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Solvability of a Four Point Nonlinear Boundary Value Problem

Goteti V R L Sarma, Mebrahtom Sebhatu Woldeab , Awet Mebrahtu Solomon
Department of Mathematics, Eritrea Institute of Technology, Mainefhi, Eritrea.

ABSTRACT
In this article a typical four point boundary value problem associated with a second order differential equation is proposed. Then its solution is developed with the help of the Green's function associated with the homogeneous equation. Using this idea and Iteration method is proposed to solve the corresponding non linear problem.
Keywords: Green's function, Schauder fixed point theorem, Vitali's convergence theorem.
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I. INTRODUCTION

The Green's function plays an important role in solving boundary value problems of differential equations. The exact expressions of the solutions for some linear ODEs boundary value problems can be expressed in terms of the Green's functions of the corresponding problem. The Greens function associated with the boundary value problem is an effect tool in numerical experiments. The Green's function method might be used to obtain an initial estimate in shooting method. Some BVPs for nonlinear integral equations the kernels of which are the Green's functions of corresponding linear differential equations. The solutions of associate integral equations are used to investigate the property of the Green's functions. The undetermined parametric method we use in this article is a universal method. The Green's functions of many boundary value problems for ODEs can be obtained by similar method. In this article we proposed and study a second-order differential equations with a typical four-point boundary conditions as

$$u'' + f(t) = 0, \quad t \in [a,b] \tag{1.1}$$

satisfying the boundary conditions

$$u(a) = k_1 u(\eta_1), \qquad u(b) = k_2 u(\eta_2); \tag{1.2}$$

where $a < \eta_1 < \eta_2 < b$ and k_1 and k_2 are real constants

The existence of positive solutions of singular boundary value problems of ordinary differential equations has been studied by many researchers such as Agarwal and Stanek who established the existence criteria for positive solutions singular boundary value problems for nonlinear second order ordinary and delay differential equations using the Vitali's convergence theorem. In 2007, Zhao investigated the positive solutions for singular three-point boundary value problems associated with $u'' + f(t) = 0, \quad t \in [a,b]$ subject to the boundary conditions $u(a) = 0, \quad u'(b) = ku(\eta)$ where $a < \eta < b$ and k is a constant

In (2013), Mohamed investigated the positive solutions to a singular second order boundary value problem. He consider the Sturm-Liouville boundary value problem

$u'' + \lambda g(t) f(t) = 0, \quad t \in [0,1]$ subject to the boundary conditions:

$\propto u(0) - \beta u'(0) = 0 \;\; \gamma u(1) + \delta u'(1) = 0$

where $\propto > 0, \; \beta > 0, \; \gamma > 0$ and $\delta > 0$ are all constants, λ is a positive parameter and f is singular at $u = 0$.

Gatical et al proved the existence of positive solution of the boundary value problem

$u'' + f(t) = 0, \quad t \in [0,1]$ satisfying the boundary conditions

$\propto u(0) - \beta u'(0) = 0$ and $\gamma u(1) + \delta u'(1) = 0$ using the iterative technique and fixed point theorem for cone for decreasing mappings.

Wang and Liu proved the existence of positive solution to the boundary value problem

$u'' + \lambda g(t) f(t) = 0, \quad t \in [0,1]$ subject to the boundary conditions $\propto u(0) - \beta u'(0) = 0$ and $\gamma u(1) + \delta u'(1) = 0$ using the Schauder fixed point theorem.

This article is organized as follows: In section 2 we construct the Green's function to the homogeneous BVP corresponding to (1.1) satisfying (1.2) and then using this we proved the existence and uniqueness of the solution of the boundary value problem (1.1) satisfying the condition (1.2). In section 3 we present the iterative method of solution to the corresponding non linear boundary value problem. We illustrated our results by constructing a suitable example.

II. THE ASSOCIATED GREEN'S FUNCTION

We have the following results:

Theorem 2.1 Assume $(1-k_1)(b-k_2\eta_2)\neq(1-k_2)(a-k_1\eta_1)$. Then the Green's function for the second-order four-point linear boundary value problem (1.1), (1.2) is given by

$$G(t,s)=K(t,s)+\frac{\left[k_1(b-t)+k_1k_2(t-\eta_2)\right]K(\eta_1,s)+\left[k_2(t-a)+k_1k_2(\eta_1-t)\right]K(\eta_1,s)}{(1-k_1)(b-k_2\eta_2)-(1-k_2)(a-k_1\eta_1)} \tag{2.1}$$

$$\text{where } K(t,s)=\begin{cases}\dfrac{(s-b)(t-a)}{b-a}, & a\le s\le t\le b\\[2mm] \dfrac{(s-a)(t-b)}{b-a}, & a\le t\le s\le b\end{cases} \tag{2.2}$$

Proof: It is well known that the Green's function is K(t, s) as in (2.2) for the second-order two-point linear boundary value problem

$$\begin{cases}u''+f(t)=0, & t\in[a,b],\\ u(a)=0, & u(b)=0\end{cases} \tag{2.3}$$

and also the solution of (2.3) is given by

$$w(t)=\int_a^b K(t,s)f(s)ds, \tag{2.4}$$

and $\quad w(a)=0,\qquad w(b)=0,\qquad w(\eta)=\int_a^b K(\eta,s)f(s)ds.$ (2.5)

The four-point boundary value problem (1.1), (1.2) can be obtained from replacing $u(a)=0$, $u(b)=0$ by $u(a)=k_1u(\eta_1)$ and $u(b)=k_2u(\eta_2)$ in (2.3). Thus, we seek the solution of the four-point boundary value problem (1.1), (1.2) in the form

$$u(t)=w(t)+(c+dt)\left[w(\eta_1)+w(\eta_2)\right] \tag{2.6}$$

where c and d are constants that will be determined.

From (2.6) we get

$$u(a)=(c+da)\left[w(\eta_1)+w(\eta_2)\right] \text{ and } u(b)=(c+db)\left[w(\eta_1)+w(\eta_2)\right]$$

$$u(\eta_1)=w(\eta_1)+(c+d\eta_1)\left[w(\eta_1)+w(\eta_2)\right] \text{ and } u(\eta_2)=w(\eta_2)+(c+d\eta_2)\left[w(\eta_1)+w(\eta_2)\right]$$

Putting this into the boundary condition (1.2) yields

$$\begin{cases}c(1-k_1)\left[w(\eta_1)+w(\eta_2)\right]+d(a-k_1\eta_1)\left[w(\eta_1)+w(\eta_2)\right]=k_1w(\eta_1)\\ c(1-k_2)\left[w(\eta_1)+w(\eta_2)\right]+d(b-k_2\eta_2)\left[w(\eta_1)+w(\eta_2)\right]=k_2w(\eta_2)\end{cases}$$

Since $(1-k_1)(b-k_2\eta_2)\neq(1-k_2)(a-k_1\eta_1)$, solving the system of linear equations on the unknown numbers c, d, using Cramer's rule we obtain

$$\begin{cases}c=\dfrac{(k_1w(\eta_1))(b-k_2\eta_2)-(k_2w(\eta_2))(a-k_1\eta_1)}{\left[(1-k_1)(b-k_2\eta_2)-(1-k_2)(a-k_1\eta_1)\right]\left[w(\eta_1)+w(\eta_2)\right]}\\[4mm] d=\dfrac{(k_2w(\eta_2))(1-k_1)-(k_1w(\eta_1))(1-k_2)}{\left[(1-k_1)(b-k_2\eta_2)-(1-k_2)(a-k_1\eta_1)\right]\left[w(\eta_1)+w(\eta_2)\right]}\end{cases}$$

Hence, the solution of (1.1) with the boundary condition (1.2) is

$$u(t)=w(t)+(c+dt)\left[w(\eta_1)+w(\eta_2)\right]$$
$$=w(t)+\frac{\left[k_1(b-t)+k_1k_2\left(t-\eta_2\right)\right]\left(w(\eta_1)\right)+\left[k_2(t-a)+k_1k_2\left(\eta_1-t\right)\right]\left(w(\eta_2)\right)}{\left(1-k_1\right)\left(b-k_2\eta_2\right)-\left(1-k_2\right)\left(a-k_1\eta_1\right)}$$

This together with (2.4) implies that

$$u(t)=\int_a^b K(t,s)f(s)ds+\frac{\left[k_1(b-t)+k_1k_2\left(t-\eta_2\right)\right]\int_a^b K(\eta_1,s)f(s)ds+\left[k_2(t-a)+k_1k_2\left(\eta_1-t\right)\right]\int_a^b K(\eta_1,s)f(s)ds}{\left(1-k_1\right)\left(b-k_2\eta_2\right)-\left(1-k_2\right)\left(a-k_1\eta_1\right)}$$

Consequently, the Green's function G(t, s) for the boundary value problem (1.1), (1.2) is as described in (2.1).

From Theorem 2.1 we obtain the following corollary.

Corollary 2.1. If $\left(1-k_1\right)\left(b-k_2\eta_2\right)\neq\left(1-k_2\right)\left(a-k_1\eta_1\right)$, then the second-order four-point linear boundary value problem

$$\begin{cases} u''+f(t)=0, & t\in[a,b], \\ u(a)=k_1u(\eta_1), & u(b)=k_2u(\eta_2) \end{cases}$$

has a unique solution $u(t)=\int_a^b G(t,s)f(s)ds$ where G(t, s) as in (2.1).

Proof: Assume that the second-order four-point linear boundary value problem (2.7) has two solutions u(t) and v(t), that is

$$\begin{cases} u''+f(t)=0, & t\in[a,b], \\ u(a)=k_1u(\eta_1), & u(b)=k_2u(\eta_2) \end{cases} \quad \text{and} \tag{2.7}$$

$$\begin{cases} v''+f(t)=0, & t\in[a,b], \\ v(a)=k_1v(\eta_1), & v(b)=k_2v(\eta_2) \end{cases} \tag{2.8}$$

Let

$$z(t)=v(t)-u(t), \quad t\in\left[a,b\right]$$

Then using (2.7) and (2.8) $z''(t)=v''(t)-u''(t)=0, \quad t\in\left[a,b\right]$

Therefore $z(t)=C_1t+C_2,$ (2.10)

where C_1 and C_2 are undetermined constants. From (2.7), (2.8) and (2.9) we have

$$z(a)=v(a)-u(a)=k_1z(\eta_1), \tag{2.11}$$

$$z(b)=v(b)-u(b)=k_2z(\eta_2). \tag{2.12}$$

Using (2.10) we obtain

$$z(a) = C_1 a + C_2, \tag{2.13}$$

$$z(b) = C_1 b + C_2, \tag{2.14}$$

$$z(\eta_1) = C_1 \eta_1 + C_2, \tag{2.15}$$

$$z(\eta_2) = C_1 \eta_2 + C_2, \tag{2.16}$$

From (2.11), (2.13) and (2.15) we know that

$$C_1(a - k_1\eta_1) + C_2(1 - k_1) = 0, \tag{2.17}$$

and from (2.12), (2.14) and (2.16) we know that

$$C_1(b - k_2\eta_2) + C_2(1 - k_2) = 0, \tag{2.18}$$

Solving the system of equations (2.17) and (2.18), we get $C_1 = 0$ and $C_2 = 0$.

Therefore $z(t) = 0, t \in [a,b]$, so $u(t) = v(t),\ t \in [a,b]$, that establishes uniqueness of the solution.

Corollary 2.2. Suppose the nonlinear function g(t, u) is continuous on $[a,b] \times \mathbb{R}$, then if $(1-k_1)(b-k_2\eta_2) \neq (1-k_2)(a-k_1\eta_1)$, the nonlinear four-point boundary value problem

$$\begin{cases} u'' + g(t,u) = 0, & t \in [a,b], \\ u(a) = k_1 u(\eta_1), & u(b) = k_2 u(\eta_2) \end{cases}$$

is equivalent to the nonlinear integral equation

$u(t) = \int_a^b G(t,s) g(s, u(s))\, ds$ where G(t,s) as in (2.1)

If the endpoints of the interval are $a = 0, b = 1$ in the boundary condition, from Theorem 2.1, Corollaries 2.1 and 2.2 we obtain the following corollary.

Corollary 2.3. If $(1-k_1)(1-k_2\eta_2) \neq (k_2-1)(k_1\eta_1)$, then the Green's function for the second-order four-point linear boundary value problem

$$\begin{cases} u'' + f(t) = 0, & t \in [0,1], \\ u(0) = k_1 u(\eta_1), & u(1) = k_2 u(\eta_2) \end{cases}$$

is

$$G(t,s) = K(t,s) + \frac{[k_1(1-t) + k_1k_2(t-\eta_2)]K(\eta_1,s) + [k_2(t) + k_1k_2(\eta_1 - t)]K(\eta_1,s)}{(1-k_1)(1-k_2\eta_2) + (1-k_2)(k_1\eta_1)}$$

where

$$K(t,s) = \begin{cases} (s-1)(t), & 0 \le s \le t \le 1 \\ (s)(t-1), & 0 \le t \le s \le 1 \end{cases}$$

Hence the problem (2.19) has a unique solution

$$u(t) = \int_a^b G(t,s) f(s)\, ds.$$

If g(t, u) is continuous on $[0,1] \times \mathbb{R}$, then the nonlinear four-point boundary value problem

$$\begin{cases} u'' + g(t,u) = 0, & t \in [0,1], \\ u(0) = k_1 u(\eta_1), & u(1) = k_2 u(\eta_2) \end{cases}$$

is equivalent to the nonlinear integral equation

$$u(t) = \int_0^1 G(t,s) g(s, u(s))\, ds.$$

Example: Solve the second-order four-point boundary value problem:

$$\begin{cases} u'' + \sin t = 0, & t \in [0,1], \\ u(0) = \frac{-1}{2} u\left(\frac{1}{5}\right), & u(1) = \frac{2}{3} u\left(\frac{1}{6}\right) \end{cases}$$

Solution

Since $(1-k_1)(1-k_2\eta_2)=\left(1+\frac{1}{2}\right)\left(1-\left(\frac{2}{3}\right)\left(\frac{1}{6}\right)\right)=\frac{4}{3}$ and $(k_2-1)(k_1\eta_1)=\left(\frac{2}{3}-1\right)\left(\frac{-1}{2}\right)\left(\frac{1}{5}\right)=\frac{1}{30}$ are not equal, from (2.20), the Green's function is:

$$G(t,s)=B(t,s)+\frac{\left[k_1(1-k_2\eta_2)-tk_1(1-k_2)\right]B(\eta_1,s)+\left[k_2(k_1\eta_1)+tk_2(1-k_1)\right]B(\eta_2,s)}{(1-k_1)(1-k_2\eta_2)+(1-k_2)(k_1\eta_1)}$$

where $k_1=\frac{-1}{2}$, $\eta_1=\frac{1}{5}$, $k_2=\frac{2}{3}$, $\eta_1=\frac{1}{6}$

$$G(t,s)=B(t,s)+\frac{\left[\left(\frac{-1}{2}\right)\left(\frac{8}{9}\right)+t\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\right]B(\eta_1,s)+\left[\left(\frac{2}{3}\right)\left(\frac{-1}{10}\right)+\left(\frac{2}{3}\right)t\left(\frac{3}{2}\right)\right]B(\eta_2,s)}{\left(\frac{3}{2}\right)\left(\frac{8}{9}\right)-\left(\frac{1}{3}\right)\left(\frac{1}{10}\right)}$$

$$=B(t,s)+\left[\frac{15t-40}{117}\right]B\left(\frac{1}{5},s\right)+\left[\frac{30t-2}{39}\right]B\left(\frac{1}{6},s\right)$$

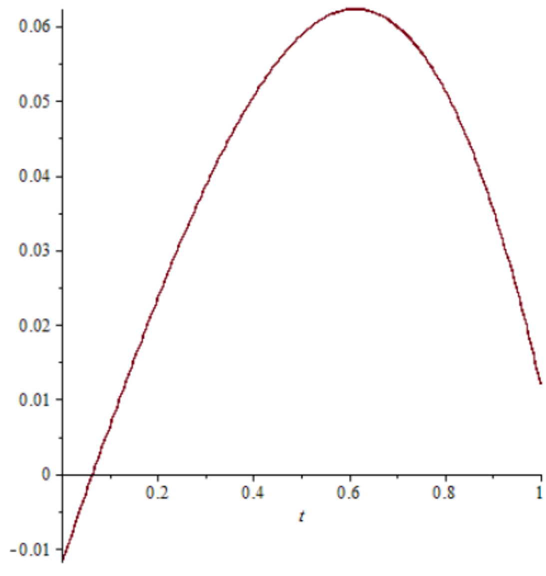
From (2.21), $B(t,s)=\begin{cases}(s-1)(t),\ 0\le s\le t\le 1\\ (s)(t-1),\ 0\le t\le s\le 1\end{cases}$

Hence the solution of second-order four-point boundary value problem is:

$$u(t)=\int_0^1 G(t,s)f(s)\,ds=\int_0^1\left[B(t,s)+\left[\frac{15t-40}{117}\right]B\left(\frac{1}{5},s\right)+\left[\frac{30t-2}{39}\right]B\left(\frac{1}{6},s\right)\right]f(s)\,ds$$

$$=\sin(t)+\left(\frac{1-15t}{13}\right)\sin(1)+\left(\frac{15t-40}{117}\right)\sin\left(\frac{1}{5}\right)+\left(\frac{30t-2}{39}\right)\sin\left(\frac{1}{6}\right)$$

Its solution curve is obtained using Maple as follows:



III. METHOD TO SOLVE NONLINEAR SINGULAR BOUNDARY VALUE PROBLEM

In this section, we study the iterative method to solve the following nonlinear four-point boundary value problem

$$\begin{cases} u'' + f(t,u) = 0, & t \in (0,1), \\ u(0) = k_1 u(\eta_1), & u(1) = k_2 u(\eta_2) \end{cases} \tag{3.1}$$

with $\eta_1, \eta_2 \in (0,1)$, $(1-k_1)(k_2\eta_2 - 1) < (1-k_2)(k_1\eta_1)$.

Let $J = (0,1)$, $I = [0,1]$, $\mathbb{R}^+ = [0,\infty)$,

$D = \{x \in C(I) \,|\, \exists M_x \geq m_x > 0, \text{ such that } m_x(1-t) \leq x(t) \leq M_x(1-t), t \in I\}$.

Concerning the function f we impose the following hypotheses:

$$\begin{cases} f(t,u) \text{ is nonnegative continuous on } J \times \mathbb{R}^+, \\ f(t,u) \text{ is monotone increasing on u, for fixed t} \in \text{J}, \\ \exists q \in (0,1) \text{ such that } f(t,ru) \geq r^q f(t,u), \forall 0 < r < 1, (t,u) \in J \times \mathbb{R}^+. \end{cases} \tag{3.2}$$

Obviously, from (3.2) we obtain

$$f(t,\lambda u) \geq \lambda^q f(t,u), \quad \forall \lambda > 1, \quad (t,u) \in J \times \mathbb{R}^+. \tag{3.3}$$

It is easy to see that if $0 < \alpha_i < 1$, $a_i(t)$ are nonnegative continuous on J, for i = 0,1,2,…,m, then

$f(t,u) = \sum_{i=1}^{m} a_i(t) u^{\alpha_i}$ satisfy the condition (3.2).

Concerning the boundary value problem (3.1), we have following conclusions:

Theorem 3.1. Suppose the function f(t,u) satisfy the condition (3.2), it may be singular at t=0 and/or t=1, and

$$0 < \int_0^1 f(t, 1-t)dt < \infty. \tag{3.4}$$

Then nonlinear singular boundary value problem (3.1) has a unique solution w(t) in $C(I) \cap C^2(J)$. Constructing successively the sequence of functions

$$h_n(t) = \int_0^1 G(t,s) f(s, h_{n-1}(s)) ds, \quad n = 1, 2, \ldots . \tag{3.5}$$

for any initial function $h_0(t) \geq 0 (\not\equiv 0), t \in I$ then $\{h_n(t)\}$ must converge to w(t) uniformly on I and the rate of convergence is

$$\max_{t \in I} |h_n(t) - w(t)| = O\left(1 - N^{q^n}\right), \tag{3.6}$$

where 0< N < 1 , which depends on the initial function $h_0(t)$, $G(t,s)$ as in (2.20).

Proof. Let

$$P = \{x(t) \,|\, x(t) \in C(I), x(t) \geq 0\},$$

$$Fx(t) = \int_0^1 G(t,s) f(s, x(s)) ds, \quad \forall x(t) \in D. \tag{3.7}$$

It easy that the operator $F : D \to P$ is increasing; From Corollary 2.3 we know that if $u \in D$ satisfies

$$u(t) = Fu(t), \quad t \in I, \tag{3.8}$$

then $u \in C^1(I) \cap C^2(J)$ is a solution of (3.1).

For any $x \in D$, there exist positive numbers $0 < m_x < 1 < M_x$ such that

$$m_x(1-s) \le x(s) \le M_x(1-s), \quad s \in I,$$

$$(m_x)^q f(s,1-s) \le f(s,x(s)) \le (M_x)^q f(s,1-s), \quad s \in J. \tag{3.9}$$

By (2.20) and (2.21) we have

$$G(t,s) = B(t,s) + \frac{\left[k_1(1-t) + k_1k_2(t-\eta_2)\right]B(\eta_1,s) + \left[k_2(t) + k_1k_2(\eta_1 - t)\right]B(\eta_2,s)}{(1-k_1)(1-k_2\eta_2) + (1-k_2)(k_1\eta_1)}$$

$$\Rightarrow \quad G(t,s) \ge (1-t)\frac{[k_1 + k_1k_2]B(\eta_1,s) + \left[\frac{k_2}{1-t} - k_2 - k_1k_2\right]B(\eta_2,s)}{(1-k_1)(1-k_2\eta_2) + (1-k_2)(k_1\eta_1)} \tag{3.10}$$

$$G(t,s) \le t(1-t) + (1-t)\frac{[k_1 + k_1k_2]B(\eta_1,s) + \left[\frac{k_2}{1-t} - k_2 - k_1k_2\right]B(\eta_2,s)}{(1-k_1)(1-k_2\eta_2) + (1-k_2)(k_1\eta_1)}$$

$$G(t,s) \le t(1-t) + (1-t)\frac{[k_1 + k_1k_2]B(\eta_1,s) + \left[\frac{k_2}{1-t} - k_2 - k_1k_2\right]B(\eta_2,s)}{(1-k_1)(1-k_2\eta_2) + (1-k_2)(k_1\eta_1)}$$

$$\Rightarrow G(t,s) \le (1-t)\left(1 + \frac{[k_1 + k_1k_2]B(\eta_1,s) + \left[\frac{k_2}{1-t} - k_2 - k_1k_2\right]B(\eta_2,s)}{(1-k_1)(1-k_2\eta_2) + (1-k_2)(k_1\eta_1)}\right) \tag{3.11}$$

Using (3.7), (3.3) and (3.9)-(3.11) and the conditions (3.2), we obtain

$$Fx(t) = \int_0^1 G(t,s) f(s,x(s))ds$$

$$\ge \int_0^1 (1-t)\frac{[k_1 + k_1k_2]B(\eta_1,s) + \left[\frac{k_2}{1-t} - k_2 - k_1k_2\right]B(\eta_2,s)}{(1-k_1)(1-k_2\eta_2) + (1-k_2)(k_1\eta_1)}\Big((m_x)^q f(s,1-s)\Big)ds \tag{3.12}$$

$$\ge (1-t)(m_x)^q \frac{[k_1 + k_1k_2] + \left[\frac{k_2}{1-t} - k_2 - k_1k_2\right]}{(1-k_1)(1-k_2\eta_2) + (1-k_2)(k_1\eta_1)}\left[\int_0^1 B(\eta_1,s)\big(f(s,1-s)\big)ds + \int_0^1 B(\eta_2,s)\big(f(s,1-s)\big)ds\right], t \in I$$

$$Fx(t) = \int_0^1 G(t,s) f(s,x(s))ds$$

$$\le \int_0^1 (1-t)\left(1 + \frac{[k_1 + k_1k_2]B(\eta_1,s) + \left[\frac{k_2}{1-t} - k_2 - k_1k_2\right]B(\eta_2,s)}{(1-k_1)(1-k_2\eta_2) + (1-k_2)(k_1\eta_1)}\right)\Big((M_x)^q f(s,1-s)\Big)ds \tag{3.13}$$

$$\le (1-t)(M_x)^q \int_0^1 \left(1 + \frac{[k_1 + k_1k_2]B(\eta_1,s) + \left[\frac{k_2}{1-t} - k_2 - k_1k_2\right]B(\eta_2,s)}{(1-k_1)(1-k_2\eta_2) + (1-k_2)(k_1\eta_1)}\right)\big(f(s,1-s)\big)ds, \quad t \in I$$

By (3.4), (3.12) and (3.13) we obtain

$$F : D \to D.$$

For any $h_o \in D$, we let

$$l_{h_o} = \sup\left\{l > 0 \middle| l h_o(t) \le (Fh_o)(t), t \in I\right\},$$

$$L_{h_o} = \inf\left\{L > 0 \middle| L h_o(t) \ge (Fh_o)(t), t \in I\right\}, \tag{3.14}$$

$$m=\min\left\{1,\left(l_{h_0}\right)^{\frac{1}{1-q}}\right\},\quad M=\max\left\{1,\left(L_{h_0}\right)^{\frac{1}{1-q}}\right\}$$

$$\begin{aligned} &u_0(t)=mh_0(t), && u_n(t)=Fu_{n-1}(t) \\ &v_0(t)=Mh_0(t), && v_n(t)=Fv_{n-1}(t), \quad n=0,1,2,\ldots \end{aligned} \tag{3.15}$$

Since the operator F is increasing, from (3.2), (3.14) and (3.15) we know that

$$u_0(t)\le u_1(t)\le\cdots\le u_n(t)\cdots\le v_n(t)\le\cdots\le v_1(t)\le v_0(t),\quad t\in I. \tag{3.16}$$

For $t_0=\frac{m}{M}$, from (3.2), (3.7) and (3.15), it can obtained by induction that

$$u_n(t)\ge(t_0)^{q^n}v_n(t),\quad t\in I, n=0,1,2,\cdots \tag{3.17}$$

From (3.16) and (3.17) we know that

$$0\le u_{n+p}(t)-u_n(t)\le v_n(t)-u_n(t)\le\left(1-(t_0)^{q^n}Mh_0(t)\right),\quad \forall n,p \tag{3.18}$$

so that there exist function $w(t)\in D$ such that

$$u_n(t)\to w(t),\qquad v_n(t)\to w(t),\quad \text{(uniformly on I)}, \tag{3.19}$$

and

$$u_n(t)\le w(t)\le v_n(t),\quad t\in I, n=0,1,2,\cdots \tag{3.20}$$

From the operator F is increasing and (3.15) we have

$$u_{n+1}(t)=Fu_n(t)\le Fw(t)\le Fv_n(t)=v_{n+1}(t),\quad n=0,1,2,\cdots$$

This together with (3.19) and uniqueness of the limit imply that $w(t)$ satisfy (3.8), hence $w(t)\in C^1(I)\cap C^2(J)$ is a solution of (3.1).

From (3.5) and (3.15) and the operator F is increasing, we obtain

$$u_n(t)\le h_n(t)\le v_n(t),\quad t\in I, n=0,1,2,\cdots \tag{3.21}$$

thus, from (3.18), (3.20) and (3.21) we know

$$\begin{aligned} |h_n(t)-w(t)| &\le |h_n(t)-u_n(t)|+|u_n(t)-w(t)| \\ &\le 2|v_n(t)-u_n(t)|\le\left(1-(t_0)^{q^n}\right)M|h_0(t)|, \end{aligned}$$

so that $$\max_{t\in I}|h_n(t)-w(t)|\le\left(1-(t_0)^{q^n}\right)M\max_{t\in I}|h_0(t)|.$$

So that (3.6) holds.

From $h_0(t)$ which is arbitrary in D we know that *w(t)* is the unique solution of the Eq. (3.8) in D. Suppose $w_1(t)$ is a $C^1(I)\cap C^2(J)$ solution of boundary value problem (3.1). Let

$$z(t)=w_1(t)-Fw(t), t\in I.$$

Similar to the proof of (2.9) in section 2 we obtain $w_1(t)=w(t)$, hence $w(t)$ is the unique solution of Eq. (3.1) in $C^1(I)\cap C^2(J)$.

Remark: If $f(t,u)$ is continuous on $I\times\mathbb{R}^+$, then it is quite evident that the condition (3.4) holds. Hence the unique solution $w(t)\in C^2(J)$.

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