RESEARCH ARTICLE OPEN ACCESS

A comparison between the Differential Transform Method and the Adomian Decomposition Method for the Burgers equation

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ABSTRACT
In this paper, we consider the Differential Transform Method (DTM)and the Adomian Decomposition Method (ADM) for finding approximate and exact solution of the Burgers equation. Moreover, the reliability and performance of DTM and ADM. Numerical results show that these methods are powerful tools for solving Burgers equation.
A simple model of turbulence is represented by Burger's equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}= \varepsilon\frac{\partial^2 u}{\partial x^2}, \qquad a<x<b, \qquad t>0 \qquad (1)$$

With the initial condition

$$u(x,0)=f(x), \qquad a<x<b \qquad (2)$$

and the boundary conditions:

$$u(a,t)=g_1(t), \qquad u(b,t)=g_2(t), \qquad t>0 \qquad (3)$$

where ε is the kinematic viscosity. Thus Burger's equation is as balance between time evolution, nonlinearity and diffusion. It may be noted that the equation is parabolic if $\varepsilon \neq 0$ and hyperbolic if $\varepsilon = 0$.

Date of Submission: 19-12-2017 Date of acceptance: 30-12-2017

I. INTRODUCTION

Burgers' equation is a fundamental partial differential equation that occur in different areas of applied mathematics, likes fluid mechanics,nonlinear acoustics, gas dynamics, traffic flow. It is named for Johannes Martinus Burgers (1895–1981). Bateman (1915) consider Burger's expensive works by putting his name up on the equation, then it is known as Burgers' equation. It is a nonlinear equation for which exact solutions are known and is therefore important as a benchmark problem for numerical methods. Navier Stokes equation was more difficult than this one, so Burger's is a simplification one in a good way, where the velocity is in one spatial dimension and the external force is neglected and without any pressure gradient.

Differential Transform Method (DTM)

we introduce the basic definition and the operation of one and two dimensional differential transformationare defined [1,2,3,4,5].

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} , \qquad (4)$$

where the t- dimensional spectrum function $U_k(x)$ is the transformed function.

So the differential inverse transform of $U_k(x)$ is defined as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k \qquad (5)$$

From equation (4) and equation (5) the function $u(x,t)$ can be described as

$$u(x,t) = \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k \qquad (6)$$

From the above, it can be found that the concept of differential transform method is derived from the power series expansion of a function.

The two dimensional differential transformation$u(x,t)$is defined as

$$U(k,h) = \frac{1}{k!\,h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h}u(x,t)\right]_{\substack{x=x_0\\ t=t_0}} , \qquad (7)$$

The inverse two-dimensional differential transform of $U(k,h)$ is defined as

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h , \qquad (8)$$

From equation (7) and equation(8) the function $u(x,t)$ can be written as the following,

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!\,h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h}u(x,t)\right]_{\substack{x=x_0\\ t=t_0}} x^k t^h ,$$

the equation (8) can be written as follow:

$$u(x,t) = \sum_{k=0}^{m}\sum_{h=0}^{n} U(k,h) x^k t^h , \quad (9) \qquad (10)$$

usually, the values of $m$and n are decided by convergence of the series coefficient. The following table that can be deduced from equation (8) and equation (9):

Table 1. Differential transform

Functional Form	Transformed Form
$w(x,t) = u(x,t) \pm v(x,t)$	$W(k,h) = U(k,h) \pm V(k,h)$
$w(x,t) = \alpha u(x,t)$	$W(k,h) = \alpha U(k,h) \quad , \alpha\ is\ a\ constant$
$w(x,t) = x^m t^n$	$W(k,h) = \delta(k-m,h-n) = \delta(k-m)\delta(h-n)$ $\delta(-m,h-n) = \begin{cases} 1, & k = m\ and\ h = n \\ 0, & otherwise \end{cases}$
$w(x,t) = u(x,t)v(x,t)$	$W(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)V(k-r,s)$
$w(x,t) = \frac{\partial}{\partial x}u(x,t)$	$W(k,h) = (k+1)U(k+1,h)$
$w(x,t) = \frac{\partial}{\partial t}u(x,t)$	$W(k,h) = (h+1)U(k,h+1)$
$w(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)$	$W(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+2)(k-r+1)U(r,h-s)V(k-r+2,s)$
$w(x,t) = u(x,t)\frac{\partial}{\partial x}u(x,t)$	$W(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)U(k-r+1,s)$

By using the basic properties of the differential transform, we can transformed form of equation (1) as:

$$(h+1)U(k,h+1) + \sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)U(k-r+1,s) = (k+2)(K+1)U(k+2,h) , \qquad (11)$$

II. ADOMIAN DECOMPOSITION METHOD (ADM)

In this section, we explain the main algorithm of ADM for Burgers equation with initial condition. Through the following references [7,5,6,8,9,10,11,12] the ADM are studied to find approximate solutions to the equation (1).

Equation (1) is approximated by an operator in the following form

$$L_t u(x,t) + N(u) = \varepsilon L_{xx} u \qquad (12)$$

Where $L_t = \frac{\partial}{\partial t}$, $L_{xx} = \frac{\partial^2}{\partial x^2}$, and $Nu = u\frac{\partial u}{\partial x}$.

Solving equation (12) by the inverse operator, and using the initial condition gives,

$$u(x,t) = u(x,0) + \varepsilon L_t^{-1} L_{xx} u + L_t^{-1} N(u). \qquad (13)$$

Where

$$L_t^{-1}(.) = \int_0^t (.)dt \ (14)$$

In ADM we represent a solution suppose that

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \ (15)$$

is a required solution of equation (1). A nonlinear term$Nu = u\frac{\partial u}{\partial x}$ into an infinite series of Adomain's polynomials

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n \qquad (16)$$

Where these polynomials A_n can be calculated for all forms of nonlinearity, A_n are given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{n=0}^{\infty}\lambda^n u_n\right)\right]_{\lambda=0} , n \geq 0 \qquad (17)$$

So the Adomain's polynomials for the nonlinear term $Nu = u\frac{\partial u}{\partial x}$ are derived in the following form,

$$A_0 = u_0 \frac{\partial u_0}{\partial x},$$
$$A_1 = u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x},$$
$$A_2 = u_0 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_0}{\partial x},$$
(18)
$$A_3 = u_0 \frac{\partial u_3}{\partial x} + u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_1}{\partial x} + u_3 \frac{\partial u_0}{\partial x}$$
$$\vdots$$

Substituting equations (15) and (16) into equation (14) gives,

$$u(x,t) = u(x,0) + \varepsilon L_t^{-1} L_{xx} \sum_{n=0}^{\infty} u_n(x,t) - L_t^{-1} \sum_{n=0}^{\infty} A_n \ . \qquad (19)$$

Form equation (19) the Adomian decomposition scheme is defined by the recurrent relation

$$u_0(x,0) = f(x) \qquad (20)$$

and $u_n(x,t) = \varepsilon L_t^{-1} L_{xx} u_{n-1}(x,t) - L_t^{-1} A_{n-1}$ for $n \geq 0$ (21)

We can estimate the approximate solution ϕ_n can be determined by

$$\phi_n = \sum_{n=0}^{n-1} u_n(x,t) \qquad (22)$$

Where the components produce as

$$\phi_1 = u_0$$
$$\phi_2 = u_0 + u_1$$
$$\phi_3 = u_0 + u_1 + u_2$$
$$\vdots$$
$$\phi_n = u_0 + u_1 + u_2 + \cdots + u_{n-1}$$

As it is clear from equations (15) and (22)

$$u(x,t) = \lim_{n \to \infty} \phi_n(x,t) \ , \qquad (23)$$

Application

Here, we will use the Differential transform method (DTM) and Adomain decomposition method (ADM) are used to find the solutions of the Burger's equations, and compared with two methods.

Example (1):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}, \qquad (24)$$

With the initial condition

$$u(x,0) = 2x, \qquad (25)$$

By Differential transform method (DTM):

Let $U(k,h)$ as the differential transform of $u(x,t)$. Applying table 1, we get the differential transform version of equation (24) as following

$$(h+1)U(k,h+1) + \sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)U(k-r+1,s) = (k+2)(K+1)U(k+2,h), \qquad (26)$$

By the first initial condition (25) we get

$$u(x,0) = \sum_{k=0}^{\infty} U(k,0)x^k = 2x, \qquad (27)$$

Where

$$u(k,0) = \begin{cases} 2, & if\ k=1 \\ 0, & otherwise \end{cases} \qquad (28)$$

Using equation (28) into equation (26), we get values of $U(k,h)$ as

$$u(k,1) = \begin{cases} -4, & if\ k=1 \\ 0, & otherwise \end{cases} \qquad (29)$$

and

$$u(k,2) = \begin{cases} 8, & if\ k=1 \\ 0, & otherwise \end{cases} \qquad (30)$$

etc. Then from Equation (8), we have

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h = \sum_{h=0}^{\infty} U(1,h)xt^h \qquad (31)$$

$$u(x,t) = 2x - 4xt + 8xt^2 - 16xt^3 + 32xt^4 - \cdots \qquad (32)$$

Which is the Taylor expansion of the

$$u(x,t) = \frac{2x}{1+2t} \qquad (33)$$

By Adomain Decomposition method (ADM):

In the following section, we discuss the solution of the Burger's equation using ADM. Equation (1) can be written in an operator form:

$$L_t u(x,t) + N(u) = L_{xx} u \qquad (34)$$

Where $L_t = \frac{\partial}{\partial t}$, $L_{xx} = \frac{\partial^2}{\partial x^2}$, and $Nu = u\frac{\partial u}{\partial x}$. Applying the inverse operator L_t^{-1} on both side of equation (34) and using the decomposition series (16) and (17) yield

$$u(x,t) = u(x,0) + L_t^{-1} L_{xx} \sum_{n=0}^{\infty} u_n(x,t) - L_t^{-1} \sum_{n=0}^{\infty} A_n \ . \qquad (35)$$

Where A_n are Adomain polynomials that represent the nonlinear term $Nu = u\frac{\partial u}{\partial x}$ and given by

$A_0 = 4x,$
$A_1 = -16xt,$

$$A_2 = 48xt^2,$$
$$A_3 = -128xt^3,$$

Other polynomials can be generated in alike manner. The first few component of $u_n(x,t)$ follows as

$$u_0 = 2x$$
$$u_1 = -4xt$$
$$u_2 = 8xt^2$$
$$u_3 = -16xt^3$$
$$u_4 = 32xt^4$$

So, the solution in a series form is given by

$u(x,t) = 2x - 4xt + 8xt^2 - 16xt^3 + 32xt^4 - \cdots$

The exact solution, in closed form is given by

$$u(x,t) = \frac{2x}{1+2t}$$

which is exactly the same as those obtained by differential transform method.

Example (2):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \qquad (36)$$

With the initial condition

$$u(x,0) = -x, \qquad (37)$$

By Differential transform method (DTM):

Taking the differential transform of equation (36), we have

$$(h+1)U(k,h+1) = -\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)U(k-r+1,s), \qquad (38)$$

where $U(k,h)$ is the differential transform of $u(x,t)$.

Taking the differential transform from initial condition equation (37), and by recursive method, would be

$$u(x,0) = \sum_{k=0}^{\infty} U(k,0)x^k = -x, \qquad (39)$$

where

$$u(k,0) = \begin{cases} -1, & if\ k = 1 \\ 0, & otherwise \end{cases}$$

substituting equation (40) into equation (38) and using the recurrence relation we have

$$U(1,1) = -1$$
$$U(1,2) = -1$$
$$U(1,3) = -1$$
$$\vdots$$
$$U(1,h) = -1$$

and so on. In general, we have

$$U(k,h) = \begin{cases} -1, & if\ k = 1 \\ 0, & otherwise \end{cases} \qquad (41)$$

Substituting all $U(k,h)$ into equation (8) yield the solution

$$u(x,t) = -x - xt - xt^2 - xt^3 - xt^4 - \cdots$$

Which leads to the closed form solution

$$u(x,t) = \frac{x}{t-1}$$

By Adomain Decomposition method (ADM):

Rewrite equation (36) in an operator from

$$L_t u(x,t) = -N(u) \qquad (42)$$

Where $L_t = \frac{\partial}{\partial t}$ and$Nu = u\frac{\partial u}{\partial x}$. Applying the inverse operator L_t^{-1} on both side of equation (42) and using the decomposition series (16) and (17) yield

$$u(x,t) = u(x,0) - L_t^{-1}\sum_{n=0}^{\infty} A_n \ . \qquad (43)$$

Where A_n are Adomain polynomials that represent the nonlinear term $Nu = u\frac{\partial u}{\partial x}$ and given by

$$u(x,t) = u(x,0) - L_t^{-1}\sum_{n=0}^{\infty} A_n \ . \qquad (35)$$

Where A_n are Adomain polynomials that represent the nonlinear term $Nu = u\frac{\partial u}{\partial x}$ and given by

$A_0 = x,$
$A_1 = 2xt,$

$$A_2 = 3xt^2,$$
$$A_3 = 4xt^3,$$
$$A_4 = 5xt^4,$$

Other polynomials can be generated in alike manner. The first few component of $u_n(x,t)$ follows as

$$u_0 = -x$$
$$u_1 = -xt$$
$$u_2 = -xt^2$$
$$u_3 = -xt^3$$
$$u_4 = -xt^4$$

So, the solution $u(x,t)$ is readily obtained in a series form by

$$u(x,t) = -x - xt - xt^2 - xt^3 - xt^4 - \cdots$$

Or in a closed form by

$$u(x,t) = \frac{x}{t-1} \qquad (40)$$

III. COMPARISON

DTM and ADM generate the same results for Burgers' equation. Solutions obtained by the techniques are infinite power series. For the appropriate initial conditions, the exact solutions could be expressed in closed form. Based on the results, it can be concluded that that the methods are effective approaches for solving Burgers' equations. Since the approaches do not involve complex computations, the methods are extremely convenient. DTM is highly effective in comparison to ADM. DTM does not require elaborate computations. The advantage of the ADM technique is that it eliminates discretization in space and time, which are common in the conventional methods of resolution of equations.The numerical results of a problem are solved by DTM generates more accurate results in comparison to ADM. This is particularly relevant for higher-order solutions. Even for the same initial conditions, DTM provides highly accurate results for higher-order solutions, whilst ADM suffers from loss of accuracy at the higher orders. Despite the finding, it is noteworthy that both methods are capable of generating solutions, which could use used in applications. This finding is extremely useful as it provides a rationale for selecting a method for each application. The analysis shows that the techniques are powerful methods for providing an efficient potential for solving physical applications, which are modelled by nonlinear differential equations. Further analysis indicates that DTM could be simplified into a reduced DTM as suggested by Srivastava *et al*. (2014). This is an alternative approach for simplifying computations in DTM. The enhanced approach reduces the size of the calculation. The main benefit of RDTM is that it produces all the Poisson series coefficients of solutions. This is an advance over DTM, which produces Taylor series coefficients of solutions. Some of the other benefits of RDTM include high levels of accuracy, rapid convergence, and easy implementation. The method could be reliably used for many complex engineering applications to solve multidimensional physical problems, without the loss of accuracy.

IV. CONCLUSIONS

Burgers' equation provides a prototype for describing the interaction between convection effects, diffusion transport, and reaction mechanisms. In physical problems, approximate solutions of nonlinear differential equations are of practical significance. DTM and ADM could be used to find the approximate solutions for Burgers' equations. DTM as well as ADM provide highly accurate numerical solutions. For nonlinear partial differential equations, DTM and ADM algorithms provide solutions without discretisation. Comparative analysis shows that ADM suffers from loss of accuracy for higher-order solutions, whilst DTM provides highly accurate solutions even for higher-order solutions. The main benefit of the analysis is the choice of method for specific applications. Depending on the characteristics of each application, the appropriate method could be used for generating solutions. Whilst both DTM and ADM could generate reliable results, the level of accuracy required is critical in the decision.

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Ahmad M. D. Al-Eybani "A comparison between the Differential Transform Method and the Adomian Decomposition Method for the Burgers equation." International Journal of Engineering Research and Applications (IJERA) , vol. 7, no. 12, 2017, pp. 43-48.