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Using a One-Dimensional Map to Determine the Buckling Load of Columns

Jiin-Po Yeh
Department of Civil and Ecological Engineering, I-Shou University, Kaohsiung, Taiwan

ABSTRACT: This paper is aimed at finding the buckling load of columns by using a different approach than the traditional methods. One-dimensional maps are often seen in dynamic systems to describe how a regular motion changes into an irregular motion through bifurcations.The totalpotential energy is one of the traditional methods to find the buckling load of a structure. At first sight, it seems that these two are irrelevant. However, due to the fact that they are bothrelated to the bifurcation theory, this paper first transform the equilibrium condition of the total potential energy of a structure into the form of a one-dimensional map.By locating the first bifurcation point of the one-dimensional map, the buckling load of the structure can then be obtained. There are three examples given in this paper to demonstrate this innovative approach. The results show that the one-dimensional map theory works well and can be used as an alternative for finding the buckling load.

Keywords: One-DimensionalMap, Total Potential Energy, Buckling Load, Bifurcation Theory

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I. INTRODUCTION

When a slender member is subjected to an axial compressive load, it may result in the loss of its stability to resist loading even though the stresses developing in the structure are well below the strength of materials. There are many ways structures can become unstable. If the elastic buckling is the cause of failure, to obtain the buckling load becomes necessary and important because the maximum load the structure can support must be determined beforehand. There are different kinds of methods to findthe buckling load, such as equilibrium methods, dynamic methods and energy methods, as discussed in Simitses[1] and Chen and Liu[2]. In a mathematical sense, buckling is a bifurcation from stable to unstable equilibrium in the solution to the equations of static equilibrium. A bifurcation occurs when a small smooth changemade to the parameter values of a system causes a sudden qualitative or topological change in its behavior [3-5], usually from simple to highly complicated behavior. Bifurcation theory is a useful and widely studied subfield of dynamical systems.

one-dimensional mapsarethe most generally accepted paradigm in chaotic dynamics to show how the system is affected by the parameter in the map and how it becomes chaotic when the parameteris changed, as illustrated in Alligood [4] and Moon[5]. The system behavior doesn't change as long as the fixed point of the map remains stable, while the system behavior changes qualitatively as soon as the fixed point becomes unstable. The boundary separating the stable and unstable fixed points corresponds to a critical value of the parameter. A point in the parameter space where one can see a change in the qualitative behavior of a system is defined as a bifurcation point,e.g., loss of stability of a solution or the emergence of a new solution with different properties. When a bifurcation occurs, more and different fixed points emerge, that is, the period of the system grows longer. The bifurcation can keep happening until the system becomes chaotic with amazingly irregular behaviors[3-5]. As a consequence, whether the system behavior changes qualitatively is much related to the stability of the fixed points. Based on the above stability concept of a one-dimensional map, this paper develops another approach to find the buckling load, which is different than the traditional methods and proves to be feasible and effective.

II. FIXED POINTS AND STABILITY OF A ONE-DIMENSTIONAL MAP

A fixed pointfor a functiong(x)is a number pfor which the value of the function does not change, i.e., g(p)=p.By simple algebraic manipulation, an equation h(x)=0 can equivalently be transformed into the form

x = g(x)(1)

According to Burden and Fairs [6], if Eq. (1) is regarded as a one-dimensional map

$$x_{n+1} = g(x_n),\ \ n = 0,\ 1,\ 2,\ 3,... \quad (2)$$

thenthe fixed point of this one-dimensional map corresponds to the solution to the equation h(x)=0. Based on this concept, this paper first transforms the equilibrium equation of the total potential energy into a one-dimensional map and then finds the fixed point

$x_f$of the map, which is exactlythe equilibrium position of the total potential energy.

To determine the stability of a map, one looks at the value of the slope evaluated at the fixed point. Suppose that x_f is the fixed point of a one-dimensional map and $x_o$is the initial value close to x_f. If

$$\left|\frac{\partial g(x_f)}{\partial x}\right| < 1 \quad (3)$$

the sequence of the one-dimensional map will converge to x_f, i.e., thestable fixed point; if

$$\left|\frac{\partial g(x_f)}{\partial x}\right| > 1 \quad (4)$$

it will diverge from x_f,i.e., theunstable fixed point. When

$$\left|\frac{\partial g(x_f)}{\partial x}\right| = 1 \quad (5)$$

the parameter of the map is at a critical value where bifurcation occurs, which is illustrated in Alligood [4] and Moon[5].

III. BIFURCATION OF A ONE-DIMENSIONAL MAP

Perhaps the simplest example of a dynamic mode that exhibits the bifurcation phenomena is thelogistic map or population growth model[7], which is given by

$$x_{n+1}=\mu x_n(1-x_n) \quad (6)$$

where$x_n$is limited to the interval [0, 1] and the parameter μto the interval [0, 4]. The equilibrium points (or fixed points) can be found by solving the equation

$$x=\mu x(1-x)=g(x) \quad (7)$$

whichhas roots atx=0 and $x=\frac{\mu-1}{\mu}$. Whenμ<1, the map has only one fixed and stable point at x=0, while for 1<μ<3, the map has two fixed points x=0 and $x=\frac{\mu-1}{\mu}$, where the former is unstable and the latter is stable. Hence the first bifurcation point occurs at μ=1. Forμ>3, a double-period orbit appears, i.e., the second bifurcation point happens at μ=3, as shown in Fig. 1.

From Eq.(7), the first derivative of g(x) with respect to x gives

$$\frac{\partial g}{\partial x}=\mu(1-2x) \quad (8)$$

Substituting the smaller equilibrium point x=0 into Eq. (8) leads to(9)

$$\frac{\partial g}{\partial x}=\mu(1-0)=\mu \quad (9)$$

According to Eq. (5), if $\frac{\partial g}{\partial x}=1$, bifurcation will happen, i.e., μ=1, which corresponds to the first bifurcation point in Fig. 1

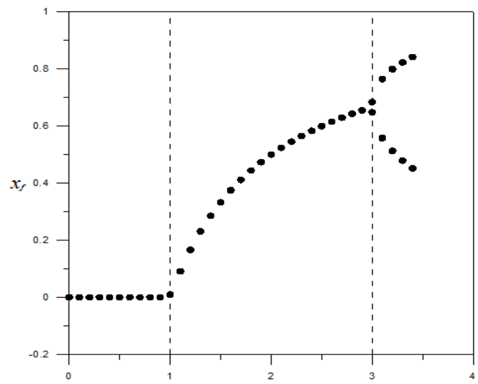

Fig. 1 The first and second bifurcation points of the logistic map at μ=1 and μ=3.

IV. STABILITY OF TOTAL POTENTIAL ENERGY

For a conservative one-degree-of–freedom system, the total potential energyV is usually expressed in term of the generalized coordinateθ, measured from a fixed datum,and external load P, i.e., V(θ,P). If the system is in equilibrium, the first derivative of Vwith respect to θmust be equal to zero, i.e.,

$$\frac{\partial V}{\partial \theta}=0 \quad (10)$$

Based on the second derivative of V evaluated at the equilibrium position, the total potential energy V(θ, P) can further be classified into three types[8]: (1) Stable equilibrium, if

$$\frac{\partial^2 V}{\partial \theta^2}>0 \quad (11)$$

i.e., the total potential energy is at a local minimum; (2) Unstable equilibrium, if

$$\frac{\partial^2 V}{\partial \theta^2}<0 \quad (12)$$

i.e., the total potential energy is at a local maximum; (3) Neutral equilibrium, if

$$\frac{\partial^2 V}{\partial \theta^2}=0 \quad (13)$$

When the system is in neural equilibrium, it is at the boundary between stability and instability, where the equilibrium branches and a bifurcation occurs. According to Simitses [1] and Gere and Goodno [9], when the external load reaches the bifurcation point, the structure is on the verge of buckling and the corresponding load is called the critical load, denoted by the symbolP_{cr}. Hence the buckling load can be determined fromEqs. (10) and (13).

Because both the bifurcation point of the one-dimensional map and the total potential energy correspond to a critical point, if the equilibrium condition of the total potential energy could be changed into a one-dimensional map, the buckling load P_{cr} might be obtained by considering this one-dimensional map instead of the equation. Due to this conjecture, this paper develops the following method to find the buckling load of a structure.

Suppose that V(θ,P) is the total potential energy of a structure. The stationary value of the total potential energy can be found by making

$$\frac{\partial V}{\partial \theta} = h(\theta, P) = 0 \quad (14)$$

which yields the equilibrium equation. After that, Eq. (14) can be transformed equivalently into

$$\theta = g(\theta, P) \quad (15)$$

The one-dimensional map

$$\theta_{n+1} = g(\theta_n, P) \quad (16)$$

is then constructed, whose fixed points correspond exactly to the static equilibrium positions of the total potential energy.According to Eq.(5), one of the critical states of the map at fixed point is

$$\frac{\partial g(\theta_f)}{\partial \theta} = 1 \quad (17)$$

whereθ_f is the fixed point. It is apparent that the condition of Eq. (17) is equivalent to that of Eq. (13). As a result, the critical state $\frac{\partial g(\theta_f)}{\partial \theta} = 1$ of the one-dimensional mapcan be used to locate the bifurcation point of a structure and find the buckling load. It should be noted here that the other critical state

$$\frac{\partial g(\theta_f)}{\partial \theta} = -1 \quad (18)$$

cannot be applied to find the buckling load, because it's not equivalent to the condition of Eq. (13) obviously.

V. ILLUSTRATIVE EXAMPLES

Buckling is one of the major causes of failures in many structures. Hence, considering the possibility of buckling occurring in the structures is very important in design. Following are three examples that could fail due to buckling, rather than compression exceeding strength of materials. The rotational displacement θ is assumed to be small in each example. For the sake of comparison, the three examples are taken form Gere and Goodno [9] and Hibbeler [10].

5.1 Single Rigid Bar Supported byaTranslational Spring

This idealized column consists of a rigid bar of length 2L, compressed by an axial load P, as shown in Fig.2. It is initially held in a vertical position by a translational spring having stiffness k. Since the system is conservative, the potential energy is then given by

$$V = -p(2L - 2L\cos\theta) + (\frac{1}{2}k)(\frac{3}{2}L\theta)^2 \quad (19)$$

The equilibrium condition for the total potential energy is

$$\begin{aligned}\frac{\partial V}{\partial \theta} &= -2PL\sin\theta + \frac{9kL^2\theta}{4} \\ &\approx -2PL(\theta - \frac{\theta^3}{6}) + \frac{9kL^2\theta}{4} = 0\end{aligned} \quad (20)$$

which can be rewritten as

$$\theta = \frac{\theta^3}{6} + \frac{9kL\theta}{8P} \quad (21)$$

The one-dimensional map takes the form

$$\theta_{n+1} = \frac{\theta_n^3}{6} + \frac{9kL\theta_n}{8P} \quad (22)$$

There are two fixed points of the map: θ=0 and $\theta = \sqrt{6(1 - \frac{9kL}{8P})}$. Let

$$g(\theta, P) = \frac{\theta^3}{6} + \frac{9kL\theta}{8P} \quad (23)$$

The derivative of the function g(θ, P) with respect to θ becomes

$$\frac{\partial g}{\partial \theta} = \frac{\theta^2}{2} + \frac{9kL}{8P} \quad (24)$$

Substituteθ=0 (the smallest value of the two fixed positions)into Eq. (24) and let $\frac{\partial g}{\partial \theta} = 1$ which leads to

$$\frac{9kL}{8P} = 1 \quad (25)$$

Therefore, the buckling load for the column is $P_{cr} = \frac{9}{8}kL$, whichcoincides with Gere and Goodno[9]. When the axial load P>9kL/8,the single rigid bar will buckle.

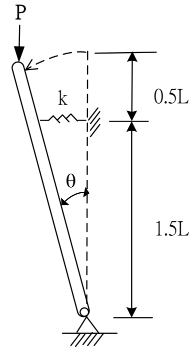

Fig. 2 Single rigid bar supported by a translational spring

5.2 Two -Bar SystemSupportedbyaTranslational Spring

The second example is a column composed of two rigid bars AB and BC, each of length L/2, which are connected with a translational spring of stiffness k, as shown in Fig. 3. Initially the two bars are perfectly aligned and held in a vertical position. The compressive force P is acting along the longitudinal axis, i.e., the dashed line in Fig.3. The total potential energy for the conservative system is expressed as

$$V(\theta, P) = -2P(\frac{L}{2} - \frac{L}{2}\cos\theta) + \frac{k}{2}(\frac{L}{2}\theta)^2 \quad (26)$$

The equilibrium condition for the this mechanism is

$$\frac{\partial V}{\partial \theta} = -2P(\frac{L}{2}\sin\theta) + \frac{kL^2\theta}{4}$$

$$\approx -PL(\theta - \frac{\theta^3}{6}) + \frac{kL^2\theta}{4} = 0 \quad (27)$$

which can be written as

$$\theta = \frac{\theta^3}{6} + \frac{kL\theta}{4P} \quad (28)$$

The one-dimensional map can be formed as

$$\theta_{n+1} = \frac{\theta_n^3}{6} + \frac{kL\theta_n}{4P} \quad (29)$$

There are two fixed points of this map:θ=0 and $\theta = \sqrt{6(1 - \frac{kL}{4P})}$. Let

$$g(\theta, P) = \frac{\theta^3}{6} + \frac{kL\theta}{4P} \quad (30)$$

Then

$$\frac{\partial g}{\partial \theta} = \frac{\theta^2}{2} + \frac{kL}{4P} \quad (31)$$

Similarly, substitute the smaller fixed point θ=0 into Eq. (31) and let, $\frac{\partial g}{\partial \theta}$ which leads to

$$\frac{kL}{4P} = 1 \quad (32)$$

The critical load for this two-bar systemis $p_{cr} = \frac{kL}{4}$, whichcoincides with the result found by Hibbeler [10]. When the compressive force P>kL/4 , the two-bar system will buckle.

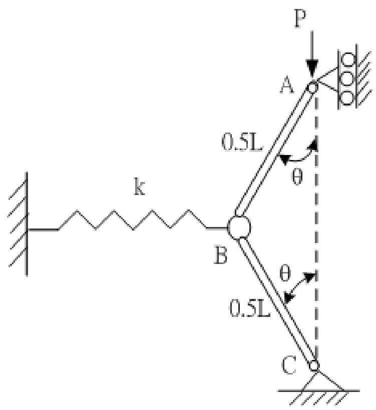

Fig. 3 Two -bar system supported by a translational spring.

5.3 Two-Bar SystemSupported bya Rotational Spring

The third example is a system of two rigid bars AB and BC, each of length L/2, connected with a rotational spring of stiffness $k_{R,}$ as shown is Fig.4.Initially the two bars are perfectly aligned and held in a vertical position. The compressive force P is acting along the longitudinal axis, i.e., the dashed line in Fig. 4. The total potential energy is given by

$$V = -2P(\frac{L}{2} - \frac{L}{2}\cos\theta) + \frac{1}{2}k_R(2\theta)^2 \quad (33)$$

If the system is in in equilibrium, then

$$\frac{\partial V}{\partial \theta} = -PL\sin\theta + 4k_R\theta$$

$$\approx -PL(\theta - \frac{\theta^3}{6}) + 4k_R\theta = 0 \quad (34)$$

which can be written as

$$\theta = \frac{\theta^3}{6} + \frac{4k_R\theta}{PL} \quad (35)$$

The one-dimensional map can be formed as

$$\theta_{n+1} = \frac{\theta_n^{\ 3}}{6} + \frac{4k_R\theta_n}{PL} \quad (36)$$

There are two fixed points of this map: θ=0 and $\theta = \sqrt{6(1-\frac{4k_R}{PL})}$. Let

$$g(\theta, p) = \frac{\theta^3}{6} + \frac{4k_R\theta}{PL} \quad (37)$$

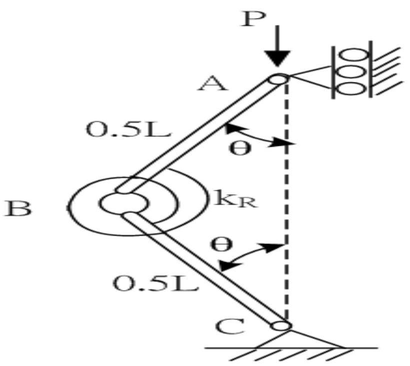

Fig. 4 Two-bar system supported by a rotational spring

Then

$$\frac{\partial g}{\partial \theta} = \frac{\theta^2}{2} + \frac{4k_R}{PL} \quad (38)$$

Likewise, substitute the smaller fixed point θ=0 into Eq. (38) and let $\frac{\partial g}{\partial \theta} = 1$, which leads to

$$\frac{4k_R}{PL} = 1 \quad (39)$$

Hence, the buckling load for this two-bar system is, $P_{cr} = \frac{4k_R}{L}$ which agrees with the result found by Gere and Goodno[9]. When the compressive force P>4k_R/L, the two-bar system will buckle.

VI. CONCLUSION

Due to the existence of equilibrium and stabilityin both the one-dimensional map and total potential energy, this paper first transform the equilibrium condition of the total potential energy of a column into the one-dimensional map and then locates the first bifurcation point of the one-dimensional map, from which the buckling load of the columncan be determined. Three kinds of columnsare given to prove that the approach presented in this paper works well. The buckling load of each column found by the one-dimensional mapagrees withthatobtained by the traditional method; therefore, the approach introduced in this paper can be considered an alternative to determining the critical load of a structure.

(37)

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