

Non-Linear Water Wave Equation Time Series Formulated Using Velocity Equation as the Result of Laplace Equation

Syawaluddin Hutahaean, and Hendra Achiari

syawaluddin@ocean.itb.ac.id, and h_achiari@ocean.itb.ac.id, Ocean Engineering Program, Faculty of Civil and Environmental Engineering – Bandung Institute of Technology (ITB) - INDONESIA

ABSTRACT

This article developed a model of time series wave where during the formulation velocity equations as the result of Laplace theory was done. As the result of this method, the continuity equation was substituted to momentum equation, or in other words momentum equation produces a velocity controlled by continuity equation, since Laplace equation comes from continuity equation. Using this method, the concept of calculation velocity at the momentum equation was also developed where calculation velocity is equal to average velocity of the depth of the Airy long wave equation.

Key word: Continuity equation, Laplace equation, Momentum equation, Nonlinear wave.

I. INTRODUCTION

Velocity at a momentum equation is a velocity at any position on a vertical axis. In a numerical calculation using discrete method, finite element and finite difference method, a calculation can be done for some velocities along the vertical axis. However, in very shallow water, it is easier to do the calculation using a representative velocity, or a calculation velocity, so that in a momentum equation, there is only velocity on one depth position. This calculation velocity is known as depth-averaged velocity on Airy long wave equation introduced by Fenton [1].

The velocity produced by momentum equation should meet continuity equation. Therefore, there should be the role of continuity equation at the momentum equation. In this research, the role of continuity equation is substituted to momentum equation through driving force by working on the vertical direction of velocity equation as a result of Laplace equation solution

II. VELOCITY RELATION AS A RESULT OF LAPLACE EQUATION SOLUTION

The velocity of particle the horizontal and vertical direction based on Hutahaean [2],[3],[4] is defined as,

$$u = Ge^{kh} \beta(z) \left(k \sin kx - \frac{1}{2h} \frac{\partial h}{\partial x} \cos kx \right) \sin \sigma t \quad (2.1)$$

The velocity of particle in vertical -direction

$$w = -Ge^{kh} k \beta_1(z) \cos kx \sin \sigma t \quad (2.2)$$

where

G = wave constant

k = wave number

σ = angle velocity

h = water depth

$\frac{\partial h}{\partial x}$ = slope of water bottom

$$\beta(z) = \alpha e^{k(h+z)} + e^{-k(h+z)},$$

$$\beta_1(z) = \alpha e^{k(h+z)} - e^{-k(h+z)}$$

$$\alpha = \frac{1}{2} \begin{vmatrix} 1 + \frac{\partial h}{\partial x} & 1 - \frac{\partial h}{\partial x} \\ 1 - \frac{\partial h}{\partial x} & 1 + \frac{\partial h}{\partial x} \end{vmatrix}$$

With the characteristics, $\frac{\partial k(h+z)}{\partial x} = 0$

and $\frac{\partial kh}{\partial x} = 0$, then

$$\frac{\partial \beta(z)}{\partial x} = \frac{\partial \beta_1(z)}{\partial x} = \frac{\partial e^{kh}}{\partial x} = 0 \text{ and } \frac{\partial k}{\partial x} = -\frac{k}{h} \frac{\partial h}{\partial x}.$$

Another characteristic is $\frac{\partial G}{\partial x} = \frac{G}{2h} \frac{\partial h}{\partial x}$.

From equation (2.1) a relation can be made,

$$u = \frac{\beta(z)}{\beta(\eta)} u_\eta \quad (2.3)$$

And also,

$$\frac{\partial u}{\partial t} = \frac{\beta(z)}{\beta(\eta)} \frac{\partial u}{\partial t} - \frac{\beta(z)}{\beta^2(\eta)} \beta_1(\eta) k u \frac{\partial \eta}{\partial t} \quad (2.4)$$

III. THE CONCEPT OF CALCULATION VELOCITY

At the continuity equation and momentum equation, velocity is a function of z , where z is the vertical axis as shown on **Fig 3.1**, so, in a water column from the bottom to the surface of the water, there will be plenty of velocity. Calculation will be much easier if it is done using a representative velocity on a water column. This velocity is called depth calculation velocity. It was developed for the first time during the formulation of Airy long wave equation developed by Dean [5], called depth-averaged velocity, i.e.

$$U = \frac{1}{\beta_u H} \int_{-h}^{\eta} u dz \quad (3.1)$$

In the equation (3.1), the value of depth-averaged velocity is determined by the value of integration coefficient β_u . The bigger value of β_u , makes the smaller value of U , where this shows that a value of integration coefficient corresponds to a velocity at a depth of z .

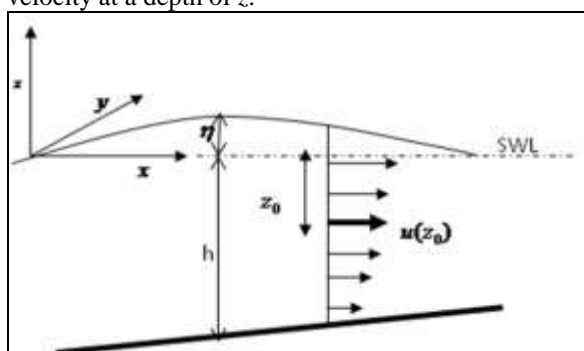


Figure 3.1. Depth-averaged Velocity and axis system

Equation (3.1) can be written as equation for integration coefficient, β_u defined as

$$\beta_u = \frac{1}{UH} \int_{-h}^{\eta} u dz . \text{ If the depth-averaged velocity of } U \text{ is a velocity at } z = z_0, \text{ or } U = u(x, z_0, t), \text{ then}$$

$$\beta_u = \frac{1}{u(z_0)H} \int_{-h}^{\eta} u dz .$$

Using the form of velocity equation in the direction of horizontal x as equation (2.1), then

$$\beta_u = \frac{1}{\beta(z_0)H} \int_{-h}^{\eta} \beta(z) dz = \frac{\beta_1(\eta) - (\alpha - 1)}{k\beta(z_0)H} \quad (3.2)$$

Table 3.1 shows the result of β_u calculation for various wave periods, for $z_0 = -0.40 h$. The result of the calculation shows that at $z_0 = -0.40 h$, the value of β_u was obtained which is very close to 1. This shows that depth-averaged velocity is a velocity at a depth of $z = z_0$.

Table 3.1. The value of β_u , at $z_0 = -0.40 h$,

$$A = 0.60 \text{ m}, \frac{dh}{dx} = 0.00$$

H (m)	T=6 (sec)	T=8 (sec)	T=10 (sec)	T=12 (sec)	T=14 (sec)	T=16 (sec)
15	1.021	0.999	0.997	0.998	0.998	0.999
14	1.017	0.999	0.998	0.998	0.998	0.999
13	1.014	0.999	0.998	0.998	0.999	0.999
12	1.011	0.999	0.999	0.999	0.999	0.999
11	1.009	1	0.999	0.999	0.999	0.999
10	1.008	1	0.999	0.999	1	1
9	1.007	1.001	1	1	1	1
8	1.007	1.001	1	1	1	1
7	1.007	1.002	1.001	1	1	1
6	1.007	1.003	1.001	1.001	1.001	1
5	1.008	1.003	1.002	1.001	1.001	1.001
4	1.009	1.004	1.003	1.002	1.001	1.001
3	1.011	1.005	1.003	1.002	1.002	1.001
2	1.013	1.007	1.004	1.003	1.002	1.002
1	1.017	1.009	1.006	1.004	1.003	1.002

In this paper, the concept of depth-averaged velocity is used with reversed procedure, i.e. the position of z_0 is set, and then integration coefficient is calculated using equation (3.2). The use of $z_0 = -0.40 h$ will give β_u integration coefficient, very close to 1.

With the inversion of the procedure, the

$$\text{velocity integration is defined as } \int_{-h}^{\eta} u dz = \beta_u UH$$

where $U = u_{z_0}$. Integration coefficient β_u , is calculated by using equation (3.2) and a new integration coefficient is defined as,

$$\beta_{uH} = \beta_u H = \frac{\beta_1(\eta) - (\alpha - 1)}{k\beta(z_0)} \quad (3.3)$$

$$\text{Therefore, } \int_{-h}^{\eta} u dz = \beta_{uH} U \quad (3.4)$$

With equation (3.4), the integration of continuity equation to vertical axis can be done easily.

IV. THE FORMULATION OF MOMENTUM EQUATION

As the basic momentum equation, Euler momentum equation developed by Dean [5] is used. The momentum equation is for the flow at (x, z) plane where x axis is horizontal axis and z axis is vertical axis, consisting of two equations, i.e. momentum equation in horizontal x direction and momentum equation in vertical z direction, respectively

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial x} \quad (4.1)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad (4.2)$$

Where $u = u(x, z, t)$ is the velocity in horizontal- x direction, $w = w(x, z, t)$ velocity in vertical- z direction, p is pressure on water particle, g is gravitational velocity, ρ is water density and t is time.

At the convective equation, the characteristic of irrotational flow is done where $\frac{\partial w}{\partial x} = \frac{\partial u}{\partial z}$. Hence based Hutaean [6], momentum x dan z equations become

$$\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} (uu + ww) = - \frac{1}{\rho} \frac{\partial p}{\partial x} \quad (4.3)$$

$$\frac{\partial w}{\partial t} + \frac{1}{2} \frac{\partial}{\partial z} (uu + ww) = - \frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad (4.4)$$

4.1 The Formulation of Driving Force

Equation (4.4) is written as driving force equation in vertical z - $\frac{1}{\rho} \frac{\partial p}{\partial z}$ direction, i.e.

$$- \frac{1}{\rho} \frac{\partial p}{\partial z} = \frac{\partial w}{\partial t} + \frac{1}{2} \frac{\partial}{\partial z} (uu + ww) + g$$

Then, it is multiplied by dz and integrated into vertical z axis,

$$- \frac{1}{\rho} \int dz dp = \int \frac{\partial w}{\partial t} dz + \frac{1}{2} \int \partial (uu + ww) + \int g dz$$

where $\eta = \eta(x, t)$ is water surface elevation. Integration is completed and dynamic surface boundary condition, $p_\eta = 0$, is done

$$\frac{p}{\rho} = \int \frac{\partial w}{\partial t} dz + \frac{1}{2} (u_\eta u_\eta + w_\eta w_\eta)$$

$$- \frac{1}{2} (uu + ww) + g (\eta - z) \quad (4.5)$$

Equation (4.5) is pressure equation experienced by water particle. The driving force is pressure difference, so to obtain driving force at horizontal- x direction; equation (4.5) is differentiated in x direction.

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial}{\partial x} \int \frac{\partial w}{\partial t} dz + \frac{1}{2} \frac{\partial}{\partial x} (u_\eta u_\eta + w_\eta w_\eta) - \frac{1}{2} \frac{\partial}{\partial x} (uu + ww) + g \frac{\partial \eta}{\partial x} \quad (4.6)$$

Substitute equation (4.6) to horizontal- x momentum equation, equation (4.3),

$$\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} (uu + ww) = - \frac{\partial}{\partial x} \int \frac{\partial w}{\partial t} dz - \frac{1}{2} \frac{\partial}{\partial x} (u_\eta u_\eta + w_\eta w_\eta) + \frac{1}{2} \frac{\partial}{\partial x} (uu + ww) - g \frac{\partial \eta}{\partial x}$$

The second term on the left-hand side and the third term on the right hand side are eliminated each other

$$\frac{\partial u}{\partial t} = - \frac{\partial}{\partial x} \int \frac{\partial w}{\partial t} dz - \frac{1}{2} \frac{\partial}{\partial x} (u_\eta u_\eta + w_\eta w_\eta) - g \frac{\partial \eta}{\partial x} \quad (4.7)$$

In equation (4.7), there is still an integral that should be completed, i.e. integral in the first term of

the right hand side. Equation $\frac{\partial}{\partial x} \int \frac{\partial w}{\partial t} dz$ is obtained

by integrating continuity equation that will be done in the next section.

4.2 Equation for $\frac{\partial}{\partial x} \int \frac{\partial w}{\partial t} dz$

Huataean [7] completed the integration of

$$\frac{\partial}{\partial x} \int \frac{\partial w}{\partial t} dz$$

by integrating continuity equation. In this paper, the integration is completed using velocity equation from the solution of Laplace equation. From the velocity equation, the equation (2.2), become

$$\frac{\partial w}{\partial t} = -Gke^{kh} \beta_1(z) \sigma \cos kx \cos \sigma t \text{ so}$$

$$\int \frac{\partial w}{\partial t} dz = -Ge^{kh} (\beta(\eta) - \beta(z)) \sigma \cos kx \cos \sigma t$$

$$\frac{\partial}{\partial x} \int_z^\eta \frac{\partial w}{\partial t} dz = Ge^{kh} (\beta(\eta) - \beta(z)) \sigma k \sin kx \cos \sigma t$$

$$- \frac{\partial G}{\partial x} e^{kh} \sigma (\beta(\eta) - \beta(z)) \cos kx \cos \sigma$$

Therefore

$$\frac{\partial}{\partial x} \int_z^\eta \frac{\partial w}{\partial t} dz = Ge^{kh} (\beta(\eta) - \beta(z)) \times \sigma \left(k \sin kx - \frac{1}{2h} \frac{\partial h}{\partial x} \cos kx \right) \cos \sigma t$$

From the velocity equation, equation (2.1) so

$$\frac{\partial u}{\partial t} = Ge^{kh} \beta(z) \sigma \left(k \sin kx - \frac{1}{2h} \frac{\partial h}{\partial x} \cos kx \right) \cos \sigma t$$

Substitute equation $\frac{\partial u}{\partial t}$ to equation $\frac{\partial}{\partial x} \int_z^\eta \frac{\partial w}{\partial t} dz$ the

last,

$$\frac{\partial}{\partial x} \int_z^\eta \frac{\partial w}{\partial t} dz = \left(\frac{\partial u}{\partial t} \right)_{z=\eta} - \frac{\partial u}{\partial t}$$

Equation (2.4) was done at $z = \eta$,

$$\left(\frac{\partial u}{\partial t} \right)_{z=\eta} = \frac{\partial u_\eta}{\partial t} - \frac{\beta_1(\eta)}{\beta(\eta)} k u_\eta \frac{\partial \eta}{\partial t}, \text{ then}$$

$$\frac{\partial}{\partial x} \int_z^\eta \frac{\partial w}{\partial t} dz = \frac{\partial u_\eta}{\partial t} - \frac{\beta_1(\eta)}{\beta(\eta)} k u_\eta \frac{\partial \eta}{\partial t} - \frac{\partial u}{\partial t} \quad (4.8)$$

Equation (4.8) is substituted to equation (4.7).

$$\frac{\partial u}{\partial t} = - \frac{\partial u_\eta}{\partial t} + \frac{\beta_1(\eta)}{\beta(\eta)} k u_\eta \frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial t} - \frac{1}{2} \frac{\partial}{\partial x} (u_\eta u_\eta + w_\eta w_\eta) - g \frac{\partial \eta}{\partial x}$$

The first term on the right hand side is moved to the left, where as the third term on the right hand side is eliminating each other with the first term on the left hand side.

$$\frac{\partial u_\eta}{\partial t} = \frac{\beta_1(\eta)}{\beta(\eta)} k u_\eta \frac{\partial \eta}{\partial t} - \frac{1}{2} \frac{\partial}{\partial x} (u_\eta u_\eta + w_\eta w_\eta) - g \frac{\partial \eta}{\partial x} \quad (4.9)$$

Considering that $\frac{\partial}{\partial x} \int_z^\eta \frac{\partial w}{\partial t} dz$ is formulated

using the result of Laplace solution equation coming from the continuity equation, it can be said that there is a role of continuity equation on driving force in

equation (4.9) or in other words the momentum equation is controlled by continuity equation.

The momentum equation in equation (4.9) is in the form of surface momentum equation that produces surface velocity of u_η based on Beji et.al [8]. The equation can be changed to momentum equation for velocity other than surface velocity. In this paper, the equation is changed into equation with calculation velocity equation, i.e. velocity of a depth $z = z_0$, where $u(z_0) = U$. Using $z_0 = -0.4h$, the velocity of calculation U is similar to the depth-averaged velocity in Airy long wave equation.

Equation (2.3) can be written as $u_\eta = \frac{\beta(\eta)}{\beta(z_0)} U$, therefore

$$\frac{\partial u_\eta}{\partial t} = \frac{\beta(\eta)}{\beta(z_0)} \frac{\partial U}{\partial t} + \frac{\beta_1(\eta)}{\beta(z_0)} k U \frac{\partial \eta}{\partial t}$$

Substituted to equation (4.9),

$$\frac{\beta(\eta)}{\beta(z_0)} \frac{\partial U}{\partial t} + \frac{\beta_1(\eta)}{\beta(z_0)} k U \frac{\partial \eta}{\partial t} = \frac{\beta_1(\eta)}{\beta(\eta)} k u_\eta \frac{\partial \eta}{\partial t}$$

$$- \frac{1}{2} \frac{\partial}{\partial x} (u_\eta u_\eta + w_\eta w_\eta) - g \frac{\partial \eta}{\partial x}$$

The equation is divided with $\frac{\beta(\eta)}{\beta(z_0)}$, u_η in the first

term on the right hand side substitutes $u_\eta = \frac{\beta(\eta)}{\beta(z_0)} U$ and the second term is moved to the

right hand side, finally:

$$\frac{\partial U}{\partial t} = - \left(\frac{1}{2} \frac{\partial}{\partial x} (u_\eta u_\eta + w_\eta w_\eta) + g \frac{\partial \eta}{\partial x} \right) \frac{\beta(z_0)}{\beta(\eta)} \quad (4.10)$$

This momentum equation will produce horizontal velocity u , in a position $z = z_0$, i.e. U .

V. THE FORMULATION OF WATER SURFACE EQUATION

Equation for water surface $\frac{\partial \eta}{\partial t}$ is formulated

by integrating continuity and energy conservation equation based on Mei [9], where the result of the integration of each equation will produce $\frac{\partial \eta}{\partial t}$. Then,

the last equation is the superposition of both $\frac{\partial \eta}{\partial t}$ equations.

5.1 Equation $\frac{\partial \eta}{\partial t}$ of continuity equation integration

This equation is formulated from continuity equation, i.e. continuity equation integrated to vertical axis z , from the bottom of the water $z = -h$ to the surface $z = \eta$.

$$\int_{-h}^{\eta} \frac{\partial u}{\partial x} dz + w_{\eta} - w_{-h} = 0 \text{ Leibniz rule is applied,}$$

$$\int_z^b \frac{\partial u}{\partial x} dz = \frac{\partial}{\partial x} \int_a^b u dz - u_a \frac{\partial a}{\partial x} + u_b \frac{\partial b}{\partial x}, \text{ according}$$

Dean [5], it becomes

$$\frac{\partial}{\partial x} \int_{-h}^{\eta} u dz - u_{\eta} \frac{\partial \eta}{\partial x} - u_{-h} \frac{\partial h}{\partial x} + w_{\eta} - w_{-h} = 0$$

$$\frac{\partial}{\partial x} \int_{-h}^{\eta} u dz + \frac{\partial \eta}{\partial t} = 0 \text{ or, } \frac{\partial \eta}{\partial t} = - \frac{\partial}{\partial x} \int_{-h}^{\eta} u dz$$

The kinematic water surface boundary and kinematic water base boundary are applied, where $w_{\eta} = \frac{\partial \eta}{\partial t} + u_{\eta} \frac{\partial \eta}{\partial x}$ and $w_{-h} = -u_{-h} \frac{\partial h}{\partial x}$, and similar terms are added and the result is,

$$\frac{\partial}{\partial x} \int_{-h}^{\eta} u dz + \frac{\partial \eta}{\partial t} = 0 \text{ or, } \frac{\partial \eta}{\partial t} = - \frac{\partial}{\partial x} \int_{-h}^{\eta} u dz$$

$$\text{From equation (2.3), } u = \frac{\beta(z)}{\beta(z_0)} U$$

$$\frac{\partial \eta}{\partial t} = - \frac{\partial}{\partial x} \int_{-h}^{\eta} u dz = - \frac{\partial}{\partial x} \int_{-h}^{\eta} \frac{\beta(z)}{\beta(z_0)} U dz$$

$$\frac{\partial \eta}{\partial t} = - \frac{\partial \beta_{uH} U}{\partial x} \tag{5.1}$$

$$\text{where } \beta_{uH} = \frac{\beta_1(\eta) - (\alpha - 1)}{k \beta(z_0)} \tag{5.2}$$

5.2 Equation $\frac{\partial \eta}{\partial t}$ of the integration of energy conservation equation

Energy conservation equation of Hutaean [2] is,

$$\frac{\partial E_k}{\partial t} + \frac{\partial u E_k}{\partial x} + \frac{\partial w E_k}{\partial z} = 0 \tag{5.3}$$

where $E_k = \frac{uu + ww}{2g}$. The product of integration of this equation is,

$$\beta_{Ek} \frac{\partial \eta}{\partial t} = - \frac{1}{2g} \left(\int_{-h}^{\eta} \frac{\beta^2(z)}{\beta^2(z_0)} dz \right) \frac{\partial UU}{\partial t}$$

$$- \frac{1}{2g} \left(\int_{-h}^{\eta} \frac{\beta_1^2(z)}{\beta_1^2(z_0)} dz \right) \frac{\partial WW}{\partial t}$$

$$- \int_{-h}^{\eta} \frac{\partial u E_k}{\partial x} dz - \int_{-h}^{\eta} dw E_k \tag{5.4}$$

Where, dimension β_{Ek} and the right side of the equation are eliminating each other so that β_{Ek} is non-dimensional number.

$$\beta_{Ek} = \frac{1}{2g} \left(\frac{(\beta(2\eta) + 2\alpha)}{\beta^2(z_0)} UU - u_{\eta} u_{\eta} \right)$$

$$+ \frac{1}{2g} \left(\frac{(\beta(2\eta) - 2\alpha)}{\beta_1^2(z_0)} WW - w_{\eta} w_{\eta} \right) \tag{5.5}$$

Where U and W are calculation velocity. Equation (5.4) is super imposed with equation (5.2), and is

written as equation for $\frac{\partial \eta}{\partial t}$,

$$(1 + \beta_{Ek}) \frac{\partial \eta}{\partial t} = - \frac{\partial \beta_{uH} U}{\partial x}$$

$$- \left(\int_{-h}^{\eta} \frac{\beta^2(z)}{\beta^2(z_0)} dz \right) \frac{\partial UU}{\partial t} - \left(\int_{-h}^{\eta} \frac{\beta_1^2(z)}{\beta_1^2(z_0)} dz \right) \frac{\partial WW}{\partial t}$$

$$- \int_{-h}^{\eta} \frac{\partial u E_k}{\partial x} dz - \int_{-h}^{\eta} dw E_k \tag{5.6}$$

Equations for integration on the right hand side of the equation (5.6) can be seen in Appendix.

VI. THE RESULT OF THE MODEL

Momentum and water surface equations are done numerically where spatial differential is done with finite difference method, whereas time differential is done using predictor-corrector method based on numeric integration from Newton-Cote developed by Hutahaean [2]. Then, the numeric model was applied in various cases

6.1 As Water Surface Equation, just Continuity Equation

The model is done using sinusoidal wave with a wave period of 8 seconds and amplitude of 0.80 m, at a constant depth of 10m and 5 m. At a constant depth of 10 m, the result of the model is presented in **Fig. 6.1**, where the wave profile is seen deforming toward Cnoidal wave. Part of the wave valley decreases by 0.20 m, where the lowest valley elevation becomes -0.60m and the top of the wave is a little over 0.80m,

i.e. 0.86 m. So at this depth of 10 m, there are some wave setups. At a depth of 5m, the valley elevation is -0.40 m, whereas the top elevation is 0.94 m. The wave setup that is occurring gets bigger and the Cnoidal profile gets clearer,

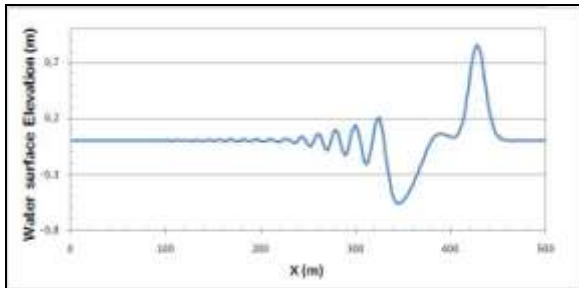


Figure 6.1. The result of the model for a wave with period of 8 seconds, amplitude 0.80 m, constant depth of 10 m.

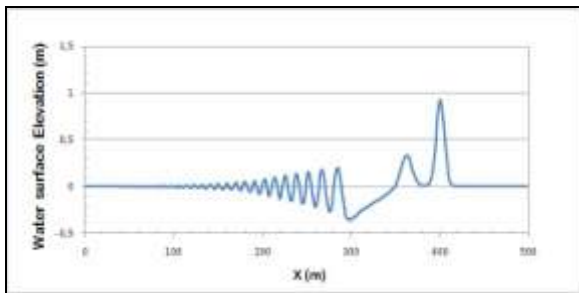


Figure 6.2. The Result of model for a wave with a period of 8 seconds, amplitude 0.80 m, and constant depth of 5 m.

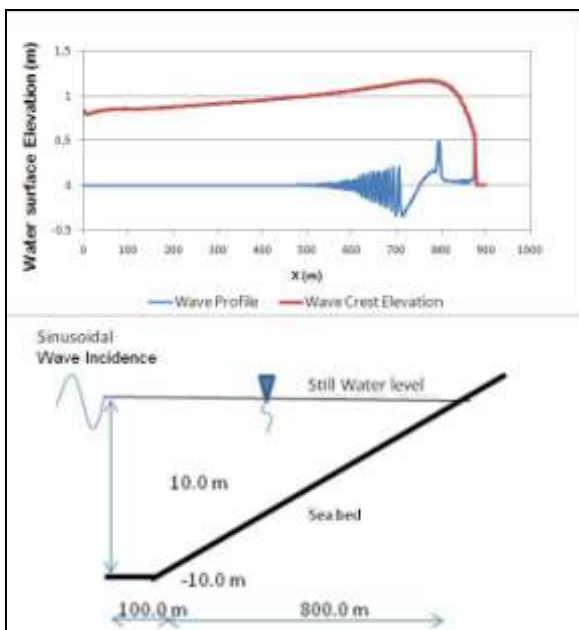


Figure 6.3. The result of model for a wave with a period of 8 seconds, amplitude 0.80 m, changing depth

Then the model was done in the water with not constant depth. Initially, the depth of the water was

constant at 10 m, at a distance span of 100 m. Then, the depth changed continuously until a depth of 0.0 m with a distance span of 800 m. The wave input used is sinusoidal with a period of 8 seconds, amplitude 0.80 m. The results of the model are presented on **Fig. 6.2** and **Fig. 6.3** which shows the increase of water surface elevation as a result of wave setup, and increase in wave amplitude (shoaling) as a result of the shoaling of the water. Then at the peak elevation of 1.18 m, at a depth of 1.77 m, the water surface elevation declines continuously or breaking taking place. Another phenomenon that can be seen in the very shallow water is that the wave profile becomes fully Cnoidal. From the work of the model, on constant as well as changing depth, a conclusion can be made that the model contains the phenomena of wave setup, shoaling and breaking, and produces Cnoidal wave profile.

A. As Water Surface Equation, Superimpose Continuity Equation with Energy Conservation Equation

With similar wave input and bottom of water, as in the previous section, the result of the model is shown on **Fig. 6.4** below. The figure shows that there are shoaling and breaking. The breaking started to occur at a depth of 2.68 m, at the peak of elevation or amplitude of 1.07 m, where initially the breaking occurred continuously and then discontinuously.

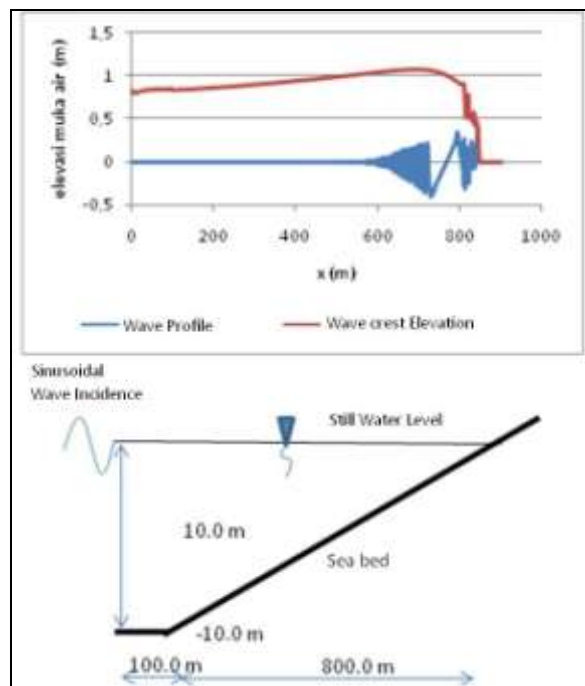


Figure 6.4. The result of the model for a wave with a period of 8 seconds, amplitude of 0.80 m, changing depth, superimpose with Momentum Eq.

Compare with the previous result, the application of the energy conservation equation on the water surface equation produces model with faster breaking where in the previous result, the breaking started at a depth of 1.77 m, amplitude 1.18 m.

Fig. 6.5 shows the result of the model at a constant depth of 2.70 m, with sinusoidal wave input similar to model at changing depth. The elevation of the top of the wave is 0.96 m with Cnoidal wave profile. At that depth, at changing depth, the elevation of the top is 1.07 m. So, the component of the increase of wave amplitude as a result of shoaling is more or less 0,11m. This clarifies that the elevation of top of the wave or the height of wave at shallow water is a result of wave setup and shoaling, with wave setup component that is not small compared with shoaling component.

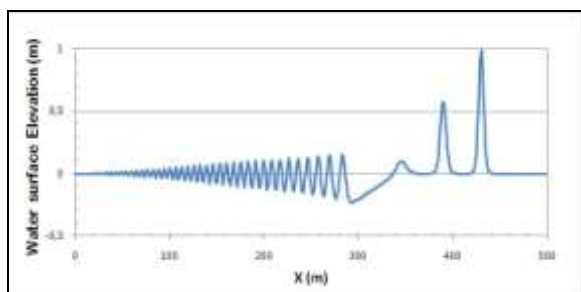


Figure 6.5. The result of the model for a wave with a period of 8 seconds, amplitude of 0.80 m, constant depth of 2.70 m

In addition, the work of energy conservation equation in the water surface equation can model wave unstableness at the shallow water after breaking.

VII. CONCLUSION

The first conclusion is that the formulation of time series water wave equation using velocity equation as the result of Laplace equation solution simplifies the process of integration of basic equation toward a depth, where equation with calculation velocity variable is produced. The use of calculation velocity concept at time series model caused the model to be done very easily at very shallow water.

The model developed in this paper can stimulate wave setup, shoaling and breaking well and produce Cnoidal wave profile. In other words, the model can stimulate various natural phenomena occurring in a wave on its journey to shallow water.

By working on the velocity equation as the result of Laplace equation solution at the formulation of momentum equation and water surface equation, there is wave number k at the momentum equation and water surface equation. In order for the number to correspond with the equation that is used, it is better to use wave number that is calculated from the equation

of kinematic water surface boundary and complete momentum equation, with methods developed by Hutahaean [3], and [4] and Lin et.al. [10].

By using velocity equation as the result of Laplace equation solution at the formulation of governing equation, the velocity produced by momentum equation will be more controlled by continuity equation, considering Laplace equation originated from continuity equation.

The work on energy conservation equation on water surface equation causes the wave to be faster in breaking than without energy conservation equation. In addition, wave instability after breaking is also visible on the water surface equation which is a superposition between continuity equation and energy conservation equation.

APPENDIX: INTEGRATION OF ENERGY CONSERVATION EQUATION

Energy conservation equation, i.e. equation (5.3), is multiplied with dz and integrated with depth,

$$\int_{-h}^{\eta} \frac{\partial E_k}{\partial t} dz + \int_{-h}^{\eta} \frac{\partial u E_k}{\partial x} dz + \int_{-h}^{\eta} \partial w E_k = 0$$

On the integration of this energy conservation equation, the concept of calculation velocity U and W is done, where $U = u(x, z_0, t)$ and $W = u(x, z_0, t)$ where the relation of this velocity

with velocity at any position of z is, $\frac{u}{U} = \frac{\beta(z)}{\beta(z_0)}$

$$\text{and } \frac{w}{W} = \frac{\beta_1(z)}{\beta_1(z_0)}$$

$$\text{a. } \int_{-h}^{\eta} \frac{\partial E_k}{\partial t} dz = \frac{1}{2g} \int_{-h}^{\eta} \frac{\partial}{\partial t} (u^2 + w^2) dz$$

$$\text{a.1. } \int_{-h}^{\eta} \frac{\partial u^2}{\partial t} dz$$

Based on the concept of calculation velocity,

where $uu = \frac{\beta(z)\beta(z)}{\beta^2(z_0)}UU$, then

$$\int_{-h}^{\eta} uudz = UU \int_{-h}^{\eta} \frac{\beta(z)\beta(z)}{\beta^2(z_0)} dz$$

The work on Leibniz rule,

$$\int_{-h}^{\eta} \frac{\partial uu}{\partial t} dz = \frac{\partial}{\partial t} \int_{-h}^{\eta} uudz - u_{\eta} u_{\eta} \frac{\partial \eta}{\partial t}$$

The work on depth-averaged velocity

$$\int_{-h}^{\eta} \frac{\partial uu}{\partial t} dz = \frac{\partial}{\partial t} \left(UU \int_{-h}^{\eta} \frac{\beta^2(z)}{\beta^2(z_0)} dz \right) - u_{\eta} u_{\eta} \frac{\partial \eta}{\partial t} + \left(\frac{(\beta(2\eta) - 2\alpha)}{\beta_1^2(z_0)} WW - w_{\eta} w_{\eta} \right) \frac{\partial \eta}{\partial t}$$

$$\int_{-h}^{\eta} \frac{\partial uu}{\partial t} dz = \left(\int_{-h}^{\eta} \frac{\beta^2(z)}{\beta^2(z_0)} dz \right) \frac{\partial UU}{\partial t}$$

$$+ UU \frac{\partial}{\partial t} \int_{-h}^{\eta} \frac{\beta^2(z)}{\beta^2(z_0)} dz - u_{\eta} u_{\eta} \frac{\partial \eta}{\partial t}$$

$$\int_{-h}^{\eta} \frac{\partial uu}{\partial t} dz = \left(\int_{-h}^{\eta} \frac{\beta^2(z)}{\beta^2(z_0)} dz \right) \frac{\partial UU}{\partial t}$$

$$+ \left(\frac{(\beta(2\eta) + 2\alpha)}{\beta^2(z_0)} UU - u_{\eta} u_{\eta} \right) \frac{\partial \eta}{\partial t}$$

where,

$$\int_{-h}^{\eta} \beta^2(z) dz = \left(\frac{1}{2k} (\beta_1(2\eta) - (\alpha^2 - 1)) + 2\alpha H \right)$$

$$\text{and } \frac{\partial}{\partial t} \int_{-h}^{\eta} \beta^2(z) dz = (\beta(2\eta) + 2\alpha) \frac{\partial \eta}{\partial t}$$

$$\text{a.2. } \int_{-h}^{\eta} \frac{\partial w^2}{\partial t} dz$$

From the concept of calculation velocity,

$$\text{then, } ww = \frac{\beta_1(z)\beta_1(z)}{\beta_1^2(z_0)} WW \text{ then}$$

$$\int_{-h}^{\eta} wwdz = WW \int_{-h}^{\eta} \frac{\beta(z)\beta(z)}{\beta^2(z_0)} dz$$

The work on Leibniz integration

$$\int_{-h}^{\eta} \frac{\partial ww}{\partial t} dz = \frac{\partial}{\partial t} \int_{-h}^{\eta} wwdz - w_{\eta} w_{\eta} \frac{\partial \eta}{\partial t}$$

Substitute relation of calculation velocity,

$$\int_{-h}^{\eta} \frac{\partial ww}{\partial t} dz = \frac{\partial}{\partial t} \left(WW \int_{-h}^{\eta} \frac{\beta(z)\beta(z)}{\beta^2(z_0)} dz \right) - w_{\eta} w_{\eta} \frac{\partial \eta}{\partial t} + \frac{3\alpha}{k} (\beta_1(\eta) - (\alpha - 1))$$

$$\int_{-h}^{\eta} \frac{\partial ww}{\partial t} dz = \left(\int_{-h}^{\eta} \frac{\beta_1^2(z)}{\beta_1^2(z_0)} dz \right) \frac{\partial WW}{\partial t}$$

$$+ WW \frac{\partial}{\partial t} \int_{-h}^{\eta} \frac{\beta_1^2(z)}{\beta_1^2(z_0)} dz - w_{\eta} w_{\eta} \frac{\partial \eta}{\partial t}$$

$$\int_{-h}^{\eta} \frac{\partial ww}{\partial t} dz = \left(\int_{-h}^{\eta} \frac{\beta_1^2(z)}{\beta_1^2(z_0)} dz \right) \frac{\partial WW}{\partial t}$$

Where

$$\int_{-h}^{\eta} \beta_1^2(z) dz = \left(\frac{1}{2k} (\beta_1(2\eta) - (\alpha^2 - 1)) - 2\alpha H \right),$$

then

$$\frac{\partial}{\partial t} \int_{-h}^{\eta} \beta_1^2(z) dz = (\beta(2\eta) - 2\alpha) \frac{\partial \eta}{\partial t}$$

$$\text{b. } \int_{-h}^{\eta} \frac{\partial u E_k}{\partial x} dz = \frac{1}{2g} \int_{-h}^{\eta} \frac{\partial}{\partial x} (u(u^2 + v^2 + w^2)) dz$$

$$\text{b.1. } \int_{-h}^{\eta} \frac{\partial uu^2}{\partial x} dz$$

$$\frac{uu^2}{UU^2} = \frac{\beta^3(z)}{\beta^3(z_0)} \text{ or } uu^2 = \frac{\beta^3(z)}{\beta^3(z_0)} UU^2 \text{ then}$$

$$\int_{-h}^{\eta} uu^2 dz = UU^2 \int_{-h}^{\eta} \frac{\beta^3(z)}{\beta^3(z_0)} dz$$

The work on Leibniz rule

$$\int_{-h}^{\eta} \frac{\partial uu^2}{\partial x} dz = \frac{\partial}{\partial x} \int_{-h}^{\eta} uu^2 dz - u_{\eta} u_{\eta}^2 \frac{\partial \eta}{\partial x} - u_{-h} u_{-h}^2 \frac{\partial h}{\partial x}$$

The work on the concept of calculation

velocity

$$\int_{-h}^{\eta} \frac{\partial uu^2}{\partial x} dz = \frac{\partial}{\partial x} \left(UU^2 \int_{-h}^{\eta} \frac{\beta^3(z)}{\beta^3(z_0)} dz \right) - u_{\eta} u_{\eta}^2 \frac{\partial \eta}{\partial x} - u_{-h} u_{-h}^2 \frac{\partial h}{\partial x}$$

Where

$$\int_{-h}^{\eta} \beta^3(z) dz = \frac{1}{3k} (\beta_1(3\eta) - (\alpha^3 - 1))$$

$$\text{b.2. } \int_{-h}^{\eta} \frac{\partial uw^2}{\partial x} dz$$

$$uw^2 = \frac{\beta(z)\beta_1^2(z)}{\beta(z_0)\beta_1^2(z_0)} UW^2$$

$$\text{then } \int_{-h}^{\eta} uw^2 dz = UW^2 \int_{-h}^{\eta} \frac{\beta(z)\beta_1^2(z)}{\beta(z_0)\beta_1^2(z_0)} dz$$

The work on Leibniz integration

$$\int_{-h}^{\eta} \frac{\partial uw^2}{\partial x} dz = \frac{\partial}{\partial x} \int_{-h}^{\eta} uw^2 dz - u_{\eta} w_{\eta}^2 \frac{\partial \eta}{\partial x} - u_{-h} w_{-h}^2 \frac{\partial h}{\partial x}$$

Substitute calculation velocity,

$$\int_{-h}^{\eta} \frac{\partial uw^2}{\partial x} dz = \frac{\partial}{\partial x} \left(UW^2 \int_{-h}^{\eta} \frac{\beta(z)\beta_1^2(z)}{\beta(z_0)\beta_1^2(z_0)} dz \right) - u_{\eta} w_{\eta}^2 \frac{\partial \eta}{\partial x} - u_{-h} w_{-h}^2 \frac{\partial h}{\partial x}$$

Where, $\beta(z)\beta_1^2(z) = \beta(3z) + \alpha\beta_1(z)$

$$\int_{-h}^{\eta} \beta(z)\beta_1^2(z) dz = \frac{1}{3k} (\beta_1(3\eta) - (\alpha^3 - 1)) + \frac{\alpha}{k} (\beta(\eta) - (\alpha + 1))$$

$$d. \int_{-h}^{\eta} dwE_k = \frac{1}{2g} (w_{\eta} (u_{\eta}^2 + v_{\eta}^2 + w_{\eta}^2) - w_{-h} (u_{-h}^2 + v_{-h}^2 + w_{-h}^2))$$

In general, the integration of energy conservation equation can be written as,

$$\beta_{Ek} \frac{\partial \eta}{\partial t} = - \frac{1}{2g} \left(\int_{-h}^{\eta} \frac{\beta^2(z)}{\beta^2(z_0)} dz \right) \frac{\partial UU}{\partial t} - \frac{1}{2g} \left(\int_{-h}^{\eta} \frac{\beta_1^2(z)}{\beta_1^2(z_0)} dz \right) \frac{\partial WW}{\partial t} - \int_{-h}^{\eta} \frac{\partial uE_k}{\partial x} dz - \int_{-h}^{\eta} dwE_k \quad (a.1)$$

where

$$\beta_{Ek} = \frac{1}{2g} \left(\frac{(\beta(2\eta) + 2\alpha)}{\beta^2(z_0)} UU - u_{\eta} u_{\eta} \right) + \frac{1}{2g} \left(\frac{(\beta(2\eta) - 2\alpha)}{\beta_1^2(z_0)} WW - w_{\eta} w_{\eta} \right) \quad (a.2)$$

In this case, β_{Ek} looks as if it had meter dimension, but the dimension is eliminating each other with the right side of the equation.

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