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Comparing Optimal Homotopy Asymptotic Method and Adomian Decomposition Method for the Solution of Integro-Differential Equations

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ABSTRACT
In this work, the author compares optimal homotopy asymptotic method with Adomian decomposition method for the solution of integro-differential equations. These methods are applied to first, second and third order problems. Both of these methods develop series solutions to a wide variety of functional equations. Optimal homotopy asymptotic method traces the solution very rapidly as it uses a homotopy and control the convergence by the virtue of auxiliary function. Numerical results are compared with the exact solution.
Keywords - Integro-differential equations, optimal homotopy asymptotic method, Adomian decomposition method, auxiliary function, linear problems

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1. INTRODUCTION

Many physical systems have been modeled to integro-differential equations (IDE's). The main area of applications includes fluid dynamics, biological models, chemical kinetics, population dynamics, nuclear reactors, wave propagation, image processing and engineering systems. Exact solutions for such equations solutions can be obtained for a limited class of equations. To get approximate solutions, many numerical/semi-numerical techniques have been developed and their applications have been extended to IDE's. Many authors [1-4] used numerical methods for the solution of variety of IDE's.

Optimal homotopy asymptotic method (OHAM) is the newest of all homotopy based methods. It uses a flexible auxiliary function which control convergence of the developing series. This method gives more accurate results than homotopy analysis method (HAM) and homotopy perturbation method (HPM).

Javed Ali et al. applied this new technique to multi-point, two-point higher order boundary value problems [5-9].

Adomian decomposition method [10] is a simple and low cost method that can be applied to functional equations. This method develops a series solution that is based on Taylor's series. Modified form of this method has proven to develop solution of the problems more effectively and rapidly [11, 12].

In this work, OHAM and ADM are compared. Three examples are taken from literature. The article is organized as follows. In section 2, basic idea of OHAM is extended to the problems under discussion and in section 3, the ADM is developed for the purpose. In section 4, numerical examples are given. Finally, in section 5, conclusion is given. Mathematica 7 is used for symbolic calculations.

II. APPLICATION OF OHAM TO NTH-ORDER IDE

Consider the nth order integro-differential equation of the form

$$v^{(n)}(x) = \phi(x) + \int_0^x \mathrm{K}(x,\tau)\Phi(\upsilon(\tau))d\tau, \qquad (1)$$

with boundary conditions $v^{(\iota)}(0) = \alpha_\iota$.

Here $v^{(n)}(x)$ is the nth derivative of the unknown function $v(x)$, and $\Phi(v(x))$ is a nonlinear function. The kernel $\mathrm{K}(x,\tau)$ and the function $\phi(x)$ are real and derivable functions on $[0,b]$ and α_i and β_i are real constants.

Constructing a family of curves that is defined by the homotopy equation

$$(1-p)L(\psi(x,p)-\phi(x)) = \mathrm{H}(p,C_i)(L(\psi(x,p))-\phi(x)- \int_0^x \mathrm{K}(x,\tau)\Phi(\psi(\tau,p))d\tau), \qquad (2)$$

along with the boundary conditions

$$\mathrm{B}(\psi,\frac{\partial\psi}{\partial\xi}) = 0, \qquad (3)$$

where L is a linear, Φ is a nonlinear and B is a boundary operator. $\mathrm{H}(p,C_i)$ is an auxiliary function with $p\in[0,1]$ as an embedding parameter and C_i's are convergence control parameters.

For $p=0$, $\mathrm{H}(p,C_i)=0$, and for $p\neq 0$, $\mathrm{H}(p,C_i)\neq 0$.

Eq.(2) satisfies

$$L(\psi(x,0))=\phi(x), \quad \text{for } p=0, \text{and} \tag{4}$$

$$L(\psi(x,1))=\phi(x)+\int_0^x \mathrm{K}(x,\tau)\Phi(\phi(\tau,1))d\tau, \tag{5}$$

for $p=1$.

Solution of Eq.(4) is denoted by $v_0(x)$ and is called initial solution which satisfies the boundary conditions. It serves as an initial or starting value for the scheme to be developed. As p moves from 0 to 1, the initial solution $v_0(x)$ traces the solution curve $v(x)=\psi(x,1)$, of the given problem (1).

For $p=0$, we have the linear ODE, $L(v(x))=\phi(x)$, and for $p=1$, we get the complete problem $L(v(x))=\phi(x)+\int_0^x \mathrm{K}(x,\tau)\Phi(v(\tau))d\tau.$

The auxiliary function $\mathrm{H}(p,C_i)$ is not a fixed function, one can choose the best among many which suits the given problem. The most famous and simple form of this auxiliary function is

$$\mathrm{H}(p,C_i)=pC_1+p^2C_2+\ldots \tag{6}$$

Next, the unknown function $\psi(x,p)$ is expanded in the usual way as

$$\psi(x,p,C_i)=v_0(x)+\sum_{k=1}^{\infty} v_k(x,C_1,\ldots,C_k)p^k. \tag{7}$$

Plugging in Eq.(6) and Eq.(7) into Eqs.(2) and (3), and then equating the like powers of p, the following linear problems are obtained:

Zeroth-Order Problem:

$$v_0^{(n)}(x)=\phi(x), \quad v_0^{(i)}(0)=\alpha_i.$$

First-Order Problem:

$$v_1^{(n)}(x)=(C_1-1)\phi(x)+v_0^{(4)}(x)$$
$$+C_1v_0^{(4)}(x)+C_1\int_0^x \mathrm{K}(x,\tau)\Phi_0(v_0(\tau))d\tau,\ v_1^{(i)}(0)=0.$$

Second-Order Problem:

$$v_2^{(n)}(x)=(1+C_1)v_1^{(4)}(x)+C_2v_0^{(4)}(x)+C_2\phi(x)$$
$$+C_1\int_0^x \mathrm{K}(x,\tau)\Phi_1(v_0(\tau),v_1(\tau))d\tau$$
$$+C_2\int_0^x \mathrm{K}(x,\tau)\Phi_0(v_0(\tau))d\tau,\ v_2^{(i)}(0)=0.$$

Where the nonlinear term $\Phi(\upsilon(x))$ has been decomposed in the following manner:

$$\Phi(v(x))=\Phi_0(v_0(x))+p\Phi_1(v_0(x),v_1(x))$$
$$+p^2\Phi_1(v_0(x),v_1(x),v_2(x))+\ldots$$

Higher order problems can be constructed easily in the similar way.

For $p=1$, if the series (7) converges then

$$\psi(x,1,C_i)=v(x,C_i)=v_0(x)+\sum_{k=1}^{\infty} v_k(x,C_1,\ldots,C_k).$$

Practically, the truncated series is taken and the approximate solution is sought. The *Mth* order approximation is:

$$\tilde{v}(x,C_i)=v_0(x)+\sum_{k=1}^{M} v_\kappa(x,C_1,\ldots,C_k),$$

where C_i's are to be determined.

In most of the cases approximate solution upto the second order i.e,

$$\tilde{v}(x,C_1,C_2)=v_0(x)+v_1(x,C_1)+v_2(x,C_1,C_2), \tag{8}$$

is enough to produce excellent results. For the rest of OHAM procedure, second order solution is considered. For computational efficiency, solution (8) is expressed in power series with order of approximation $O(x^{14})$.

To determine C_i's, we plug in Eq.(8) into Eq.(1), and obtain the following residual:

$$R(x,C_i)=\tilde{v}^{(n)}(x,C_i)-\phi(x)-\int_0^x \mathrm{K}(x,\tau)\Phi(\tilde{v}(\tau,C_i))d\tau$$

Generally, $R\neq 0$, but it can be minimized over the domain of the problem. The authors reported many methods to achieve this goal. For this purpose, the method of least squares or the Galerkin's method is considered. According to these methods the following system is to be solved for C_1 and C_2.

For the method of least squares:

$$\int_0^b R\frac{\partial R}{\partial C_1}dx=0, \qquad \int_0^b R\frac{\partial R}{\partial C_2}dx=0.$$

For Galerkin's procedure:

$$\int_0^b R\frac{\partial \tilde{v}}{\partial C_1}dx=0, \qquad \int_0^b R\frac{\partial \tilde{v}}{\partial C_2}dx=0.$$

Having these values the second order approximate solution is determined.

III. ADOMIAN DECOMPOSITION METHOD(ADM)

Consider equation (1) again

$$v^{(n)}(x)=\phi(x)+\int_0^x \mathrm{K}(x,\tau)\Phi(v(\tau))d\tau,$$

with boundary conditions: $v^{(\iota)}(0)=\alpha_\iota$.

Eq.(1) can be written as

$$L(v(x))=\phi(x)+\int_0^x \mathrm{K}(x,\tau)\Phi(v(\tau))d\tau, \tag{9}$$

where L is linear operator and Φ is nonlinear operator.

In Adomian decomposition method, solution $v(x)$ is given as

$$v(x)=v_0(x)+v_1(x)+v_2(x)+\ldots=\sum_{m=0}^{\infty} v_m(x) \tag{10}$$

The nonlinear operator Φ is decomposed as

$$\Phi_k=\sum_{k=0}^{\infty} A_k\ , \tag{11}$$

where A_k are Adomian polynomials that are given as

$$A_k = \frac{1}{k!}\frac{d^k}{d\lambda^k} F\left(\sum_{i=0}^{\infty}\lambda^j v_j\right)\bigg|_{\lambda=0} \quad j = 0,1,2,...,k \qquad (12)$$

Using Eqs. (1), (9), (10), (11) and (12), an iteration is established:

$$L(v_0(\xi)) = \phi(x),\ v_0^{(\iota)}(0) = \alpha_\iota \qquad (13)$$

$$v_{k+1}(x) = L^{-1}\int_0^x \mathrm{K}(x,\tau)\Phi_k\, d\tau, \qquad k = 0,1,2,\ldots$$

where $L^{-1} = \int_0^x (.)\,d\tau$, is n-fold integral.

Solution of problem (13) is:

$$v_0(x) = \alpha_0 + \sum_i^{n-1}\alpha_i\, x^i + L^{-1}\phi(x)$$

The above equation plays the role of starting value for the above scheme.

IV. NUMERICAL IMPLEMENTATIONS

Example1: $\frac{dv}{dx} + v = e^{-x}\int_0^x v\,e^x dx$, $v(0) = 1$.

Exact solution of this problem is: $v = e^{-x}\cosh x$

Solution using OHAM: Following the procedure described in section 2 and confining it to only second order approximate solution, the following problems are obtained:

Zeroth Order Problem:

$$v_0'(x) + v_0(x) = 0,\ v_0(0) = 1.$$

First Order Problem:

$$v_1'(x) = -C_2\int_0^x e^x v_0(x)\,dx - C_1 e^{-x}\int_0^x e^x v_0(x)\,dx + v_0(x)$$
$$+C_1 v_0(x) + C_2\ e^x v_0(x) - v_1(x) + v_0'(x) + C_1\ v_0'(x)$$
$$+C_2\ e^x\ v_0'(x), v_1(0) = 0.$$

Second Order Problem:

$$v_2'(x) = -C_2\int_0^x e^x v_1(x)\ - C_1 e^{-x}\int_0^x e^x v_1(x)\ + v_1(x)$$
$$+C_1 v_1(x) + C_2 e^x v_1(x) - v_2(x) + v_1'(x) + C_1 v_1'(x)$$
$$+C_2\ e^x v_1'(x),\ v_2(0) = 0.$$

Solutions of the above problems are given in respective order:

$$v_0(x) = e^{-x}$$
$$v_1(x) = -0.5e^{-x}(2C_2 - 2C_2 e^x + 2C_2 x e^x + C_1 x^2)$$
$$v_2(x) = -(1/24)\ e^{-x}(24C_2 - 48C_1C_2 - 24C_2 e^x$$
$$+48C_1C_2\,e^x - 24\ C_2^2\,e^x\ + 24C_2^2\,e^{2x} - 48C_1C_2\,x$$
$$+24C_2\,e^x x - 24\ C_2^2\,e^x\,x + 12C_1\,x^2 + 12\ C_1^2\,x^2$$
$$-12C_1C_2x^2 + 12C_1C_2e^x x^2 - 4C_1C_2e^x x^3 - C_1^2 x^4)$$

Second order approximate solution to be sought is:

$$\tilde{v}(x) = v_0(x) + v_1(x) + v_2(x) + O(x^{14}).$$

Following the procedure for finding C_1 and C_2, values for C_1 and C_2 are:

$$C_1 = -0.95942910$$
$$C_2 = -0.03308793.$$

Using these values, the approximate solution is:

$$\tilde{v} = \begin{cases} 1 - x + 0.999972\,x^2 - 0.666474\,x^3 + 0.332321\,x^4 \\ -0.130974x^5 + 0.0416759x^6 - 0.0108206x^7 \\ +\ 0.00232884x^8 - 0.000423353\ x^9 \\ +\ 6.6180\times10^{-6}\,x^{10} - 9.0402\times10^{-6}\,x^{11} \\ +\ 1.0940\times10^{-6}\,x^{12} - 1.1865\times10^{-7}\,x^{13} + O(x^{14}) \end{cases}$$

Solution using ADM: The numerical scheme is:

$$\begin{cases} L(v_0(\xi)) = 0,\ v_0(0) = 1, \\ v_{k+1}(x) = L^{-1}(-v) + L^{-1}\left(e^{-x}\int_0^x \mathrm{K}(x,\tau)\Phi_k\,d\tau\right), \end{cases}$$

$k = 0,1,2,\ldots$, where $L^{-1} = \int_0^x (.)\,d\tau.$

Using the above scheme, the following components of the solution are obtained:

$$v_0 = 1,\ v_1 = e^{-x} - 1,\ v_2 = 1 - e^{-x}(1\ +\ x),$$
$$v_3 = e^{-x}(1\ +\ x\ +\ x^2/2) - 1,$$
$$v_4 = 1/6\ e^{-x}(-6 + 6\ e^x - 6\,x - 3\,x^2 - x^3),$$
$$v_5 = 1/24e^{-x}(24 - 24e^x + 24x + 12x^2 + 4x^3 + x^4), \ldots$$

Combining the components for the approximate solution of order five:

$$\tilde{v} = v_0 + v_1 + v_2 + v_3 + v_4 + v_5,$$

and now expressing it in power series with order of approximation $O(x^{14})$, the following series solution is obtained:

$$\tilde{v} = \begin{cases} 1 - x + x^2 - 2x^3/3 + x^4/3 - 2x^5/15 \\ +31x^6/720 - 19x^7/1680 + 11x^8/4480 \\ -163x^9/362880 + x^{10}/14175\ - \\ 9.6701\times10^{-6}\,x^{11} + 1.1691\times10^{-7}\,x^{12} \\ -\ 1.2751\times10^{-7}\,x^{13} \end{cases}$$

Table 1: Numerical results for example 1

x	Error-OHAM	Error-ADM
0.0	0.0000	0.0000
0.1	1.6712E-7	1.25694E-9
0.2	5.9483E-7	7.28281E-8
0.3	1.4288E-6	7.51285E-7
0.4	2.6201E-6	3.82429E-6
0.5	3.7860E-6	1.32215E-5
0.6	4.4121E-6	3.57925E-5
0.7	4.1623E-6	8.18567E-5
0.8	3.1600E-6	1.65479E-4
0.9	2.1691E-6	3.04477E-4
1.0	2.64903E-6	5.20181E-4

Error=Exact-Approx.

Example 2: Consider the second order equation:

$$\frac{d^2v}{dx^2} = e^x - x + x\int_0^1 x\,v\,dx\,,\quad v(0)=1,\ v'(0)=1$$

Exact solution of this problem is: $v = e^x$

Solution using OHAM: Following the procedure of OHAM, the second order approximate to be sought is: $\tilde{v}(x) = v_0(x) + v_1(x) + v_2(x) + O(x^{15})$

The optimal values for the convergence control parameters are:

$C_1 = 1.01709526,\ C_2 = -4.06867327$

Having these values, the approximate solution is:

$$\tilde{v} = \begin{cases} 1 + x + 0.49999999999202993x^2 \\ +0.16666666667772267x^3 \\ +0.04166666666002494x^4 \\ +\ 0.008333333333200499\,x^5 \\ +\ 0.00138888888886675\,x^6 \\ +\ 0.00019841269840953569\,x^7 \\ +\ 0.00002480158730119x^8 \\ +2.75573192235466\times10^{-6}\,x^9 \\ +2.755731922354662\times10^{-6}\,x^{10} \\ +2.50521083850424\times10^{-6}\,x^{11} \\ +2.08767569875353\times10^{-6}\,x^{12} \\ +1.6059043836566\times10^{-6}\,x^{13} \\ +1.1470745597546877\times10^{-6}\,x^{14} \end{cases}$$

Solution using ADM: Starting with $v_0 = e^x - x^3/6$, fifth order approximate solution $\tilde{v} = v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + O(x^{15})$ becomes:

$$\tilde{v} = \begin{cases} 1 + x + 0.5x^2 + 0.166666659807956x^3 \\ +0.0416666666666666664x^4 \\ +0.00833333333333333x^5 \\ +0.00138888888888889\,x^6 \\ +0.0001984126984126984\,x^7 \\ +0.0000248015873015873\,x^8 \\ +2.755731922398589\times10^{-6}\,x^9 \\ +2.75573192239859\times10^{-7}\,x^{10} \\ +2.505210838544172\times10^{-8}\,x^{11} \\ +2.0876756987868\times10^{-9}\,x^{12} \\ +1.605904383682161\times10^{-10}\,x^{13} \\ +1.147074559772972\times10^{-11}\,x^{14} \end{cases}$$

Table 2: Numerical results for example 2

x	Error-OHAM	Error-ADM
0.0	0.0000	0.0000
0.1	6.8612E-14	6.8585E-12
0.2	2.3137E-13	5.4870E-11
0.3	4.2433E-13	1.8519E-10
0.4	5.8612E-13	4.3896E-10
0.5	6.5659E-13	8.5734E-10
0.6	5.7931E-13	1.4815E-09
0.7	3.0109E-13	2.3525E-09
0.8	-2.0917E-13	3.5117E-09
0.9	-9.0950E-13	5.0002E-09
1.0	-1.4477E-12	6.8595E-09

Error=Exact-Approx.

Example 3: Consider the third order equation:

$$\frac{d^3v}{dx^3} = \sin x - x - x\int_0^{\pi/2} x\,v'\,dx\,,$$

$v(0)=0,\ v'(0)=0,\ v''(0)=-1.$

Exact solution of this problem is: $v = \cos x$

Solution using OHAM: Following the procedure of OHAM, and considering second order approximate solution: $\tilde{v}(x) = v_0(x) + v_1(x) + v_2(x) + O(x^{15})$

$C_1 = -0.87079378$ and $C_2 = -0.01669417.$

Having these values, the approximate solution is:

$$\tilde{v} = \begin{cases} 1 - x^2/2 + 0.0416667\,x^4 - 0.00138889\,x^6 \\ +0.0000248016\,x^8 - 2.75573\times10^{-7}\,x^{10} \\ +2.08768\times10^{-9}\,x^{12} - 1.14707\times10^{-11}\,x^{14} \end{cases}$$

Solution using ADM: Starting with $v_0 = \cos x - x^4/24$, fifth order approximate solution $\tilde{v} = v_0 + v_1 + v_2 + v_3 + v_4 + v_5 + O(x^{15})$ becomes:

$$\tilde{v} = \begin{cases} 1 - 0.5\,x^2 + 0.0418038\,x^4 \\ -0.00138889x^6 + 0.0000248016x^8 \\ -2.75573\times10^{-7}\,x^{10} + 2.08768\times10^{-9}\,x^{12} \\ -1.14707\times10^{-11}\,x^{14} \end{cases}$$

V. CONCLUSION

In this work, the author compared OHAM with ADM for the solution of integro-differential equations. It is observed that second order solution of OHAM produced more accurate results than fifth order solution of ADM. This accuracy is linked with the use of homotopy and auxiliary function in OHAM. On other hand, ADM is simpler and its computational cost is low as compared to OHAM. Both methods are effective and reliable for the solution of a wide class of functional equations.

Table 3: Numerical results for example 3

x	Error-OHAM	Error-ADM
0.0	0.0000	0.0000
0.1	4.1300E-14	-1.3714E-8
0.2	6.5581E-13	-2.1943E-7
0.3	3.2750E-12	-1.1109E-6
0.4	1.0151E-11	-3.5109E-6
0.5	2.4157E-11	-8.5716E-6
0.6	4.8512E-11	-1.7774E-5
0.7	8.6433E-11	-3.2928E-5
0.8	1.4071E-10	-5.6174E-5
0.9	2.1322E-10	-8.9981E-5
1.0	3.0442E-10	-1.3714E-4
1.1	4.1274E-10	-2.0080E-4
1.2	5.3420E-10	-2.8438E-4
1.3	6.6214E-10	-3.9170E-4
1.4	7.8788E-10	-5.2686E-4
1.5	9.0332E-10	-6.9430E-4

Error=Exact-Approx.

REFERENCES

[1]. R. Glowinski and L. Shiau, M. Sheppard, Numerical methods for a class of nonlinear integro-differential equations, Calcolo, 50(1), 2013, 17-116.

[2]. D. Bahuguna, A. Ujlayan and D.N. Pandey, A comparative study of numerical methods for solving an integro-differential equation, Computers &

[3]. Mathematics with Applications, 57(9), 2009, 1485-1493.

[4]. Ishak Hashim, Adomian decomposition method for solving BVPs for fourth-order integro-differential equations, Journal of Computational and Applied Mathematics, 193(2), 2006, 658-664.

[5]. A. M. Wazwaz, A reliable algorithm for solving boundary value problems for higher-order integro-differential equations, Applied Mathematics and Computation, 118(2-3), 2001, 327-342.

[6]. Javed Ali, S.Islam, Siraj-UL-Islam and Gulzaman, The Solution of Multi Point Boundary Value Problems by the Optimal Homotopy Asymptotic Method, Computer and Mathematics with Application, 59(6), 2010, 2000-2006.

[7]. Javed Ali, Saeed Islam, Hamid Khan and Gul Zaman, Solution of a Parameterized Sixth-Order Boundary Value Problem by the Optimal Homotopy Asymptotic Method, Proceedings of the Romanian Academy, Series A, 12(3), 2011,167–172.

[8]. Javed Ali, S. Islam, M. Tariq Rahim and Gul Zaman, Solution of Special Twelfth-Order Boundary value Problems by the Optimal Homotopy Asymptotic Method, World Applied Science Journal,11(3), 2010, 371-378.

[9]. Javed Ali , S.Islam, Hamid Khan, and Syed Inayat Ali Shah, Application of Optimal Homotopy Asymptotic Method to higher order boundary value problems, Abstract and Applied Analysis, Volume 2012, 1-14.

[10]. Javed Ali , S.Islam, Syed Inayat Ali Shah ,and amid Khan, Application of Optimal Homotopy Asymptotic Method to 5th & 6th order boundary value problems, World Applied Science Journal, 15(8), 2011,1120-1126.

[11]. G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, vol. 60 (Kluwer Academic, Dordrecht, The Netherlands, 1994).

[12]. A. M. Wazwaz and S. M. El-Sayed, A new modification of the Adomian decomposition method for linear and nonlinear operators, Applied Mathematics and Computation, 122(3), 2001, 393–405.

[13]. Lie-jun Xie, A New Modification of Adomian Decomposition Method for Volterra Integral Equations of the Second Kind, Journal of

[14]. Applied Mathematics, Volume 2013 , Article ID 795015, 7 pages.

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