

time of active transportation routes (i,j) , as next
 $T(x) = \sum_{ij} t_{ij} x_{ij}$
 $i \in I, j \in J$
 if $x_{ij} = 0$ if $x_{ij} > 0$

Indicative and clear example of this kind of total transportation time is in a military problem where is of primary importance to analyze the total time of all means of transportation which may be exposed to danger of the enemy attacks. This two types of

measure of the transportation efficiency (2) and (3) will be called Variant A (linear function) and Variant B (nonlinear function) of the total time transportation problem, respectively.

If multiple optimal solutions exist with T^* as minimal value of (3), it is recommended to optimize an another criteria retain T^* , like the transportation efficiency (2), the time of the longest

$$q^{(k)} = t_{ij} \quad \text{is}$$

The characteristic q_{ij} is the change of the transportation time in problem (3). Then the solution $x^{(k+1)}$ has:

$$\begin{aligned} & \left(\begin{array}{l} T(x^{(k)}) \\ T(x^{(k+1)}) \end{array} \right) = T(x^{(k)}) + \sum_{ij} q_{ij} x_{ij} \\ & \left(\begin{array}{l} T(x^{(k)}) \\ T(x^{(k+1)}) \end{array} \right) = T(x^{(k)}) + \sum_{ij} q_{ij} x_{ij} \\ & \text{if } q_{ij}^{(k)} \leq 0 \\ & \text{if } q_{ij}^{(k)} > 0 \end{aligned}$$

Let T^* is minimum value of $T(x)$, x^* is the optimal solution of (3) and X_T is a set of multiple optimal solutions of (3):

th

objective

active transportation operation, the number of units on transportation operation with longest time, the total transportation cost e.t.c.

SOLUTION METHODS

Let $x^{(k)}$ and $x^{(k+1)}$ are two basis neighbouring feasible solutions, where $x^{(k)}$ is entering basis variable and $x^{(k)}$ is leaving basis variable for $x^{(k)}$:

$$\begin{aligned} x^{(k)} \text{ is contain: } & x^{(k)} = 0 \text{ and } x^{(k)} > 0 \quad x^{(k+1)} \\ & \text{contain: } x^{(k+1)} > 0 \text{ and } x^{(k+1)} = 0 \\ \text{there is: } & x^{(k+1)} = x^{(k)} \end{aligned}$$

In moving from $x^{(k)}$ to $x^{(k+1)}$ the total time $T(x)$ given as (3) will be changed with the following values:

$$t_{ij} \quad (5)$$

$$T(x^{(k+1)}) = T(x^{(k)}) + q_{ij}^{(k)} \quad (6)$$

Clearly, the total time $T(x^{(k+1)})$ is determined by values $q_{ij}^{(k)}$ as following:

$$\text{if } q_{ij}^{(k)} \leq 0$$

$$(7)$$

$$T^* = \min$$

$$(x) = \sum_{i=1}^I \sum_{j=1}^J x_{ij} \quad (8)$$

$$X =$$

$$i = 1, 2, \dots, I \quad j = 1, 2, \dots, J$$

$$x^* = x \quad T^* = T$$

$$(9)$$

$$T = \min T(x)$$

$$\text{Min}$$

$$F(x) = \sum_{i=1}^I \sum_{j=1}^J t_{ij} x_{ij}$$

$$T(x) = T^*$$

Minimize the time of the longest active transportation operation

$$\text{Min } t(x) = \max t$$

$$T(x) = T^*$$

$$x_{ij} \geq 0$$

Minimize the number of units on transportation operation with longest time

$$\sum_{i=1}^I \sum_{j=1}^J x_{ij}$$

$$X = \begin{matrix} & j=1 & j=2 & \dots & j=J \\ i=1 & x_{11} & x_{12} & \dots & x_{1J} \\ i=2 & x_{21} & x_{22} & \dots & x_{2J} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ i=I & x_{I1} & x_{I2} & \dots & x_{IJ} \end{matrix}$$

$$X T^* = x T^* \quad (10)$$

Above discussion makes possible to develop the solving methods for defined transportation problem (3). If this problem has multiple optimal solution (10), undoubtedly, maybe required to optimize some of next criteria:

Minimize the transpiration efficiency from (2)

$$(11)$$

$$i = 1, 2, \dots, I \quad j = 1, 2, \dots, J$$

$$(12)$$

$$ij$$

Min
Q(x)

(13)

$T(x) \leq T^*$
 $t_{ij} \leq t(x)$
 $i \in I, j \in J$

Minimize the total transportation cost

Min
F(x) = cx

(14)

$\{ \quad \}$
 $\square \in ij$

$T(x) \leq T^*$
 $i \in I, j \in J$

where c_{ij} = the units transportation costs.

Algorithm 1 finds the optimal solution and minimum total transportation time (3). Algorithm 2 continue the solving process using the multiple optimal solution of problem (3), if exist, and minimize chosen criteria (11) to (14).

Algorithm 1.

Step 0: Find the basic feasible solution $x^{(1)}$. Set number of iteration $k = 1$.

Step 1: Determine the indicators $h^{(k)}$ of active transportation routes $x^{(k)} > 0$, and

$i \in I, j \in J$
the total time $T^{(k)} = T(x^{(k)})$.

$h^{(k)} \leq 1$
if $x^{(k)} \leq 0$

(15)

≤ 0
if $x^{(k)} \leq 0$

$T^{(k)} \leq$
 $\square \leq$
 $i \in I, j \in J$

(k)
 ij
(16)

Step 2: Determine the characteristics $q^{(k)}$ for all nonbasic variables $x^{(k)} = 0$ using

$ij \quad ij$
(5). Use the changing path of the basic solution (as in a Stepping-Stone method) and corresponding leaving basic variable, e.g. $x^{(k)} > 0$ become $x^{(k+1)} = 0$, if entering basic variable would be $x^{(k+1)} > 0$.

Step 3: Check the optimality of the total time (3), using (7). If all $q^{(k)} \leq 0$, the ij optimal solution x^* is found. Stop. Otherwise, go to Step 4.

Step 4: Determine next basic solution, using entering variable x_{ij} with minimum $q^{(k)}$, regarding $q^{(k)} < 0$. Set $k = k + 1$ and goto Step 1.

$ij \quad ij$
If the optimal solution x^* in last Step 3 has $q^{(k)} = 0$ for nonbasic variables $x^{(k)} = 0$, there is

$ij \quad ij$
no unique optimal solution and exist a set of multiple optimal solutions X_T^* . Each of these variables gives an alternative optimal solution for (3), go to optimize other criteria.

Algorithm 2.

Step 0: Chose one of criteria from (11) to (14) and calculate his increase $\square^{(k)}$ for each nonbasic variable $x^{(k)} = 0$ with $q^{(k)} = 0$ in multiple optimal solution on end of

$ij \quad ij$
Algorithm 1. For $\square^{(k)}$ use the known solving process for regarded criteria.

Step 1: If there are negative increase, $\square^{(k)} < 0$, for regarded criterion, chose minimum of them and minimize this criterion in set of multiple optimal solution for (3).

Step 2: Repeat Step 1 with each of negative increase for regarded criterion and choose solution with minimum criterion value.

ANEXAMPLE

Let us consider the following transportation problem with $m = 4$ sources A_i , $i \in I = \{1, 2, 3, 4\}$, and $n = 5$ destinations B_j , $j \in J = \{1, 2, 3, 4, 5\}$. The initial data are presented in TABLE 1. Each row corresponds to a supply point and each column to a demand point. The total supply 65 is equal to the total demand. In each cell (i, j) , top left corner represents the time t_{ij} required

for transporting x_{ij} units from source A_i to destination B_j . The basic variables x_{ij} are presented in the middle of corresponding cells and the increase q_{ij} of time in bottom right corner of each cell (i,j) with nonbasic variable.

First basic feasible solution in the Step 0 of Algorithm 1 maybe determined using north-west corner method or another method. In TABLE 1 is chosen the optimal solution $x^{(1)}$ of criteria $F(x)$ with minimum value $F^* = F^{(1)} = 222$ and increases $d^{(1)}$

11 11
 $d^{(1)} = t$
 $-t + t - t$
 11 11 15 45 41

$x^{(2)} > 0, x$
 $(2) = \min \square x$

$(1), x$
 $(1) \square = \min \square 1, 2 \square = 1 = x$

$(1), x$

$(2) = 0$

in TABLE 2 which verify the unique optimal solution. The corresponding total time $T^{(1)} = 32$ is calculated in Step 1 with indicator $sh^{(1)}$ (TABLE 1 shows only indicators with

value 1 for active transportation routes).

In Step 2 are calculated increases $q^{(1)}$ of the total transportation time $T(x)$ for nonbasic variables. Let demonstrate them for cell $(1,1)$.

$x^{(1)} = 0, L^{(1)} = \square (1,1), (1,5), (4,5), (4,1) \square$

$= 11 - 5 + 5 - 9 = 2$

11 11
15 41
15 15

$$q^{(1)} = t - t \\ = 11 - 5 = 6.$$

11 1115

The changing path L

⁽¹⁾for Stepping-Stone method is used for calculate d
(1)

and indicate that x

11

⁽¹⁾is leaving variable if x

⁽¹⁾is entering variable. Then is q

11

$$(1) = t -$$

15 11

$$t_{15} = 11 - 5 = 6.$$

11 11

On changing path L

25

is no optimal, too, because for no basic variable x

(4,5)□ exists x

$$^{(3)} = \min \square x$$

(2), x

44

$$^{(2)}\square = \min \square 10, 1\square = 1 = x$$

44 14 45

45 44

-2. After entering this variable, the optimal solution $x^* = x^{(3)}$ is finding (TABLE 4).²³

solution $x^{(3)}$ with d

(3)

21

optimize some another criteria using Algorithm 2 regarding corresponding changing paths with their entering and leaving variables:

$$x^{(3)} = 0, L^{(3)} = \square (2,1), (2,5), (1,5), (1,4), (4,4)\square$$

23 23 21 21

$$x^{(5)} > 0, x$$

$$^{(5)} = \min \square x$$

(3), x

23 23

33 12

12 12

$$^{(1)} = \square (2,5), (4,5), (4,1), (2,1)\square \text{ for } x$$

$$^{(1)} = 0 \text{ there are } x$$

(2)

$$= \min \square x$$

(1), x

25

$$^{(1)}\square = \min \square 14, 13\square = 13 = x$$

(1) and x

25

$$^{(2)} = 0. \text{ Value } q$$

$$^{(1)} = t - t$$

25

$$= 1 - 2$$

15 21

21 21

2525 21

= -1 shown that $x^{(1)}$ is no optimal solution for $T(x)$.

Entering basic variable x_{25} with

$$\text{solution } x^{(2)} \text{ decrease } T^{(1)} = 32 \text{ to } T^{(2)} = T^{(1)} + t^{(1)} =$$

$$32 - 1 = 31 \text{ (TABLE 3). This solution}$$

⁽²⁾on path L

$$^{(2)} = \square (4,4), (1,4), (1,5),$$

44

⁽²⁾and t

$$^{(2)} = t - t$$

$$= 3 - 5 =$$

44 45

Minimum of total transportation time $T^* = T^{(2)} = 29$ has multiple optimal

^{(3)} = 0. Keeping minimum of $T(x)$ maybe it is possible to}

$$x^{(4)} > 0, x^{(4)} = \min \square x^{(3)}, x^{(3)}, x^{(3)}\square = \min \square 13, 9, 15\square$$

$$= 9 = x^{(3)}, x^{(4)} = 0$$

$$x^{(3)} = 0, L^{(3)} = \square (2,3), (3,3), (3,2), (1,2)\square$$

$$^{(3)}\square = \min \square 15, 3\square = 3 = x$$

(3), x

$$^{(5)} = 0.$$

Table 1: Initial data, optimal solution $x^{(1)}$ for $F(x)$ and indicators $q^{(1)}$ for $T(x)$

	i \ j	Destinations					Supplies, a_i
		B_1	B_2	B_3	B_4	B_5	
Sources	A_1	11 6	3 3	10 7	2 10	5 1	14
	A_2	2 (-) 13	7 4	3 0	8 6	1 (+) -1	13
	A_3	12 7	2 7	4 15	5 3	7 2	22
	A_4	9 (+) 2	4 1	6 3	3 1	5 (-) 14	16
Demands, b_j		15	10	15	10	15	65 \ 65

$F^{(1)} = 3 \times 3 + 2 \times 10 + 5 \times 1 + 2 \times 13 + 2 \times 7 + 4 \times 15 + 9 \times 2 + 5 \times 14 = 221$
 $h^{(1)} = h^{(1)} = h^{(1)} = h^{(1)} = h^{(1)} = h^{(1)} = 1$
 $12 \quad 14 \quad 15 \quad 21 \quad 32 \quad 33 \quad 41 \quad 45$
 $T^{(1)} = 3 \times 1 + 2 \times 1 + 5 \times 1 + 2 \times 1 + 2 \times 1 + 4 \times 1 + 9 \times 1 + 5 \times 1 = 32$

Let analyze the longest time on the separable active transportation routes (12) and corresponding number of transported units (13):

$$t^{(3)} \square$$

$$\max x_{ij} \square 0$$

$$t_{ij}$$

$$= \max (t_{12}, t_{14}, t_{15}, t_{25}, t_{32}, t_{33}, t_{41}, t_{44})$$

$$= \max (3, 5, 1, 2, 4, 9, 3) = 9 = t_{41}$$

$$Q^{(3)} \square x = 15.$$

$$t_{ij} \square 9$$

$$\text{Solution } x^{(4)} \text{ kept } t^{(4)} = t^{(3)} = 9 \text{ and decrease } Q^{(3)} \text{ to value } Q^{(4)} = Q^{(3)} \square x$$

$$^{(4)} = 15 - 6$$

$$= 9 \text{ (TABLE 5). So, } x^{(4)} \text{ is better than } x^{(3)}. \text{ With } x^{(4)}$$

$$^{(5)} \text{ in } x^{(5)} \text{ (TABLE 6) both of criteria has}$$

$$23$$

$$\text{same values as } x^{(3)}, \text{ and } x^{(5)} \text{ is not better solution.}$$

ij

Table 2.

Nonbasic variables, x_{ij}	Indicators $d_{ij}^{(1)}$ for $F(x)$	Indicators $q_{ij}^{(1)}$ for $T(x)$
$x_{11}^{(1)} = 0$	$d_{11}^{(1)} = 2$	$q_{11}^{(1)} = t_{11} - t_{15} = 11 - 5 = 6$
$x_{13}^{(1)} = 0$	$d_{13}^{(1)} = 5$	$q_{13}^{(1)} = t_{13} - t_{12} = 10 - 3 = 7$
$x_{22}^{(1)} = 0$	$d_{22}^{(1)} = 11$	$q_{22}^{(1)} = t_{22} - t_{12} = 7 - 3 = 4$
$x_{23}^{(1)} = 0$	$d_{23}^{(1)} = 5$	$q_{23}^{(1)} = t_{23} - t_{13} = 3 - 3 = 0$
$x_{24}^{(1)} = 0$	$d_{24}^{(1)} = 13$	$q_{24}^{(1)} = t_{24} - t_{14} = 8 - 2 = 6$
$x_{25}^{(1)} = 0$	$d_{25}^{(1)} = 3$	$q_{25}^{(1)} = t_{25} - t_{21} = 1 - 2 = -1$

25	25	
$x^{(1)} = 0$	$d^{(1)} = 4$	$q_{31}^{(1)} = t_{31} - t_{14} = 12 - 5 = 7$
31	31	
$x^{(1)} = 0$	$d^{(1)} = 4$	$q_{31}^{(1)} = t_{31} - t_{14} = 12 - 5 = 7$
34	34	
$x^{(1)} = 0$	$d^{(1)} = 3$	$q_{35}^{(1)} = t_{35} - t_{15} = 7 - 5 = 2$
35	35	
$x^{(1)} = 0$	$d^{(1)} = 1$	$q_{42}^{(1)} = t_{42} - t_{12} = 4 - 3 = 1$
42	42	
$x^{(1)} = 0$	$d^{(1)} = 1$	$q_{43}^{(1)} = t_{43} - t_{12} = 6 - 3 = 3$
43	43	
$x^{(1)} = 0$	$d^{(1)} = 1$	$q_{44}^{(1)} = t_{44} - t_{14} = 3 - 2 = 1$
44	44	
	Unique optimal solution, all $d^{(1)} > 0$ ij	No optimal solution, $q^{(1)} = -1$ 25

Table 3: Solution $x^{(2)}, T^{(2)} > T^*$

11	3	10	2 (-)	5 (+)
6	3	7	10	1
2	7	3	8	1
1	4	0	6	13
12	2	4	5	7
7	7	15	3	2
9	4	6	3 (+)	5 (-)
15	-1	1	-2	1

Table 4: Solution $x^{(3)}, T^*$

11	3	10	2 (-)	5 (+)
9	3	7	9	2
2 (+)	7	3	8	1 (-)
0	4	0	6	13
12	2	4	5	7
10	7	15	3	2
9 (-)	4	6	3 (+)	5
15	1	3	1	2

$$T^{(2)} = T^{(1)} + t^{(1)} = 32 - 1 = 31 \quad T^{(3)} = T^{(2)} + t^{(2)} = 31 - 2 = 29$$

Regarding in $x^{(3)}$ on total cost $C(x)$ defined as (14) and unique cost in following matrix C, the total cost are $C^{(3)} = 428$. The increase of total cost of variables for multiple

optimal solution in minimum total time $T^* = 29$ are
 $\square C^{(3)} = -1$ (for $x^{(3)} = 0$ with $q^{(3)} = 0$)

21 21 21

and $\square C^{(k)} = 4$ (for $x^{(3)} = 0$ with $q^{(3)} = 0$). If x is entering basic variable, solution is $x^{(4)}$

23 23 23 21
 with minimum total cost $C^{(5)} = 419$ for minimum total time $T^* = 29$. However, the minimum total cost $C^* = 383$ without minimum total time $T(x)$ would be obtained with $x^{(6)}$ when the total time is $T^{(6)} = 46$.

The optimal solutions of some basic single criteria cost and time in transportation problems are compared in TABLE 7. Using the optimal solution of each of them, the values of others criteria are calculated. All solutions with time contain an

identical time of the longest route (12) in this trivial example. Clearly, it is not a general rule.

After conditional optimization for objective (12), the objective (13) was minimized for the total quantity of goods which is transported with longest time t^* , in the sense of Hammers "real" solution for optimal solution (12). However, objective (13) is

optimized against two conditions, keeping minimal total time $T^* = 29$ and the time of the longest transport operation $t^* = 9$ for $T^* = 29$.

Table 5: Solution $x^{(4)}, T^*$ **Table 6:** Solution $x^{(5)}, T^*$

11	3	10	2	5
	3	7		11
2	9	7	3	8
				1
12	2	4	5	7
	7	15		
9	4	6	3	5
6			10	

$$T^{(4)} = T^{(3)} + t^{(3)} = 29 + 0 = 29 \quad T^{(5)} = T^{(3)} + t^{(3)} = 29 + 0 = 29$$

$$C = \begin{bmatrix} 4 & 7 & 10 & 9 & 5 \\ 6 & 8 & 16 & 9 & 8 \\ 3 & 0 & 10 & 0 \end{bmatrix}, \quad x^{(6)} = \begin{bmatrix} 14 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 4 & 6 & 10 & 7 \\ 5 & 8 & 9 & 10 & 6 \end{bmatrix} \leq f$$

$$\begin{bmatrix} 0 & 7 & 15 & 0 & 0 \\ 1 & 0 & 0 & 0 & 15 \end{bmatrix} \leq f$$

Table 7: Optimal solutions of basic single criteria transportation problems

Minimization criteria	Solutions			
	$x^{(1)} =$	$x^{(3)} =$	$x^{(4)} =$	$x^{(6)} =$
	$\begin{bmatrix} 0 & 3 & 0 & 10 & 1 \\ 13 & 0 & 0 & 0 & 0 \\ 0 & 7 & 15 & 0 & 0 \\ \leq 0 & 0 & 0 & 14 \end{bmatrix} f$	$\begin{bmatrix} 0 & 3 & 0 & 0 & 11 \\ 9 & 0 & 0 & 0 & 4 \\ 0 & 7 & 15 & 0 & 0 \\ \leq 0 & 0 & 10 & 0 \end{bmatrix} f$	$\begin{bmatrix} 0 & 3 & 0 & 0 & 11 \\ 9 & 0 & 0 & 0 & 4 \\ 0 & 7 & 15 & 0 & 0 \\ \leq 0 & 0 & 10 & 0 \end{bmatrix} f$	$\begin{bmatrix} 14 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 10 & 0 \\ 0 & 7 & 15 & 0 & 0 \\ \leq 0 & 0 & 0 & 15 \end{bmatrix} f$
F(x)	$F_{\min} = 222$	262	244	333
T(x)	31	$T_{\min} = 29$	$T_{\min} = 29$	48
t(x)	$t_{\min} = 9$	$t_{\min} = 9$	$t_{\min} = 9$	11
Q(x)	$Q_{\min} = 2$	15	6	14
C(x)	406	428	419	$C_{\min} = 383$

II. CONCLUSIONS

The time of transport might be significant factor in several transportation problems. The efficiency of transportation have been introduced as an aggregate of the time and the quantity of goods on active transport operations, longest time on single active transport operation, total time on all active transport operations, total quantity of goods with a longest time of transport etc. The question arises from this whether to perform optimization at the same time for more objectives, with eventual priorities for each, or to treat single objective problems, which are of the great significance for particular problem. In this paper, the single

objective problem of total time minimization has been exposed having in mind active transport operations and is defined as Algorithm

1 for its solution. To start from the point of multiple optimal solution for specific total time of transport, it is further suggested the minimization of some other objective while keeping minimal total time of transport (Algorithm 2). In that way, it is passed to the two objective problems with strong lexicographic order for the determined objectives where the absolute priority is given to the total time of transport. Furthermore, if there is a multiple solution, the third objective can be optimized which

bears the next level of significance etc.

Defined process of solving the problem of minimization of total time on active transport operations is necessary even for solving each multi objective transport problem which involves mentioned objective. At the first step of solving any multi objective problem, it is necessary to determine optimal solutions for each objective separately, that is to treat single objective problems and then to keep searching for pareto-optimal solutions for each multi objective problem.

In hypothetical example of small dimensions two variants of problem were illustrated for the total time of transport with objectives of "total efficiency of transport from the time view" $F(x)$ and total time of transport $T(x)$. Meanwhile, in the set of optimal solutions for $T(x)$, the conditional minimization was performed separately for the longest time on active transport operations with $t(x)$ and separately for total cost of transport with $C(x)$.

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