D^{-1} A \|_{\infty} = {}^{\max}{}_{i}\{ \sum_{j} |a_{ij}| / |a_{ii}\} < 1$.

Approximate inverse of A, The corresponding scheme $x^{(m+1)} = x^m + D^{-1}\,(b - A\,x^{(m)})$; m = 0, 1, 2, 3, …… is Jacobi iteration.

Formulas discussing Jacobi iteration as

$X_i^{(m)} = 1/a_{ii}[\, b_i - \sum_{j=i} a_{ij} x_j^{(k)}]$, I = 1, 2, 3, ……. Also

Gauss Seidel iteration is defined as $x^{(k+1)} = 1/a_{ii}[\, b_i - \sum_{j=i} a_{ij}\, x_i^{(k+1)} - \sum_{j \ge i} a_{ij}\, x_i^{(k+1)}]$ □ I

For the linear System :

9x + y +z = 11; x +9 y +z = 13: x + y +9z = 11

Iterations starting with $x^{(0)} = 0$, after applying both methods we get the values

	Jacobi Method			Gauss Seidel Method			
M	x	y	z	X	y	z	
0	0	0	0	0	0	0	
1	1.153	1.6423	1.167	1.2	1.2	0.967	
2	0.945	0.891	0.946	0.9567	1.033	1.0001	
3	1.078	1.0788	1.067	0.976	1.0025	1.0003	
4	0.9975	0.9873	0.986				
5	1.003	1.0034	1.0031				
6	0.975	0.9787	0.965				

Then, we conclude that the sequence converges to the solution (x, y, z) = (1, 1, 1) of the system.

Now $\| 1 - D^{-1} A \|_{\infty} = {}^{\max}{}_{i}\{ \sum_{j \ne i} |a_{ij}| / |a_{ii}\} = {}^{\max}{}_{i}\{ 1/10 + 1/10 \} = 0.1999$

Expecting a solution in error by a factor of 0.1999 per step, seen in the table above.

IV. Conclusion

This paper shows that the Jacobi Iteration method converges for any given matrix A, then Gauss Seidel method also converges, and it generally converges faster than the Jacobi method but is associated that the method is strictly diagonally dominant. If ∣ S ∣ <1, then the iteration

matrix S is convergent and we use Jacobi iteration method. But if ρ(s) is nearer to unity then convergence is very slow.(Implying that the spectral radius of the iteration matrix S should be small) .
And we see in each step that the error reduced by the factor of
$K = \| 1 - D^{-1} A \|$.

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