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Some common Fixed Point Theorems for compatible ψ -contractions in G-metric Spaces

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ABSTRACT
We prove some common fixed point theorems for compatible self mappings satisfying some kind of contractive type conditions on complete G-metric spaces and obtain results of Kumara Swamy and Phaneendra[6] and Sushanta Kumar Mohanta[16] as corollaries.
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I. INTRODUCTION

The study of metric fixed point theory plays an important role because the study finds applications in many important areas as diverse as differential equations, operation research, mathematical economics and the like. Different generalizations of the usual notion of a metric space were proposed by several mathematicians such as Gahler [3,4] (called 2-metric spaces) and Dhage [1,2] (called D-metric spaces). K.S.Ha et al. [5] have pointed out that the results cited by Gahler are independent, rather than generalizations, of the corresponding results in metric spaces. Moreover, it was shown that Dhage's notion of D-metric space is flawed by errors and most of the results established by him and others are invalid. These facts determined Mustafa and Sims [11] to introduce a new concept in the area, called G-metric space. Recently, Mustafa et al. studied many fixed point theorems for mappings satisfying various contractive conditions on complete G-metric spaces; see [8-13]. Subsequently, some authors like Renu Chugh et al.[14], W.Shatanawi [15] have generalized some results of Mustafa et al. [7-8] and studied some fixed point results for self-mappings in a complete G-metric space under some contractive conditions related to a non-decreasing

$$\phi : [0,+\infty) \to [0,+\infty) \text{ with } \lim_{n\to\infty} \phi^n(t) = 0 \text{ for all } t \in (0,+\infty).$$

Kumara Swamy and Phaneendra[6] and Sushanta Kumar Mohanta[16] proved some fixed point theorems for self-mappings on complete G-metric spaces.

In this paper we introduce ψ -contractions in G-metric spaces, prove fixed point results for such maps and obtain results of Kumara Swamy and Phaneendra [6].

II. PRELIMINARIES

We begin by briefly recalling some basic definitions and results for G-metric spaces that will be needed in the sequel.

Definition 2.1 :-(Mustafa and Sims [7]) Let X be a non–empty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:

$(G1)$ $G(x,y,z) = 0$ if $x = y = z$,

$(G2)$ $0 < G(x,x,y)$, for all $x, y \in X$, with $x \neq y$,

$(G3)$ $G(x,x,y) \le G(x,y,z)$, for all $x,y,z \in X$, with $z \neq y$,

$(G4)$ $G(x,y,z) = G(\pi(x,z,y))$ where π is a permutation in $\{x,y,z\}$,

$(G5)$ $G(x,y,z) \le G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$, (rectangle inequality).

Then the function G is called a generalized metric, or, more specifically a G- metric on X , and the pair (X, G) is called a G-metric space.

Example 2.2:- (Mustafa and Sims [7]) Let R be the set of all real numbers define.

$G : R \times R \times R \to R^{+}$ by

$$G(x, y, z) = |x - y| + |y - z| + |z - x|, \text{ for all } x, y, z \in X.$$

Then it is clear that (R, G) is a G-metric space.

We use the following proposition in the Sequel without explicit mention.

Proposition 2.3:- (Mustafa and Sims [7]) Let (X, G) be a G-metric space. Then for any $x, y, z, \text{ and } a \in X$, it follows that

(1) if $G(x, y, z) = 0$ then $x = y = z$,

(2) $G(x, y, z) \le G(x, x, y) + G(x, x, z)$,

(3) $G(x, y, y) \le 2G(y, x, x)$, $\longrightarrow$ $(2.3.1)$

(4) $G(x, y, z) \le G(x, a, z) + G(a, y, z)$,

(5) $G(x, y, z) \le \frac{2}{3}\left(G(x, y, a) + G(x, a, z) + G(a, y, z)\right)$,

(6) $G(x, y, z) \le G(x, a, a) + G(y, a, a) + G(z, a, a)$.

Definition 2.4: (Mustafa and Sims [7]) Let (X, G) be a G-metric space, let $\{x_n\}$ be a sequence of points of X , we say that $\{x_n\}$ is G-convergent to X if $\lim_{n,m \to \infty} G(x, x_n, x_m) = 0$; that is, for any $\in > 0$, there exists $n_0 \in N$ such that $G(x, x_n, x_m) < \in$, for all $n, m, \ge n_0$. We refer to x as the limit of the sequence $\{x_n\}$ and write $x_n \longrightarrow x$.

The following proposition is used in the section 3.

Proposition 2.5:- (Mustafa and Sims [7]) Let (X, G) be a G-metric space. Then, the following are equivalent:

(1) $\{x_n\}$ is G-convergent to x.

(2) $G(x_n, x_n, x) \to 0$, as $n \to \infty$.

(3) $G(x_n, x, x) \to 0$, as $n \to \infty$.

(4) $G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.

Definition 2.6:- (Mustafa and Sims [7]) Let (X, G) be a G-metric space, sequence $\{x_n\}$ is called G-Cauchy if given $\in > 0$, there is $n_0 \in N$ such that $G(x_n, x_m, x_l) < \in$, for all $n, m, l \ge n_0$ that is if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 2.7:- (Mustafa and Sims [7]) In a G-metric space (X, G), the following are equivalent:

(1) The sequence $\{x_n\}$ is G-Cauchy.

(2) For every $\in > 0$, there exists $n_0 \in N$ such that $G(x_n, x_m, x_m) < \in$ for all $n, m \geq n_0$.

Definition 2.8:- (Mustafa and Sims [7]) Let (X, G) and (X', G') be G-metric spaces and let $f : (X, G) \to (X', G')$ be a function, then f is said to be G-continuous at a point $a \in X$ if given $\in > 0$, there exists $\delta > 0$ such that $x, y \in X; G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \in$. A function f is G-continuous on X if and only if it is G-continuous at all $a \in X$.

Proposition 2.9:- (Mustafa and Sims [7]) Let (X, G) and (X', G') be G-metric spaces, then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.

Proposition 2.10:- (Mustafa and Sims [7]) Let (X, G) be a G-metric space. Then, the function $G(x, y, z)$ is continuous in all variables.

Definition 2.11:- (Mustafa and Sims [7]) A G-metric space (X, G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Definition 2.12:- Two self-maps f and g on a G-metric space (X, G) are said to be compatible if $\lim_{n \to \infty} G(fgx_n, gfx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = p$ for some $p \in X$.

3. Main Result:-

Notation:-

$$\Psi = \left\{ \psi : [0, \infty) \to [0, \infty) / \psi \text{ is increasing, continuous and } \sum_{n+1}^{\infty} \psi^n(t) < \infty \text{ for } t > 0 \right\}.$$

We observe that $\psi(0) = 0$ and $\psi(t) < t$. (see Sastry,KPR et al.[15]) We also observe that, if we define $\psi(t) = k\, t$ for $t > 0$, where $0 < k < 1$ then $\psi \in \Psi$.

Lemma 3.1:- Suppose $\lambda > 1$, and

$\psi : [0, \infty) \to [0, \infty)$ is increasing and $\psi(t) \leq \dfrac{t}{\lambda} \longrightarrow (3.1.1)$. Suppose $\{\alpha_n\}$ is a sequence of non-negative real numbers such that $\alpha_{n+1} \leq \psi(a\alpha_n + b\alpha_{n+1}) \longrightarrow (3.1.2)$

Where a,b are positive real numbers such that a+b= λ .

Then $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{k=n}^{m} \alpha_k \to 0$ as $m, n \to \infty$.

Proof:- First we observe that $\psi^n(t) \le \frac{t}{\lambda^n}$ for $n = 1, 2, 3, \ldots . n$

Now

$$\alpha_{n+1} \le \psi \frac{(a\alpha_n + b\alpha_{n+1})}{\lambda} \le \frac{(a\alpha_n + b\alpha_{n+1})}{\lambda}. \text{Hence } \alpha_n < \alpha_{n+1} \Rightarrow \alpha_{n+1} \le \frac{(a\alpha_n + b\alpha_{n+1})}{\lambda} < \frac{(a+b)\alpha_{n+1}}{\lambda} = \alpha_{n+1},$$

a contradiction.

$\therefore \alpha_{n+1} \le \alpha_n \forall n.$

Thus $\{\alpha_n\}$ is a decreasing sequence

$$\therefore \alpha_{n+1} \le \psi \frac{(a\alpha_n + b\alpha_{n+1})}{\lambda} \le \psi \frac{((a+b)\alpha_n)}{\lambda} \quad (\text{since } \psi \text{ is increasing})$$

$$= \psi(\lambda\alpha_n) \le \psi(\alpha_n)$$

$$\therefore \alpha_{n+1} \le \psi(\alpha_n) \le \psi(\psi(\alpha_{n-1})) \le \ldots\ldots.. \le \psi^{n+1}(\alpha_0) \le \frac{\alpha_0}{\lambda^{n+1}}.$$

$$\therefore \sum_{n=1}^{\infty} \alpha_n \le \sum_{n=1}^{\infty} \frac{\alpha_0}{\lambda^n} < \infty \quad (\text{since } \lambda > 1)$$

Hence $\sum_{k=1}^{m} \alpha_k \to 0$ as $m, n \to \infty$.

Notation:- Let $\lambda > 1$.

Write $\Psi_\lambda = \left\{ \psi_\lambda : [0, \infty) \to [0, \infty) / \psi_\lambda \text{ is increasing, continuouus and } \psi_\lambda(t) \le \frac{t}{\lambda} \right\}.$

We observe that

$$\psi_\lambda \in \Psi \text{ since } \sum \psi_\lambda^n(t) \le \sum \frac{t}{\lambda^n} < \infty$$

Thus $\Psi_\lambda \subset \Psi$ for $\lambda > 1$

$\left(\text{Here } \psi^n(t) = \psi(\psi^{n-1}(t)), n = 1, 2, \ldots \text{ with } \psi^0(t) = \psi(t)\right)$

We prove the following common fixed point theorem.

Theorem 3.2:- Suppose that f and g are self-maps on a complete G-metric space (X, G) such that

(a) $f(X) \subset g(X)$

(b) $\psi \in \Psi$

(c) f or g is continuous,

(d)
$$G(fx, fy, fz) \le \psi(\max\{G(gx, fx, fx) + G(gy, fy, fy) + G(gz, fz, fz),$$
$$G(gx, fy, fy) + G(gy, fx, fx) + G(gz, fy, fy),$$
$$G(gx, fz, fz) + G(gy, fz, fz) + G(gz, fx, fx)\})$$
for all $x, y, z \in X$ ....(3.2.1)

(e) f and g are compatible.

Then f and g have a unique common fixed point.

Proof:- Let $x_0 \in X$. In view of (a), we can choose points

$x_1, x_2, \dots, x_n \dots$ such that $fx_{n-1} = gx_n$ for $n = 1, 2, \dots\dots (3.2.2)$.

Writing $x = x_n$ and $y = z = x_{n+1}$ in $(3.2.1)$ and then using $(3.2.2)$, we have

$$\begin{aligned} G(fx_n, fx_{n+1}, fx_{n+1}) \le \psi\big(\max\{&G(fx_{n-1}, fx_n, fx_n) + G(fx_n, fx_{n+1}, fx_{n+1}) + G(fx_n, fx_{n+1}, fx_{n+1}),\\ &G(fx_{n-1}, fx_{n+1}, fx_{n+1}) + G(fx_n, fx_n, fx_n) + G(fx_n, fx_{n+1}, fx_{n+1}),\\ &G(fx_{n-1}, fx_{n+1}, fx_{n+1}) + G(fx_n, fx_{n+1}, fx_{n+1}) + G(fx_n, fx_n, fx_n)\}\big)\\ \le \psi\big(\max\{&G(fx_{n-1}, fx_n, fx_n) + 2G(fx_n, fx_{n+1}, fx_{n+1}),\\ &G(fx_{n-1}, fx_{n+1}, fx_{n+1}) + G(fx_n, fx_{n+1}, fx_{n+1})\}\big) \rightarrow (3.2.3) \end{aligned}$$

From (G_5), we have

$G(fx_{n-1}, fx_{n+1}, fx_{n+1}) \le G(fx_{n-1}, fx_n, fx_n) + G(fx_n, fx_{n+1}, fx_{n+1})$

From (2.3.1) we get
$$\begin{aligned} G(fx_{n-1}, fx_{n+1}, fx_{n+1}) + G(fx_n, fx_{n+1}, fx_{n+1}) &\le G(fx_{n-1}, fx_n, fx_n) + G(fx_n, fx_{n+1}, fx_{n+1})\\ &\quad + G(fx_n, fx_{n+1}, fx_{n+1})\\ &= G(fx_{n-1}, fx_n, fx_n) + 2G(fx_n, fx_{n+1}, fx_{n+1}) \rightarrow (3.2.4) \end{aligned}$$

From $(3.2.3)$ and $(3.2.4)$ we have

$$\begin{aligned} G(fx_n, fx_{n+1}, fx_{n+1}) \le \psi\big(\max\{&G(fx_{n-1}, fx_n, fx_n) + 2G(fx_n, fx_{n+1}, fx_{n+1}),\\ &G(fx_{n-1}, fx_n, fx_n) + 2G(fx_n, fx_{n+1}, fx_{n+1})\}\big)\\ = \psi\big(&G(fx_{n-1}, fx_n, fx_n) + 2G(fx_n, fx_{n+1}, fx_{n+1})\big) \rightarrow (3.2.5) \end{aligned}$$

Write $\alpha_n = G(fx_{n-1}, fx_n, fx_n)$. Then from $(3.2.5)$, we have

$\alpha_{n+1} \le \psi(\alpha_n + 2\alpha_{n+1})\ \forall n \quad \rightarrow (3.2.6)$

By taking $\lambda = 3, a = 1$ and $b = 2$ in lemma 2.1 it follows that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and hence

$$\sum_{k=n}^{m} \alpha_k \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

Now, by (G_5),

$G(fx_n, fx_m, fx_m) \le G(fx_n, w, w) + G(w, fx_m, fx_m)$

Put $w = fx_{n+1}$

$$\begin{aligned} \therefore G(fx_n, fx_m, fx_m) &\le G(fx_n, fx_{n+1}, fx_{n+1}) + G(fx_{n+1}, fx_m, fx_m)\\ &\le G(fx_n, fx_{n+1}, fx_{n+1}) + G(fx_{n+1}, fx_{n+2}, fx_{n+2}) + G(fx_{n+2}, fx_m, fx_m) \end{aligned}$$

$\therefore$ By induction,

$$G(fx_n, fx_m, fx_m) \le \sum_{i=0}^{m-n-1} G(fx_{n+i}, fx_{n+i+1}, fx_{n+i+1})$$

$$= \sum_{i=0}^{m-n-1} \alpha_{n+i+1} \to 0 \text{ as } m, n \to \infty \quad (\text{by lemma } 3.1)$$

$$\therefore G(fx_n, fx_m, fx_m) \to 0 \text{ as } m, n \to \infty$$

and hence $G(gx_{n-1}, gx_{m-1}, gx_{m-1}) \to 0$ as $m, n \to \infty$

Thus $\{gx_n\}$ is Cauchy and $\{fx_n\}$ is Cauchy.

Since X is G-complete, we can find a point $p \in X$ such that

$$\lim_{n\to\infty} fx_{n-1} = \lim_{n\to\infty} gx_n = p. \qquad (3.2.7)$$

Suppose that g is continuous.

Then $\lim_{n\to\infty} gfx_{n-1} = \lim_{n\to\infty} ggx_n = gp. \qquad (3.2.8)$

Since f and g are compatible,

$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$ which implies that,

$$\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} fgx_n = gp. \qquad (3.2.9)$$

But, from $(3.2.1)$, we see that

$$\begin{aligned} G(fgx_{n+1}, fx_n, fx_n) = G(ffx_n, fx_n, fx_n) \le \psi\big(\max\{&G(gfx_n, ffx_n, ffx_n) + G(gx_n, fx_n, fx_n) + G(gx_n, fx_n, fx_n), \\ &G(gfx_n, fx_n, fx_n) + G(gx_n, ffx_n, ffx_n) + G(gx_n, fx_n, fx_n) \\ &G(gfx_n, fx_n, fx_n) + G(gfx_n, fx_n, fx_n) + G(gx_n, ffx_n, ffx_n)\}\big) \end{aligned}$$

On taking the limit as $n \to \infty$, in view of (3.2.7),(3.2.8) and (3.2.9), it follow that

$$\begin{aligned} G(gp, p, p) \le \psi\big(\max\{&G(gp, gp, gp) + G(p, p, p) + G(p, p, p), \\ &G(gp, p, p) + G(p, gp, gp) + G(p, p, p), \\ &G(gp, p, p) + G(p, p, p) + G(p, gp, gp)\}\big) \end{aligned}$$

$= \psi\big(G(gp, p, p) + G(p, gp, gp)\big)$

$\le \psi\big(G(gp, p, p) + 2G(gp, p, p)\big)$ from $(2.3.1)$

$= \psi\big(3G(gp, p, p)\big)$

$\therefore G(gp, p, p) = 0$

$\therefore gp = p$ (from (G_2))

$\therefore p$ is a fixed point of g.

Now, $G(fx_n, fp, fp) \le \psi\big(\max\{G(gx_n, fx_n, fx_n) + G(gp, fp, fp) + G(gp, fp, fp),$

$G(gx_n, fp, fp) + G(gp, fx_n, fx_n) + G(gp, fp, fp),$

$G(gx_n, fp, fp) + G(gp, fp, fp) + G(gp, fx_n, fx_n)\}\big)$

$\therefore$ Proceeding to the limit as $n \to \infty$, we get

$$G(p, fp, fp) \le \psi\big(\max\{G(p,p,p) + G(gp, fp, fp) + G(gp, fp, fp),$$
$$G(p, fp, fp) + G(gp, p, p) + G(gp, fp, fp),$$
$$G(p, fp, fp) + G(gp, fp, fp) + G(gp, p, p)\}\big)$$

$$= \psi\big(\max\{0, 2G(p, fp, fp), 2G(p, fp, fp), 2G(p, fp, fp)\}\big)$$

$$= \psi\big(2G(p, fp, fp)\big)$$

$$= \psi\left(3.\frac{2}{3}G(p, fp, fp)\right) \le \frac{2}{3}G(p, fp, fp)$$

$\therefore G(p, fp, fp) = 0.$

$\therefore$ By (G_2), $fp = p$

$\therefore$ p is a common fixed point of f and g.

Suppose p and q are common fixed points of f and g.

so that $fp = qp = p$ and $fq = gq = q$.

Now from (3.2.1), we have

$$G(p,q,q) = G(fp, fq, fq) \le \psi\big(\max\{G(gp, fp, fp) + G(gq, fq, fq) + G(gq, fq, fq),$$
$$G(gp, fq, fq) + G(gq, fp, fp) + G(gq, fq, fq),$$
$$G(gp, fq, fq) + G(gq, fq, fq) + G(gq, fp, fp)\}\big)$$

$$= \psi\big(\max\{G(p,p,p) + G(q,q,q) + G(q,q,q),$$
$$G(p,q,q) + G(q,p,p) + G(q,q,q),$$
$$G(p,q,q) + G(q,q,q) + G(q,p,p)\}\big)$$

$$= \psi\big(\max\{0 + 0 + 0, G(p,q,q) + G(q,p,p) + 0,$$
$$G(p,q,q) + 0 + G(q,p,p)\}\big)$$

$$= \psi\{G(p,q,q) + G(q,p,p)\}$$

$$\le \psi\big(G(p,q,q) + 2G(q,q,p)\big) \quad (\text{from } (2.3.1))$$

$$= \psi\big(3(G(p,q,q)\big)$$

$$\therefore G(p,q,q) \le \psi\big(3(G(p,q,q)\big)$$

$$\therefore G(p,q,q) = 0.$$

$$\therefore p = q.$$

Thus f and g have unique common fixed point.

From theorem (3.2), we obtain the following result of K.Kumara Swamy and T.Phaneendra [6] as corollary.

Theorem 3.3:- (K.Kumara swamy and T.Phaneendra [6]) Suppose that f and g are self-maps on a complete G-metric space (X, G) such that

(a) $f(X) \subset g(X)$

(b) f or g is continuous, and $0 < k < \frac{1}{3}$.

(c) f and g are compatible.

$$\text{Suppose } G(fx, fy, fz) \le k\left(\max\{G(gx, fx, fx) + G(gy, fy, fy) + G(gz, fz, fz),\right.$$
$$G(gx, fy, fy) + G(gy, fx, fx) + G(gz, fy, fy),$$
$$\left.G(gx, fz, fz) + G(gy, fz, fz) + G(gz, fx, fx)\}\right)$$
$$\text{for all } x, y, z \in X \text{ ....} (3.3.1)$$

Proof:- Take $\psi(t) = kt$ where $0 < k < \frac{1}{3}$.

The following result of Sushanta Kumar Mohanta([16], Theorem 3.9)follows as a corollary of theorem 3.2, by taking $\psi(t) = kt$.

Theorem 3.4:- (Sushanta Kumar Mohanta([16], Theorem 3.9)) Let (X, G) be a complete G-metric space, and let $T : X \to X$ be a mapping which satisfies the following condition

$$G(T(x), T(y), T(z)) \le k\left(\max\{G(x, T(y), T(y)) + G(y, T(x), T(x)) + G(z, T(z), T(z)),\right.$$
$$G(y, T(z), T(z)) + G(z, T(y), T(y)) + G(x, T(x), T(x)),$$
$$\left.G(z, T(x), T(x)) + G(x, T(z), T(z)) + G(y, T(y), T(y))\}\right)$$
$$\forall\ x, y, z \in X, \text{ and } 0 \le k < \frac{1}{3}.$$

Then T has a unique fixed point in X.

Proof:- Take $f = g = T$ and $\psi(t) = kt$ in theorem 3.2.

Theorem 3.5:- Suppose that f and g are self-maps on a complete G-metric space (X, G) such that

(a) $f(X) \subseteq g(X)$

(b) $\psi \in \Psi$

(c) f or g is continuous,

(d) $G(fx, fy, fz) \le \psi\left(\max\{G(gx, gy, gz), G(gx, fx, fx), G(gy, fy, fy),\right.$
$$\left.G(gz, fz, fz)\}\right)$$
$$\text{for all } x, y, z \in X \text{ ....} (3.5.1)$$

(e) f and g are compatible.

Then f and g have a unique common fixed point.

Proof:- Let $x_0 \in X$. In view of (a), we can choose points

$x_1, x_2, \ldots, x_n \ldots$ such that $f(x_n) = gx_{n+1},\ n = 0,1,2,\ldots$

Write $x = x_n$ and $y = z = x_{n+1}$ in $(3.5.1)$ we get

$$G(fx_n, fx_{n+1}, fx_{n+1}) \le \psi\left(\max\left\{G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_n, fx_n, fx_n), G(gx_{n+1}, fx_{n+1}, fx_{n+1}),\right.\right.$$
$$\left.\left. G(gx_{n+1}, fx_{n+1}, fx_{n+1})\right\}\right)$$

$$= \psi\left(\max\left\{G(fx_{n-1}, fx_n, fx_n), G(fx_{n-1}, fx_n, fx_n), G(fx_n, fx_{n+1}, fx_{n+1}),\right.\right.$$
$$\left.\left. G(fx_n, fx_{n+1}, fx_{n+1})\right\}\right)$$

$$= \psi\left(\max\left\{G(fx_{n-1}, fx_n, fx_n), G(fx_n, fx_{n+1}, fx_{n+1})\right\}\right)$$

$$= \psi\left(\max\left\{\alpha_{n-1}, \alpha_n\right\}\right), \text{ where } \alpha_n = G(fx_n, fx_{n+1}, fx_{n+1})$$

$$\therefore \alpha_n \le \psi\left(\max\left\{\alpha_{n-1}, \alpha_n\right\}\right) \longrightarrow (3.5.2)$$

Now $\alpha_{n-1} < \alpha_n \Rightarrow \alpha_n \le \psi(\alpha_n)$ a contradiction.

$\therefore \alpha_n \le \alpha_{n-1}$. Thus $\{\alpha_n\}$ is a decreasing sequence and $\alpha_n \le \psi(\alpha_{n-1})$ $\left(\text{from } (3.5.2)\right)$

Suppose $\alpha_n \downarrow r$. Thus $\psi(\alpha_n) \downarrow \psi(r)$

$\therefore \alpha_n \le \psi(\alpha_{n-1}) \forall n \Rightarrow r \le \psi(r) \Rightarrow r = 0.$

Since $\alpha_n \le \psi(\alpha_{n-1})$, by induction we have

$$\alpha_n \le \psi^n(\alpha_0)$$

Hence $\sum \alpha_n \le \sum \psi^n(\alpha_0) < \infty$

So that for $n < m$,

$$\sum_{i=0}^{m-n-1} \alpha_{n+i+1} \to 0 \quad \text{as } n, m \to \infty \longrightarrow (3.5.3)$$

Suppose $n < m$. Put $x = x_n, y = z = x_m$ in $(3.5.1)$

We get, by(G_5)

$$G(fx_n, fx_m, fx_m) \le G(fx_n, w, w) + G(w, fx_m, fx_m)$$

Put $w = fx_{n+1}$

$$\therefore G(fx_n, fx_m, fx_m) \le G(fx_n, fx_{n+1}, fx_{n+1}) + G(fx_{n+1}, fx_m, fx_m)$$
$$\le G(fx_n, fx_{n+1}, fx_{n+1}) + G(fx_{n+1}, fx_{n+2}, fx_{n+2}) + G(fx_{n+2}, fx_m, fx_m)$$

$\therefore$ By induction,

$$G(fx_n, fx_m, fx_m) \le \sum_{i=0}^{m-n-1} G(fx_{n+i}, fx_{n+i+1}, fx_{n+i+1})$$
$$= \sum_{i=0}^{m-n-1} \alpha_{n+i+1} \to 0 \text{ as } m, n \to \infty \ \left(\text{from } (3.5.3)\right)$$

$$\therefore G(fx_n, fx_m, fx_m) \to 0 \text{ as } m, n \to \infty$$

and hence $G(gx_{n-1}, gx_{m-1}, gx_{m-1}) \to 0$ as $m, n \to \infty$

But, from $(3.5.1)$, we see that

$$G(fgx_{n+1}, fx_n, fx_n) = G(ffx_n, fx_n, fx_n)$$

$$\le \psi\left(\max\{G(gfx_n, gx_n, gx_n), G(gfx_n, fx_n, fx_n), G(gx_n, fx_n, fx_n), G(gx_n, fx_n, fx_n)\}\right)$$

$$\le \psi\left(\max\{G(gfx_n, fx_{n-1}, fx_{n-1}), G(gfx_n, fx_n, fx_n), G(fx_{n-1}, fx_n, fx_n), G(fx_{n-1}, fx_n, fx_n)\}\right).$$

On letting $n \to \infty$,

$$G(gp, p, p) \le \psi\left(\max\{G(gp, p, p), G(gp, gp, gp), G(p, p, p), G(gp, gp, gp)\}\right).$$

$\therefore G(gp, p, p) \le \psi(G(gp, p, p))$

$\therefore G(gp, p, p) = 0.$

$\therefore gp = p$

$\therefore$ p is a fixed point of g.

Now,

$$G(fx_n, fp, fp) \le \psi\left(\max\{G(gx_n, fp, fp), G(gx_n, fx_n, fx_n), G(gp, fp, fp), G(fx_n, gp, gp)\}\right).$$

On letting $n \to \infty$, we get

$$G(p, fp, fp) \le \psi\left(\max\{G(p, fp, fp), G(p, p, p), G(gp, fp, fp), G(p, gp, gp)\}\right)$$

$$= \psi\left(\max\{G(p, fp, fp), G(p, p, p), G(p, fp, fp), G(p, p, p)\}\right)$$

$$G(p, fp, fp) = \psi(G(p, fp, fp))$$

$\therefore G(p, fp, fp) = 0$

$\therefore fp = p$

$\therefore$ p is a common fixed point of f and g.

Suppose p and q are common fixed point of f and g

so that $fp = gp = p$ and $fq = gq = q$

Now from $(3.5.1)$, we have

$$G(p, p, pp) = G(p, fq, fq) = G(fp, fq, fq) \le \psi\left(\max\{G(gp, gq, gq), G(gp, fp, fp), G(gq, fq, fq), G(gq, fq, fq)\}\right)$$

$$\le \psi\left(\max\{G(p, q, q), G(p, p, p), G(q, q, q), G(q, q, q)\}\right)$$

$$= \psi\left(\max\{G(p, q, q)\}\right)$$

$$= \psi(G(p, q, q))$$

$$\therefore G(p, q, q) = 0$$

$$\therefore p = q.$$

Thus f and g have unique common fixed point.

Corollary 3.6:- Suppose that f and g are self-maps on a complete G-metric space (X, G) such that

(a) $f(X) \subset g(X)$

(b) $0 < k < 1$

(c) f or g is continuous,

(d) $G(fx, fy, fz) \leq k \max\{G(gx, gy, gz), G(gx, fx, fx), G(gy, fy, fy),$

$G(gz, fz, fz)\})$

for all $x, y, z \in X$ ....(3.5.1)

(e) f and g are compatible.

Then f and g have a unique common fixed point.

Proof:- Take $\psi(t) = kt$ in Theorem 3.5

Theorem 3.7:- Suppose that f and g are self-maps on a complete G-metric space (X, G) such that

(a) $f(X) \subset g(X)$

(b) $\psi \in \Psi$

(c) f or g is continuous,

(d) $G(fx, fy, fz) \leq \psi(\max\{G(gx, gy, gz), G(gx, fx, fx), G(gy, fy, fy),$

$G(gz, fz, fz)\})$

for all $x, y, z \in X$ ....(3.7.1)

(e) f and g are compatible.

Then f and g have a unique common fixed point.

Proof:- The proof of the theorem is similar to that of 3.5

Corollary 3.8:- Suppose that f and g are self-maps on a complete G-metric space (X, G) such that

(a) $f(X) \subset g(X)$

(b) $0 < k < 1$.

(c) f or g is continuous,

(d) $G(fx, fy, fz) \leq k \max\{G(gx, gy, gz), G(gx, fx, fx), G(gy, fy, fy),$

$G(gz, fz, fz)\})$

for all $x, y, z \in X$ ....(3.8.1)

(e) f and g are compatible.

Then f and g have a unique common fixed point.

Proof:- Take $\psi(t) = kt$ in Theorem 3.7

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