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r-Regular Near-Rings

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ABSTRACT
In this paper the terms, regular near-rings,r- regular near-rings, symmetric near-ring, weakly regular near-ring, completely prime ideal, 1-prime ideal, 1-semiprime ideals are introduced. We investigated some basic properties for r- regular near-rings. We use completely prime ideal, maximal ideal, 1-prime ideal, 1-semiprime ideals to characterize r- regular near-rings. Finally, we proved that the following conditions concerning for r-regular near-ring with identity and has IFP are equivalent (1) N is regular near-ring. (2) $A = \sqrt{A}$for every N-subgroup A of N. (3) N is left bipotent. (4) N is strongly regular near-ring. And also it is proved that the following conditions concerning for a near-ring $N \in \eta_0$with identity are equivalent (1) N is r - regular and has IFP and (2) N is reduced and every completely prime ideal is maximal.

Key words: completely prime ideal, Insertion Factor Property (IFP), reduced near-ring,r-regularnear-rings.

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I. INTRODUCTION:

The history of the concept near-ring is highly influenced by the knowledge of ring theory.Regular (von-neumann regular) ring plays an important role in structure theory of rings which was first introduced by Von-Neumann. Generalization of rings (Near-rings) plays a vital role in the development of Mathematics. Several mathematicians studied and developed various types of near-rings such as Boolean near-rings, IFP near-rings, left bipotent near-rings, P-strongly regular near-ring and strong IFP near-rings

The first step in this subject was taken by Dickson and Zassenhaus [1] with their studies of "near-fields" and by Wielandt with his classification of an important class of abstract near-rings. Various characterization of near-rings are developed by Betsch [2], Laxton [3], Steve Ligh [4], [5], Beidleman [6], Bell [7], Ramakotaiah [8], Gunter Pilz [9]. Especially, in ideal theory of near-rings, Groenewald [10], Holcombe [11], Birkenmeier [12] researched on different types of prime ideals in near-rings, types of primitivity, radicals in near-rings. Later, Dheena [13], Elavarasan [14] developed the regularity concept by introducing s-weakly regular near-rings and strong IFP near-rings. This regularity concept was researched by Mason [15], [16], Y.V.Reddy and C.V.L.N.Murthy[17], Groenewald and Argac [18]. Recently Yong Uk Cho [19], T. Tamizh Chelvam, S. Uma, and R. BalaKrishnan [20] developed some characterization on near-rings by introducing semi central idempotents and α1,α2 near-rings.

Definition1.1 : [9] A **rightnear-ring** is a nonempty set N together with two binary operations '+' and '.' such that

i. (N, +) is a group.
ii. (N, .) is a semigroup.
iii. For all $n_1, n_2, n_3 \in$ N,$(n_1 + n_2).n_3 = n_1.n_3 + n_2.n_3$ (right distributive law).

Example1.2 : [9]Let Γ be an additive group. Let $M(\Gamma)$ be the set of mappings of Γ into itself. We define addition and multiplication in $M(\Gamma)$ as follows:

i. $(f+g)(\gamma) = f(\gamma) + g(\gamma)$.

ii. $fg(\gamma)=f(g(\gamma))$, for all f, $g\in M(\Gamma)$and $\gamma\in\Gamma$.

We show that $M(\Gamma)$ together with the above binary operations of addition and multiplication forms a near-ring. Consider the two mappings **Ō :** $\gamma\rightarrow 0$ (zero mapping of Γ) and **i :** $\gamma\rightarrow\gamma$ (identity mapping of Γ).If $f\in M(\Gamma)$, define **-f** to be the mapping given by $(-f)(\gamma)=-f(\gamma)$, forall $\gamma\in\Gamma$.Obviously, $f+(-f)=\bar{O}$ and $(-f)+f=\bar{O}$, for all $f\in M(\Gamma)$. Furthermore, it can be easily shown that for all $f,g,h\in M(\Gamma)$,
$(f+g)+h=f+(g+h)$, $(fg)h=f(gh)$,
$(f+g)h=fh+gh$.
Hence ($M(\Gamma)$, +, .) is a near-ring.

Example1.3 : Let (N,+) where N = {0, a, b, c} be the klein's four group with the addition and product table1 and table2 repsectively.

Table1: Addition table

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Table2: Producttable

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

Then (N, +, .) is a near-ring.

Definition1.4 :[9]Let N be a near-ring. Then N_0 = $\{n\in N/n0=0\}$ is called the **zerosymmetricpart** of N.

Definition1.5 : [9] A near-ring N is called**zero symmetric near-ring** if n0 = 0, for all $n\in N$ i.e. N= N_0.

Definition1.6 : [9]An element 'a' of a near-ring N is said to be **left identity** of N, if $ax=x$ for all $x\in N$.

Definition 1.7: [9]An element 'a' of a near-ring N is said to be **right identity** of N, if $xa=x$ for all $x\in N$.

Definition1.8 : [9] An element 'a' of a near-ring N is said to be **two sided identity or an identity element** of N, if it is both a left identity and a right identity of N.

Definition1.9 : [9]An element 'a' of a near-ring N is said to be **left invertible**of N, if there exist an element $b\in N$ such that $ba=1$. The element 'b' is called **left inverse** of a.

Definition1.10 : [9] An element 'a' of a near-ring N is said to be **right invertible**of N, if there exist an element $c\in N$ such that $ac=1$. The element 'c' is called **right inverse** of a.

Definition1.11 : [9]An element 'a' of a near-ring is said to be **invertible** of N, if 'a' is both left invertible and right invertible of N.

Definition1.12 : [9]A subgroup M of a near-ring N with MM $\subseteq$ M is called a **subnear-ring**of N.

Definition1.13 : [9]A subgroup M of a near-ring N is said to be a **N-subgroup,**if NM $\subseteq$ M.

Notation1.14 : [9]If S, T $\subseteq$ N, then we define ST = { st / s∈ S, t∈ T }.

Definition1.15 : [9] A normal subgroup I of (N, +) is called a **leftideal** of a near-ring N, if for all n,m∈ N, for all i∈ I, n(m+i)-nm∈ I.

Definition1.16 : [9]A normal subgroup I of (N, +) is called a **rightideal** of a near-ring N, if IN $\subseteq$ I .

Definition1.17 : [9]A normal subgroup I of (N, +) is both a left ideal and a right ideal of a near-ring N is called a **two sided ideal or ideal**of N.

Theorem 1.18 : [9] If N is a zero-symmetric near-ring, then every ideal of N is an N-subgroup of N.

Definition1.19: [9]Let X be a non-empty subset of a near-ring N.Let { L_i / i $\in$ I } be the family of all left ideals which contain X. $L=\bigcap_{i\in I} L_i$ is the smallest left ideal containing X is called the **left ideal generated** by X.

Definition1.20 : [9] The intersection of all right ideals of N containing a non-empty set X is called the **right ideal generated** by X.

Definition1.21 : [9]An ideal A of a near-ring N is called **prinicipal ideal** of N if A is an ideal generated by a single element.
If an ideal A is generated by an element 'a', then A is denoted by $\langle a\rangle$.

Definition1.22 : [9]A near-ring N is said to be **simple** if N has no non-trivial ideals.

Definition1.23 : [9]Let N be a near-ring. If (N-{0}, .) is a group, then N is called a **near-field**.

Definition1.24 : [9]A **zero divisor** of a near-ring N is an element a ≠ 0 of N which satisfies ab = 0 for some nonzero b in N.

Definition1.25: [9] A near-ring N is said to be **integral** if it has no non zero divisors.

Definition1.26:[9]If N is a near-ring, then the **center**of N is the set C = { x∈ N / nx = xn, for all n∈ N}. Elements of C are called central.

Definition1.27 :[9]An element 'a' is said to be **idempotent element** of a near-ring N if $a^2 = a$, for a ∈ N.

Definition1.28 : [9] A non empty subset A of a semi group S is said to be **subsemi group** of S if for any a, b ∈ A implies ab ∈ A.

Definition1.29 : [9, 13]A nonempty set M ⊆ N is called **m-system** if for all a,b∈M there exist $a_1 \in < a >, b_1 \in < b >$ suchthat $a_1 b_1 \in$ M.

Definition1.30 : [9]An ideal M in a near-ring N is said to be **maximal ideal**if M ≠ N and for every ideal I such that M ⊆ I ⊆ N, either M = I or I = N.

Definition1.31 : [9] An ideal M in a near-ring N is said to be **minimal ideal** of N if it is minimal in the set of all non zero ideals of N.

Definition1.32 : [9] An ideal P in a near-ring N is said to be **prime ideal** if P ≠ N and for any ideals I, J in N, IJ ⊆ P ⟹ I ⊆ P or J ⊆ P.

Definition1.33 : [9] A near-ring N is said to be **primenear-ring** if {0} is a prime ideal.

Definition1.34 : [9] An ideal S in a near-ring N is **semiprime ideal** if for every ideal I of N such that $I^2 \subseteq S \Rightarrow I \subseteq S$.

Theorem1.35 : [9] Each prime ideal of a near-ring is semi prime.

Theorem1.36 : [9] Let N be a near-ring with N^2 = N. Then every maximal ideal in N is prime.

Definition1.37 : [9]If I is an ideal in a near-ring N, then $\wp(I) = \bigcap_{P\ prime\ ideal\ P \supseteq I} P$ is the **primeradical** of I.

Definition1.38 : [9] The intersection of all prime ideals of a near-ring N is called the **primeradical** of N and denoted by $\wp(N)$.

Definition1.39 : [9]Thequasi radical of a near-ring N is the intersection of all the maximal right ideals of N.

Definition1.40 : An element n in a near-ring N is called **nilpotent**, if there existk∈N such that n^k = 0.

Definition1.41 : A subset S in a near-ring N is called **nilpotent** if there existk∈N such that S^k = {0}.

Definition1.42 : A subset S in a near-ring N is called **nil** if all s∈S are nilpotent.

Definition1.43 : An N-subgroup S of a near-ring N is called a **Nil N-Subgroup**if all elements of S are nilpotent.

Definition1.44 : The sum of all nil ideals of a near-ring N is called the **nilradical** of N and is denoted by $\eta(N)$.

Definition1.45 : Let Δ be a subset of a near-ring N. Then the set(**0 : Δ) = { n∈ N / nx = 0, for all x∈Δ}** is called the **annihilator of** Δ .

Note1.46 : If Δ = { δ }, then (0 : Δ) is denoted by (0 : δ).

Theorem1.47 : [9] For any δ ∈ N, (0 : δ) is a left ideal of a near-ring N.

Theorem1.48 : [9]If Δ is a N - subgroup of Γ, then (0 : Δ) is an ideal in a near-ring N.

Definition1.49 : [7, 9, 14]A near-ring N is said to fulfill the **insertionoffactorsproperty (IFP),** if ab = 0 ⟹ anb = 0, for all a, b, n∈ N.

Example1.50 : Let (N, +) where N = {0, a, b, c} be the klein's four group. Define multiplication in N as shown in the table 3.

Table.3: Product table

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

Then (N, +, .) is a IFP near-ring.

Theorem1.51 : [9] The following assertions are equivalent :

i. **N has the IFP - property.**
ii. **(0 : n) is an ideal of N, for all n∈ N.**
iii. **(0 : S) is an ideal of N, for all subsets S of N.**

Definition1.52 : [9]A near-ring N is said to be **reduced near-ring** if it is without non- zero nilpotent elements.

Theorem1.53 : [9] If N $\in \eta_0$ is reduced near-ring. For any a, b in N, if e is an idempotent in N then $abe = aeb$.

Theorem1.54 : If N is a zero-symmetric near-ring without nonzero nilpotent elements, then N has the IFP- property.

Proof : If ab = 0 (a, b∈ N), then $(ba)^2 = baba$ = b0a = 0.
Hence ba = 0 ------- (1).
For all n∈ N, (nb) a = n (ba) = n0 = 0.
anb = a (nb) = 0 (by (1)).

Therefore N has the IFP.
The above theorem is not true for arbitrary near-rings, in general.
For this consider the following example.

Example1.55 : Let Z_2 be the group of integers modulo 2. M (Z_2) is a near-ring without nonzero nilpotent elements. But M (Z_2) does not possess the IFPproperty.
For, if f $\in$ M (Z_2) is given by f (0) = 1 and f (1) = 0;g $\in$ M (Z_2) is given by g (0) = 1 and g (1) = 1, then fg = 0, but ffg $\neq$ 0.

Definition1.56 : A **homomorphism** of a near-ring N is a mapping ϕ of N into a near-ring M such that for all a, b$\in$ N, $\phi(a+b)=\phi(a)+\phi(b)$ and $\phi(ab)=\phi(a)\phi(b)$.

Definition1.57 : An **epimorphism** of a near-ring N is a homomorphism which is onto.

Definition1.58 : A **monomorphism** of a near-ring N is a one-one mapping and homomorphism.

Definition1.59 : An **isomorphism** of a near-ring is both epimorphism and monomorphism.

Definition1.60 : [9]Let { N_i}$_{i\in I}$ be a family of near-rings. The Cartesian product of the $N_i$is the set of all functions $f : I \rightarrow \bigcup_{i\in I} N_i$ such that f (i) $\in N_i$ for all i$\in$I. It is denoted by $\prod_{i\in I} N_i$. $N_i$with the component-wise defined operations '+' and '.' (i.e $(f+g)(i)=f(i)+g(i)$ and $fg(i)=f(i)g(i)$ for all f, g$\in \prod_{i\in I} N_i$) is called the **directproductofthenear-rings** {N_i} $_{i\in I}$.

Definition1.61 : [9]Let {N_i}$_{i\in I}$ be a family of near-rings. The Cartesian product of the $N_i$is the set of all functions $f : I \rightarrow \bigcup_{i\in I} N_i$ such that f (i) $\in N_i$ for all i$\in$ I. It is denoted by $\prod_{i\in I} N_i$. For each i$\in$ I, define a map $\prod_i : \prod_{j\in I} N_j \rightarrow N_i$ by $\prod_i(f)=f(i)$. Then $\prod_i$ is an epimorphism and it is called the **projection** of $\prod_{j\in I} N_j$ onto N_i.

Definition1.62 : [9]A near-ring N is said to be a **subdirectproduct** of the family of near-rings {N_i}$_{i\in I}$ if N is a subnear-ring of the direct product $\prod_{i\in I} N_i$ such that $\prod_j(N)=N_j$, for every j$\in$I , where $\prod_i : \prod_{i\in I} N_i \rightarrow N_j$, is the projection map.

Definition1.63 : [9]A near-ring N is called **subdirectlyirreducible**if the intersection of all non-zero ideals of N is non-zero.

Theorem1.64 : [9] Every near-ring is isomorphic to a subdirectproduct of subdirectly irreducible near-rings.

Definition1.65 : [9]A relation R on a set A is called **partial order relation**if R is reflexive, anti-symmetric and transitive. Then (A, R) is called the **partial ordered set** or a **poset**.

Definition1.66 : [9]A partial ordered set A such that any two elements are comparable is called a **total ordered set**.

Lemma (Zorn's lemma) 1.67 : [9] If every totally ordered set C in a poset (S, $\leq$) has a lower bound (upper bound), then (S, $\leq$) has a minimal element(maximal element).

Theorem1.68 : [13] If N $\in \eta_0$ is a reduced near-ring then for any 0 $\neq$ a$\in$N, (1) $N/A(a)$ is reduced and the residue class $\bar{a}$ of a mod A(a) is a nonzero divisor where A(a) = {x$\in$N / xa = 0}.
(2) $x_1x_2...x_n=0$ implies $\langle x_1\rangle\langle x_2\rangle.....\langle x_n\rangle=0$ for any $x_1, x_2...x_n$ in N.

Theorem1.69: **[13] Let N be a reduced near-ring. If M is a nonvoid multiplicative subsemigroup of N such that 0 $\notin$ M, then thereexist a completely prime ideal P of N such that P$\cap$M= ϕ .**

Definition1.70 : [9]A near-ringis said to be **Leftbipotent** if Na = Na^2 for each a in N.

Definition1.71 : [9]A near-ring N is said to be **regular near-ring** if for every a in N there existx in N such thata = axa.

II. r-REGULAR NEAR-RINGS

Definition 2.1 : A near-ring N is called **r -regular near-ring**, if for each $a \in N$ there exist $e^2=e\in N$ such that $a=ae, e\in\langle a|$, where $<a|$ is the left ideal generated by a.

Example 2.2 : Any regular near-ring is a r - regular near-ring.
Consider a near-ring on the group Z_6 ={ 0, 1, 2, 3, 4, 5 } with addition and product tables are given below Table 4and Table.5

Table 4: Addition table

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Table 5:product table

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	5	5	3	1	1
2	0	4	4	0	2	2
3	3	3	3	3	3	3
4	0	2	2	0	4	4
5	3	1	1	3	5	5

This near-ring is regular near-ring and also r - regular near-ring.

Theorem2.3 : If a near-ring N is r - regular near-ring then N = aN.

Proof : If a near-ring N is r - regular near-ring then a = ae, e $\in < a|$.

$\Rightarrow$ a $\in$ aN $\forall$ a $\in$ N.

Therefore N = aN.

Theorem2.4 : If a near-ring N is r - regular near-ring then (0 : a) = (0 : aN) = (0 : N) , for all a $\in$ N.

Proof : Since N is r - regular near-ring, a $\in$ aN.

Now xaN = 0 $\Rightarrow$ xa = 0 $\Rightarrow$ (0 : aN) $\subseteq$ (0 : a).

Let x $\in$ (0 : a) then xa = 0 $\Rightarrow$ xaN = 0N = 0 $\Rightarrow$ x $\in$ (0 : aN) $\Rightarrow$ (0 : a) $\subseteq$ (0 : aN). Therefore (0 : a) = (0 : aN) .

By the theorem 2.3, (0 : a) = (0 : aN) = (0 : N) .

Theorem2.5 : Let a near-ring N is r - regular near-ring.Then every principal ideal is generated by an idempotent.

Proof : Let a $\in$ N.Consider a principal ideal generated by a, $\langle a \rangle$.If N is r - regular near-ring, then $a = ae, e \in \langle a| \subseteq \langle a \rangle \Rightarrow \langle e \rangle \subseteq \langle a \rangle$

$a = ae \in \langle e \rangle \Rightarrow \langle a \rangle \subseteq \langle e \rangle$.Therefore $\langle a \rangle = \langle e \rangle$

Theorem2.6 : Let a near-ring N be r - regular near-ring.Then N has no nonzero nil ideals.

Proof : Suppose A be a nonzero nil ideal in N.

Let 0 $\neq$ a $\in$ A and $a = ae, e \in \langle a|, e^2 = e.$

By the theorem 2.5, $e \in \langle e \rangle = \langle a \rangle \subseteq$ A.

$\Rightarrow$ e is a nilpotent element, which is a contradiction to the fact that e is an idempotent.Therefore N has no nonzero nil ideals.

Theorem2.7 : Let N a near-ring is r - regular near-ring and every N-subgroup is an ideal of N then N is regular near-ring.

Proof : Suppose that N is r - regular near-ring and every N-subgroup is an ideal of N

Then Na is an ideal, for all a $\in$ N.

By the theorem 1.18, $a = ae, e \in \langle a| \subseteq \langle a \rangle = Na.$

$\Rightarrow$ e = na for some n $\in$ N.

Na is an ideal contains a $\Rightarrow \langle a \rangle \subseteq$ Na.

But Na = $\langle a \rangle$

$\Rightarrow$ a = ae = ana.

Hence N is regular near-ring.

Theorem2.8 : If N is r - regular near-ring with 1 and has IFP thena = ae implies a = ea.

Proof : Suppose N is r - regular near-ring.

Now a $\in$ N there existe ^{2} = e $\in$ N such that a =ae, e $\in < a| \subseteq \langle a \rangle$.

Since (1 - e) e = 0 $\Rightarrow$ (1 - e) ae = 0 $\forall$ a $\in$ N.

$\Rightarrow$ ae -eae = 0 $\Rightarrow$ ae = eae = ea.

Therefore a = ae implies a = ea.

Theorem2.9 : If N is r - regular near-ring with 1 and has IFP then N is reduced.

Proof : Suppose a near-ring N is r - regular near-ring.Let a $\in$ N and a ^{2} = 0 $\Rightarrow$ a $\in$ (0 : a) $\Rightarrow \langle a \rangle \subseteq$ (0 : a) .

By the theorem 2.8, we have, a = ae = ea $\in \langle a \rangle$ a $\subseteq$ (0 : a) a = 0.

Therefore N is reduced.

Definition2.10 : A near-ring N is said to be **symmetric near-ring** if $\langle a \rangle^2$ = $\langle a^2 \rangle$ for each a in N.

Theorem2.11 : If a near-ring N is r - regular near-ring, N $\in \eta_0$ and is symmetric then N is reduced.

Proof : Assume 0 $\neq$ a $\in$ N and a ^{2} = 0.

$a = ae, e \in \langle a| \subseteq \langle a \rangle$

e = e ^{2} = $\langle a \rangle^2$ = $\langle a^2 \rangle$ = 0 $\Rightarrow$ a = 0.It is a contradiction.

Hence N is reduced.

Notation 2.12 : For any subset A of N, write $\sqrt{A}$ = { x $\in$ N / x^k $\in$ A for some positive integer k }.

Definition 2.13 : A near-ring N is said to be **strongly regular near-ring** iffor each a in N there existx $\in$ N such that a = xa ^{2}.

Theorem 2.14 :[9] Every strongly regular near-ring is regular near-ring.

The converse of the above theorem need not be true.For example,

Example 2.15 : (M(Γ), + , ·) is a regular near-ring but not strongly regular near-ring.In particular, Let Z_4 ={0, 1, 2, 3} be the group of integers modulo 4. (M(Z_4), + , ·) is regular near ring.

Let f $\in$ M (Z_4) be given by f = $\begin{cases} 0 \to 0 \\ 1 \to 2 \\ 2 \to 0 \\ 3 \to 2 \end{cases}$

$f^2 = 0$, implies that f is nilpotent. Since M (Z_4) contains non-zero nilpotent elements, M (Z_4) is not strongly regular near-ring.

Theorem2.16 : **Let a near-ring N $\in \eta_0$ be r -regular near-ring with identity and has IFP. Then the following are equivalent.**

(1) N is regular near-ring.

(2) A = $\sqrt{A}$ for every N-subgroup A of N.

(3) N is left bipotent.

(4) N is strongly regular near-ring.

Proof : (1) $\Rightarrow$ (2) : Suppose N is regular near-ring. Let $a \in A \Rightarrow a^1 \in A \Rightarrow a \in \sqrt{A} \Rightarrow A \subseteq \sqrt{A}$ .Now $a \in \sqrt{A} \Rightarrow a^k \in A$ for some positive integer k.

Since N is regular near-ring, then a = aya for some y in N and ya is an idempotent.

By the theorem 2.8 and theorem 1.53, a = ae= ea = eaya = eyaa = eya^2= ey (eya^2) a = $eyey\,a^3$ = $e^2yy\,a^3 = ey^2 a^3$ = ... = $z a^k$ for some z in N.

$a = z a^k \in NA \subseteq A \Rightarrow a \in A$.

Therefore $\sqrt{A} \subseteq A$.Hence A = $\sqrt{A}$ for every N-subgroup A of N.

(2) $\Rightarrow$ (3) : Let $0 \neq a \in N$ and $a \in Na$.

Let $0 \neq a \in N$, Na^2 is an N-subgroup of N.

$a^3 \in Na^2 \Rightarrow a \in \sqrt{Na^2} = Na^2$.

$\Rightarrow Na \subseteq Na^2$ and $Na^2 \subseteq Na$.

Therefore $Na = Na^2$.Hence N is left bipotent.

(3) $\Rightarrow$ (4) : Suppose N is left bipotent.

By the theorem 2.8, a = ae,$e \in \langle a|$,e^2 = e= ea.

$a \in Na = Na^2 \Rightarrow a = ya^2$for some y in N.

Therefore N is strongly regular near-ring.

(4) $\Rightarrow$ (1) : Suppose N is strongly regular near-ring.

By the theorem 2.14, Then N is regular near-ring.

Definition 2.17: A near-ring N is said to be **weakly regular near-ring** if $A^2 = A$ for every ideal A ofN.

Theorem2.18 : **Let a near-ring N be r - regular near-ring.Then N is weakly regular near-ring.**

Proof : Let I be an ideal of N and $a \in I$.

Since a = ae, e^2 = e, $e \in \langle a| \subseteq \langle a \rangle \subseteq I \subseteq I\,I = I^2$.

Therefore $I \subseteq I^2$But $I^2 \subseteq I$, therefore $I = I^2$.

Thus N is weakly regular near-ring.

Theorem2.19 : **Let a near-ring N be r - regular near-ring. Then N has no nonzero nilpotent ideal.**

Proof : Suppose J be a nonzero nilpotent idealin N. Then there exist a positive integer k such that J^k = (0).By the theorem 2.18, every ideal in N is idempotent i.e., $J = J^2$.

$J^k = J^{k-2}\ J = J^{k-4}\ J^2\ J = J^{k-4}\ J\ J = J^{k-4}\ J^2 = J^{k-4}\ J$ = …Continuing in this way, we get J = (0) .It is a contradiction.Thus N has no nonzero nilpotent ideal.

III. IDEAL THEORY IN r-REGULAR NEAR-RINGS

Definition3.1 : An ideal P of near-ring N is called **completely prime ideal** if $ab \in P$ implies $a \in P$ or $b \in P$.

Theorem3.2 : **Let a near-ring N be r - regular near-ring with 1 then every completely prime ideal P is maximal.**

Proof : Let P be completely prime ideal of N.

Suppose $P \subseteq M \subseteq N$, then exist$a \in M \setminus P$.

Now a = ae, $e \in \langle a| \subseteq M \Rightarrow e \in M$.

$\Rightarrow ex \in M$.

$(1 - e)\, e = 0 \in P$ and $e \notin P \Rightarrow 1 - e \in P \subset M$

$\Rightarrow 1 - e \in M$.

Now $x \in N \Rightarrow x = 1.x = (1 - e + e)\, x = (1 - e)\, x + ex \in M$.

Therefore N = M.

Hence P is maximal ideal of N.

Theorem3.3 : Leta near-ring N be r - regular near-ring with unity and let I be a maximal ideal in N then I is prime ideal.

Proof : Let A, B are ideals of N and let $AB \subseteq I$.

Assume $B \not\subseteq I$. Since I is maximal, we have B + I = N $\Rightarrow$ AN = A (B + I) = [A (B + I) – AB] +AB $\subseteq I + I \subseteq I \Rightarrow AN \subseteq I$.

Since a = ae, $e \in |a \rangle$, e^2 = e, by the theorem 2.3, we have, $a \in aN\ \forall\, a \in N$.

Let a be arbitrary element in A.

$a \in aN \subseteq AN \subseteq I$.

Therefore $A \subseteq I$.

Hence I is prime ideal.

Definition 3.4 : An ideal P in a near-ring N is said to be 1- **prime ideal** if $P \neq N$ and for any left ideals I, J in N, $IJ \subseteq P \Rightarrow I \subseteq P$ or $J \subseteq P$.

Theorem3.5 : Let a near-ring N be r - regular near-ring with unity. let I be a maximal ideal in N then I is 1- prime ideal.

Proof : Let A, B are the left ideals of N and let $AB \subseteq I$. Assume $B \not\subseteq I$.Since I is maximal, we have B + I = N.

$\Rightarrow$ AN = A (B + I) = [A (B + I) – AB] + AB $\subseteq I + I \subseteq I \Rightarrow AN \subseteq I$.

Since a = ae, e $\in |a>$, e^2 = e, by the theorem 2.3, we have, $a \in aN$ $\forall$ $a \in$ N.
Let a be arbitrary element in A.
$a \in aN \subseteq AN \subseteq I$.
Therefore A $\subseteq$ I.
Hence I is 1- prime ideal.

Theorem3.6 : For a r - regular near-ring N with 1, completely prime radical ($\wp_c(N)$) of N is quasi radical (Q (N)) of N.

Proof : $\wp_c(N)$ = $\bigcap$ { I / I is completely prime ideals of N }, by the theorem 3.2.
= $\bigcap$ { I / I is maximal ideals of N }.
= $\bigcap$ { I / I is maximal right ideals of N }.
= quasi radical (Q (N)) of N.

Theorem3.7 : If a near-ring N is r - regular near-ring, then every ideal ofN is semiprime.

Proof : Let A be an ideal and I $\trianglelefteq$ N such that $I^2 \subseteq$ A. Let $a \in$ I then there existe^2 = $e \in |a>$ such that a = ae.
a = ae $\in$ I I $\subseteq$ A. Therefore I $\subseteq$ A.
Hence every ideal is semiprime.

Definition 3.8 : A left ideal S in a near-ring N is **1-semiprime ideal** if for all left ideals I of N such that $I^2 \subseteq S \Rightarrow I \subseteq S$.

Theorem3.9: If a near-ring N is r - regular near-ring, then every left ideal of N is 1-semiprime.

Proof : Let A be left ideal in N, such that $I^2 \subseteq$ A.
Let $a \in$ I then there existe^2 = e $\in |a>$ such that a = ae.
a = ae$\in$ I I $\subseteq$ A. Therefore I $\subseteq$ A.
Hence every ideal is 1-semiprime.

Note3.10 : I is a semiprime ideal of a near-ring N then I is the intersection of all prime ideals containing I.
I = $\bigcap_P P$, P is prime.
Take I = (0) then $\bigcap_P P$ = (0) $\Rightarrow$ Prime radical of N is zero.

Theorem3.11 : If near-ring N is r - regular near-ring, then $\eta(N)$ ={ 0 }.

Proof : Let I be an ideal in a near-ring N.
By the theorem 3.7, we have, I is semiprime.
Let I be nonzero nilpotentideal of N.
Suppose I^2 = (0).
Since I is semiprime, I = (0).It is a contradiction.
Therefore N has no nonzero nilpotent ideal.
Hence $\eta(N)$ ={ 0 }.

Theorem3.12 : If a near-ring N is r - regular near-ring, the sum of all nilpotent ideals are zero.

Proof : Let I be the left ideal in a near-ring N.
By the theorem 3.9, we have, I is 1-semiprime.
Let I be nonzero nilpotent left ideal of N.
Suppose I^2 = (0).
Since I is semiprime, I = (0).It is a contradiction.
Therefore N has no nonzero nilpotent left ideal.
Hence, the sum of all nilpotent ideals are zero.

Theorem3.13 : If a near-ring N is r - regular near-ring with 1 and has IFP then N is simple if and only if N is integral.

Proof : Suppose N is simple.
Let a, b$\in$ N and ab = 0 and if a$\neq$ 0 $\Rightarrow a \in$ (0 : b) .
Since N has IFP, (0 : b) is two sided ideal.
By supposition, N is simple, we have (0 : b) = N.
b$\in$ N = (0 : b) $\Rightarrow b^2 = 0 \Rightarrow b = 0$.
Therefore N is integral.
Conversely suppose that N is integral.
Let 0 $\neq$ I $\trianglelefteq$ N, a$\neq$ 0, a$\in$ I.
a = ae, e$\in |a> \subseteq$ I $\Rightarrow$a = ea.
$\Rightarrow$ (1 - e) a = 0$\Rightarrow$1 - e = 0 $\Rightarrow 1 = e \in$ I.
Therefore N = I.
Hence N is simple.

Theorem3.14 : If a near-ring N is r - regular near-ring then every ideal I of N is r - regular near-ring.

Proof : Suppose N is r - regular near-ring.
Let I be an ideal in a near-ring N.
Let a$\in$ I then a = ae, e$\in <a| \subseteq$ I.
Therefore I is r - regular near-ring.

Note3.15 : A subring of a r - regular near-ring need not be a r - regular near-ring.

Example3.16 : Q is regular implies Q is r - regular near-ring.
But Z is subring of Q which is not r - regular.
2 = 2.1, 1 $\notin <2| \subset <2>$ = 2Z

Theorem3.17 : The following are equivalent for a near-ring N $\in \eta_0$ with identity.

(1) N is r - regular and has IFP
(2) N is reduced and every completely prime ideal is maximal.

Proof : (1) $\Rightarrow$ (2) Suppose N is r - regular near-ring.
By the theorem 2.9, N is reduced and by the theorem 3.2, every completely prime ideal is maximal.
(2)$\Rightarrow$(1) Suppose N $\in \eta_0$ is reduced and every completely prime ideal is maximal.
Since N $\in \eta_0$ is reduced, then ab = 0 $\Rightarrow$ba = 0.

Consider nba = n (ba) = n0 = 0.
$\Rightarrow$ (nb) a = 0 $\Rightarrow$ anb = 0 $\forall$ n $\in$ N.
Therefore N has IFP.
Let $0 \neq a \in N$, by the theorem 1.68, $\overline{N} = N / A(a)$ is reduced and $\bar{a}$ is not a zero divisor.
Also every completely prime ideal of $\overline{N}$ is a maximal ideal in $\overline{N}$.
Let M be the multiplicative subsemigroup generated by an element $\bar{a}$ - $\bar{e}\,\bar{a}$,where $\bar{e} \in <a|$.
If not, by the theorem 1.69, there exist a completely prime ideal $\overline{P}$ with $\overline{P} \cap M = \phi$.
Suppose $<\bar{a}> \subseteq \overline{P}$ then $\bar{a} \in \overline{P}$.
$\Rightarrow \bar{a} - \bar{e}\,\bar{a} \in \overline{P}$.
$\Rightarrow \bar{a} - \bar{e}\,\bar{a} \in \overline{P} \cap M$, it is a contradiction to the fact that $\overline{P} \cap M = \phi$.
Suppose $<\bar{a}> \not\subseteq \overline{P}$ and $\overline{P}$ is maximal, we have $\overline{N} = \overline{P} + <a|$.
$\bar{1} = \bar{\alpha} + \bar{e}$ where $\bar{\alpha} \in \overline{P}$, $\bar{e} \in <a|$.
$\bar{a} = \bar{\alpha}\,\bar{a} + \bar{e}\,\bar{a}$.
$\Rightarrow \bar{a} - \bar{e}\,\bar{a} = \bar{\alpha}\,\bar{a} \in \overline{P}$.
$\Rightarrow \bar{a} - \bar{e}\,\bar{a} \in \overline{P} \Rightarrow \bar{a} - \bar{e}\,\bar{a} \in \overline{P} \cap M$.It is a contradiction.
Thus $\bar{0} \in M$.
Now $\bar{0} = (\bar{a} - \bar{e}_1\bar{a}) (\bar{a} - \bar{e}_2\,\bar{a}) \ldots (\bar{a} - \bar{e}_n\,\bar{a})$
$= (\bar{1} - \bar{e}_i) \bar{a}$ $\bar{e}_i \in <a|$.
Since $\bar{a}$ is not zero divisor, $(\bar{1} - \bar{e}_i) = 0 \Rightarrow \bar{1} = \bar{e}_i$, $e \in <a|$, hence (1 - e) $\in$ A(a)
$\Rightarrow$ (1 - e) a = 0, $e \in <a|$, $e^2 = e \Rightarrow$ a = ea, $e \in <a|$.
(a - ae) a = $a^2 - a^2 = 0$ and (a - ae) ae = $a^2e - a^2e$ = 0.
Since N $\in \eta_0$ is reduced, if ab = 0 then ba = 0.
We have a (a - ae) = 0 and ae (a - ae) = 0.
Now $(a - ae)^2$ = a (a - ae) - ae (a - ae) =0 $\Rightarrow$ a - ae =0 $\Rightarrow$ a = ae, $e \in <a|$.
Therefore N is r - regular near-ring.

Theorem3.18 : Homomorphic image of r -regular near-ring is r - regular.
Proof : Letf : N $\rightarrow$ N^1 be an epimorphism of the r - regular near-ring N onto the near-ring N^1.
We now prove that N^1 is r - regular.
Since N is r - regular, a = ae, $e \in <a| \subseteq <a>$.
f (a) = f (ae) = f (a) f (e) .
f (e) = f (e e) = f (e) f (e) .
f (e) $\in$ f<a > =< f<a >>.
Therefore homomorphic image of r - regular near-ring is r - regular.

Theorem3.19 : If a near-ring N is r - regular near-ring with 1 and has IFP then every ideal I of N is the intersection of all maximal ideals containing I.
Proof : Suppose N is r - regular near-ring.
By the theorem 3.7, every ideal I in N is semiprime.
Then I is the intersection of all prime ideals containing I.
By the theorem 3.2, I is the intersection of all maximal ideals containing I.

Theorem3.20 : If N is a r - regular near-ring with 1 and has IFP then N is subdirect product of simple reduced near-rings.
Proof : Since N is r - regular near-ring with 1 and has IFP then N is reduced.
By the theorem 1.64, every near-ring is isomorphic to subdirect product of subdirectly irreducible near-rings.
By theorem 3.19, { 0 } is the intersection of all maximal ideals.
Hence, N is isomorphic to subdirect product of simple reduced near-ring.

Theorem3.21 : **If a near-ring N is r - regular near-ring then N is right weakly regular near-ring.**
Proof : Suppose N is r - regular near-ring.
Then a = ae, $e \in <a| \Rightarrow a \in a<a| \subseteq a<a>$.
$\Rightarrow$ N is right weakly regular near-ring.
But the converse part need not be true.For example,
Example3.22 : Consider the zero symmetric near-ring N on Z_4 = { 0, 1, 2, 3 }with addition and product tables are given below as table6 and table7:

Table 6:Addition table

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

table7 : product table

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	3	0	3
3	0	2	0	2

N is a simple near-ring with I ={ 0, 2 } the only proper left ideal.
1 = 1.3 ∈ 1< 1> = 1N.
2 = 3.1 = 2.3.1 ∈ 2N = 2< 2>.
3 = 2.3 = 3.3.3 ∈ 3< 3>.
Therefore (N, +, .) is right weakly regular but not r - regular.
Since 1 is the only idempotent i.e., there exist no $e^2 = e \in N$ such that ae = afor a = 2 or 3.

IV. CONCLUSIONS

In Mathematics, study on near-rings becomes an object of the exercise for several researchers. In this paper, we made an attempt to study the concepts regularity of near-rings, generalized regular near-rings and their characterizations.

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