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Fuzzy arithmetic and its application in problems of optimal choice

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ABSTRACT
The article examines the rules for performing basic arithmetic operations on fuzzy gradations. General relations are obtained for estimating the certainty of results when performing arithmetic operations. The solution of the optimal choice problem based on fuzzy models is given.

***Keywords*:** fuzzy set, fuzzy arithmetic, fuzzy gradation, optimal choice, decision making.

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I. INTRODUCTION

When solving system problems in conditions of uncertainty and ambiguity, it is useful to use a fuzzy representation of the data. Idempotent operations are most often used to obtain the best solution on fuzzy sets, which is due to their convenience and reliability of estimates. However, if there is a need to perform arithmetic operations on fuzzy sets, then the advantages of fuzzy representation are lost. In this case, as a rule, the Zadeh generalization principle is used, but the calculations are cumbersome, the results are not very clear, and they are difficult to interpret [2, 3, 5]. Approximation of fuzzy values by different distributions (uniform, exponential, triangular, etc.) also does not allow using the advantages of fuzzy representation [1, 4, 9]. It should be borne in mind that in many system tasks of the optimal choice related to the analysis and synthesis of systems, decision making, control and evaluation, the specific numerical content of the quantities does not matter, at least at the stage of algorithmization of the task and solution search, but only the order relation between them. Therefore, it becomes necessary to operate on quantities without being tied to a numerical context. The author suggests an approach based on the use of fuzzy gradations, in which the numbers are replaced by quantities. In [6 – 8], the advantages of this approach in solving various problems were shown. The purpose of this paper is the application of fuzzy gradations in problems of optimal choice.

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II. ARITHMETIC OPERATIONS ON FUZZY GRADATIONS

To describe the object area we use the fuzzy gradations in the range VL…VH. The range comprising gradations VL, L, M, H, VH (see below) we call the basic scale and with the adding of the intermediate gradations – extended scale [8]. Introduce also two marginal gradations out of range: VVL (lowest value) and VVH (highest value). Depending on condition of the task gradation VVL can be interpreted as zero, lower bound, exact lower bound etc. and gradation VVH – as unit, infinity, upper bound, exact upper bound etc. For any two gradations x and y of the scale we introduce a measure of distance. The relationship of similarity has the form

$$d(x, y) = \alpha(x, y) \tag{1}$$

where $\alpha(x, y)$ – the degree of proximity of the two gradations. Then the relationship of difference has the form

$$d(x, y) = \overline{\alpha}(x, y) \tag{2}$$

where $\overline{\alpha}(x, y)$ – the opposite value, i.e. if $\alpha(x, y)$ = VH, then $\overline{\alpha}(x, y)$ = VL, etc. Values $\alpha(x, y)$ are given in table 1, where due to symmetry $\alpha(x, y) = \alpha(y, x)$

Table 1 The matrix of correspondences

Fuzzy gradations	VL	VL-L	L	L-M	M	M-H	H	H-VH	VH
	$\alpha(x, y)$								
VL	VH	H-VH	H	M-H	M	L-M	L	VL-L	VL
VL-L		VH	H-VH	H	M-H	M	L-M	L	VL-L
L			VH	H-VH	H	M-H	M	L-M	L
L-M				VH	H-VH	H	M-H	M	L-M
M					VH	H-VH	H	M-H	M
M-H						VH	H-VH	H	M-H
H							VH	H-VH	H
H-VH								VH	H-VH
VH									VH

Note. VL – very low value, (VL-L) – between very low and low, L – low, (L-M) – between low and middle, M – middle (medium), (M-H) – between middle and high, H – high, (H-VH) – between high and very high, VH – very high.

For most applications the accuracy of table 1 is sufficient; if necessary table 1 can be made in increments of half or a quarter of the gradation, however the simplicity and reliability of the calculations are lost. Consider the arithmetic operations (summation and multiplication) on the set of fuzzy gradations. The results for multiplication of fuzzy gradations defined in the extended range (scale) are given in table 2 and for summation – in table 3.

When compiling the tables, it was assumed that the entire range of gradations corresponds to a numerical range of 0 ... 1 (any other range can be reduced to a single range by a linear transformation), which was divided into five equal intervals in the number of basic gradations. This establishes a one-to-one correspondence between each fuzzy gradation of the scale and the corresponding numerical interval. It is assumed that the value of the gradation is concentrated in the center of the interval. When performing the calculations, fuzzy gradations VL, L, M, H, VH correspond with the values 0.1; 0.3; 0.5; 0.7; 0.9 respectively. Intermediate gradations VL-L, L-M, M-H, H-VH correspond to values 0.2; 0.4; 0.6; 0.8 respectively. The value VVL in the table 2 means that the result is outside the left boundary of the range; the value VVH means that the result is outside the right boundary of the range. It should be noted that in calculations it makes no sense to introduce small gradation shares, and rounding should be used towards the nearest gradation, since this does not affect the accuracy of the final result (see below). When the number of factors (summands) is greater than two, the result is also determined using tables. The process quickly converges as the number of components (factors or terms) increases, in the sense that, for three to four components, the extreme limits of the range are reached. From the tables, you can determine the results for inverse operations (subtraction and division). Of course, if necessary, fuzzy gradations can be interpreted in the form of named values corresponding to the object area.

Table 2
The calculation of the product of two fuzzy gradations

Fuzzy gradations of factors	VL	VL-L	L	L-M	M	M-H	H	H-VH	VH
VL	VVL	VVL	VVL	VVL	VL	VL	VL	VL	VL
VL-L		VVL	VL	VL	VL	VL	VL	VL-L	VL-L
L			VL	VL	VL-L	VL-L	VL-L	VL-L	L
L-M				VL-L	VL-L	VL-L	L	L	L-M
M					L	L	L-M	L-M	M
M-H						L-M	L-M	M	M
H							M	M-H	M-H
H-VH								M-H	H
VH									H-VH

Note. The values are given with rounding to nearest gradation of extended scale

Table 3
The calculation of sum of two fuzzy gradations

Fuzzy gradations of summands	VL	VL-L	L	L-M	M	M-H	H	H-VH	VH
VL	VL-L	L	L-M	M	M-H	H	H-VH	VH	VVH
VL-L		L-M	M	M-H	H	H-VH	VH	VVH	VVH
L			M-H	H	H-VH	VH	VVH	VVH	VVH
L-M				H-VH	VH	VVH	VVH	VVH	VVH
M					VVH	VVH	VVH	VVH	VVH
M-H						VVH	VVH	VVH	VVH
H							VVH	VVH	VVH
H-VH								VVH	VVH
VH									VVH

Note. The values are given with rounding to nearest gradation of extended scale
The operation of bounded summation is also used in practical applications; the results for it are given in table 4.

Table 4
The calculation of restricted sum of two fuzzy gradations: $x\ [+]\ y = x + y - xy$

Fuzzy gradations of summands	VL	VL-L	L	L-M	M	M-H	H	H-VH	VH
VL	VL-L	L	L-M	M	M-H	M-H	H	H-VH	VH
VL-L		VL-L	VL-L	M	M-H	H	H-VH	H-VH	VH
L			M	M-H	H	H	H-VH	VH	VH
L-M				M-H	H	H-VH	H-VH	VH	VH
M					H-VH	H-VH	VH	VH	VVH
M-H						H-VH	VH	VH	VVH
H							VH	VH	VVH
H-VH								VVH	VVH
VH									VVH

Note. The values are given with rounding to nearest gradation of extended scale
The results of calculations for traditional fuzzy set idempotent operations $\min$ and $\max$ are given in table 5 and table 6 respectively.

Table 5
The calculation of operation $\min$ for two fuzzy gradations

Fuzzy gradation of components	VL	VL-L	L	L-M	M	M-H	H	H-VH	VH
VL	VL	VL	VL	VL	VL	VL	VL	VL	VL
VL-L		VL-L	VL-L	VL-L	VL-L	VL-L	VL-L	VL-L	VL-L
L			L	L	L	L	L	L	L
L-M				L-M	L-M	L-M	L-M	L-M	L-M
M					M	M	M	M	M
M-H						M-H	M-H	M-H	M-H
H							H	H	H
H-VH								H-VH	H-VH
VH									VH

Table 6
The calculation of operation max for two fuzzy gradations

Fuzzy gradation of components	VL	VL-L	L	L-M	M	M-H	H	H-VH	VH
VL	VL	VL-L	L	L-M	M	M-H	H	H-VH	VH
VL-L		VL-L	L	L-M	M	M-H	H	H-VH	VH
L			L	L-M	M	M-H	H	H-VH	VH
L-M				L-M	M	M-H	H	H-VH	VH
M					M	M-H	H	H-VH	VH
M-H						M-H	H	H-VH	VH
H							H	H-VH	VH
H-VH								H-VH	VH
VH									VH

It should be noted that for most applications the accuracy of tables 2 – 6 is sufficient; if necessary, tables can be compiled with a step of half or even a quarter of the gradation, although at the same time the simplicity of computation and reliability are lost. In a certain sense, we can speak of fuzzy arithmetic, since all operations are performed directly on fuzzy gradations. In ordinary arithmetic, the axiom of generating numbers is valid: n (+) 1 = n + 1, where n = 1, 2, ... In fuzzy arithmetic, it should be replaced by the axiom of generating quantities (amounts), since we are dealing with sets: u (+) VL = u + VL, where u = VL, VL-L, L etc. From the results given in tables 2-6, it follows that for two arbitrary gradations x and y there is a chain of inequalities (in view of rounding)

$xy < \min(x, y) \le \max(x, y) \le x[+]y \le x + y$. (3)

Note that operations of multiplication and bounded summation belong to Archimedean operations and the summation operation within the range under consideration belongs to nilpotent operations. This operation can be also defined outside the range, if it is necessary [7]. Fuzzy gradations form an Abelian addition group and an Abelian semigroup for multiplication.

III. THE CERTAINTY OF CALCULATIONS IN FUZZY ARITHMETIC

Since the degree of certainty is determined by the index of fuzziness, we consider the change in the index of fuzziness when performing operations of multiplication, bounded summation and summation. Designate x, y etc. – the values represented as fuzzy gradations; ν_x, ν_y etc. − the indexes of fuzziness corresponding to values are also represented as fuzzy gradations. Each gradation is a function value of which is concentrated in the center of the corresponding interval, and the value of the membership function is 1. In this sense the accuracy of determining the gradation is equal to 1(the index of fuzziness is 0), if it is not stipulated special conditions. We consider the distribution on the gradations. Then the accuracy (certainty) of the result is defined as for fuzzy set elements of which are individual gradations. This case is of interest for practice. For the index of fuzziness we use two expressions. The first of these has the form

$\nu_x = 2\min(x, \bar{x})$, (4)

where $\bar{x}$ – the opposite value to x; for example, if x = VL, then $\bar{x}$ = VH etc. Multiplication by the number 2 is understood as the summation of two equal values represented as fuzzy gradations. The expression (4) corresponds to the strong condition $\beta > \nu$, or

$\beta >$ H (0.7), (5)

where β is an estimate of the degree of certainty (reliability) of the result r or $\bar{r}$. In brackets here and below is given numeric value corresponding to the maximum of the gradation. The second expression for the index of fuzziness has the form

$\nu = \min(x, \bar{x})$, (6)

which corresponds to a softer condition $\beta > \nu/2$ or

$\beta >$ M (0.5), (7)

Consider the derivation of general relations for (4) and then discuss how they change for (6). *Operation of multiplication*. For product of two graduations we obtain four relations for index of fuzziness.

1. $\nu_{xy} \le \nu_x \cdot \nu_y$. (8a)

Ratio (8a) is true under the conditions: if {x and $y <$ H and simultaneously x or $y \le$ M} or {x and $y <$ M-H (0.6); ({x or $y <$ H} for extended scale).

2. $\nu_{xy} < \max(\nu_x, \nu_y)$. (8b)

Ratio (8b) is true under the conditions: if {x or $y <$ H and simultaneously $xy <$ L-M (0.4)} or {x or $y \le$ M}.

3. $\nu_{xy} < \nu_x + \nu_y$. (8c)

Ratio (8c) is true under the conditions: if {x or $y \ge$ H}. If {x or y = VVH}, then $\nu_{xy} = \nu_x + \nu_y$.

4. $\nu_{xy} \le \min(\nu_x, \nu_y)$. (8d)

Ratio (8d) is true under the conditions: if {x and $y < \mathrm{H}$} or {x and $y \le \mathrm{H}$ and simultaneously x or $y \le \mathrm{M}$}.

Operation of bounded summation. In this case we have four relations.

1. $\nu_{x\,[+]\,y} \le \nu_x\ [+]\ \nu_y$. (9a)

The ratio (9a) is correct for the basic scale, if {x or $y \ge \mathrm{M}$}. For the extended scale (9a) is correct under the conditions: if {$x = \mathrm{VL}$ or VL-L and simultaneously $y \ge$ M} or {$x = \mathrm{L}$ or L-M and simultaneously $y \ge$ L-M} or {x and $y \ge \mathrm{M}$}.

2. $\nu_{x\,[+]\,y} < \nu_x + \nu_y$. (9b)

The ratio (9b) is always correct, since the left part is always less than the right part.

3. $\nu_{x\,[+]\,y} \le \max(\nu_x, \nu_y)$. (9c)

The ratio (9c) is true for both scales, if {x or $y \ge \mathrm{M}$}; it is true for the extended scale, if {$x =$ L-M and simultaneously $y \ge$ L-M}.

4. $\nu_{x\,[+]\,y} \le \min(\nu_x, \nu_y)$. (9d)

The ratio (9d) is true for the basic scale under following conditions: if {x or $y > \mathrm{H}$} or {x and $y \ge \mathrm{L}$ and simultaneously x or $y > \mathrm{M}$} or {x and $y \ge \mathrm{M}$}. These conditions can be also written in alternative form, namely (9d) is true for the basic scale if {$x = \mathrm{VL}$ and simultaneously $y > \mathrm{H}$} or {$x = \mathrm{L}$ and simultaneously $y > \mathrm{M}$} or {x and $y \ge \mathrm{M}$}. For the extended scale ratio (9d) is true under the conditions: if {x and $y \ge \mathrm{VL}$ and simultaneously x or $y >$ H-VH(0.8)} or {x and $y > \mathrm{VL}$ and simultaneously x or $y \ge$ H-VH} or {x and $y \ge \mathrm{L}$ and simultaneously x or $y \ge$ M-H(0.6)} or {x and $y > \mathrm{L}$ and simultaneously x or $y \ge$ L-M} or {x and $y \ge \mathrm{M}$}. These conditions can be also written in alternative form, namely (9d) is true for the extended scale if {$x = \mathrm{VL}$ and simultaneously $y >$ H-VH} or {$x =$ VL-L(0.2) and simultaneously $y \ge$ H-VH} or {$x = \mathrm{L}$ and simultaneously $y \ge$ M-H} or {$x =$ L-M and simultaneously $y \ge$ L-M} or {x and $y \ge \mathrm{M}$}.

Operation of summation. In this case we have three ratios.

1. $\nu_{x+y} \le \nu_x + \nu_y$. (10a)

The ratio (10a) is always true.

2. $\nu_{x+y} \le \max(\nu_x, \nu_y)$. (10b)

The ratio (10b) is true, if {$x + y > \mathrm{M}$ and simultaneously x or $y < \mathrm{L}$} or {x and $y \ge \mathrm{L}$ and simultaneously $x + y \ge \mathrm{H}$}.

3. $\nu_{x+y} \le \min(\nu_x, \nu_y)$ (10c)

The ratio (10c) is true for the basic scale, if {$x + y \ge$ VH and simultaneously x or $y < \mathrm{L}$} or {x and $y \ge \mathrm{L}$ and simultaneously $x + y \ge \mathrm{H}$}. These conditions can be also written in alternative form, namely (10c) is true for the basic scale, if {$x = \mathrm{VL}$ and simultaneously $y = \mathrm{VH}$} or {$x \ge \mathrm{L}$ and simultaneously $y \ge \mathrm{M}$}. For the extended scale ratio (10c) is true under the conditions: if {x or $y \ge$ H-VH} or {x and $y \ge$ VL-L and simultaneously x or $y \ge$ M-H} or {x and $y \ge \mathrm{L}$ and simultaneously x or $y \ge$ L-M} or {x and $y \ge \mathrm{M}$}. These conditions can be also written in alternative form, namely (10c) is true for the extended scale under the conditions: if {$x = \mathrm{VL}$ and simultaneously $y \ge$ H-VH ($x + y \ge \mathrm{VH}$)} or {$x =$ VL-L and simultaneously $y \ge$ M-H ($x + y \ge$ H-VH)} or {$x \ge \mathrm{L}$ and simultaneously $y \ge$ L-M ($x + y \ge \mathrm{H}$)} or {x and $y \ge \mathrm{M}$}. We also write general ratios for operations **min**, **max** on fuzzy gradations that have the form

$$\nu_{\min(x,y)} \le \max(\nu_x, \nu_y), \qquad (11a)$$

$$\nu_{\max(x,y)} \le \max(\nu_x, \nu_y). \qquad (11b)$$

These ratios are correct unconditionally (absolutely). They can be defined more exactly, if we make additional assumptions about the change of initial values. In particular, we have

$$\nu_{\min(x,y)} \le \min(\nu_x, \nu_y), \qquad (11c)$$

if {x or $y < \mathrm{M}$ and simultaneously $\min(x, y) < \overline{\max(x, y)}$}.

$$\nu_{\max(x,y)} \le \min(\nu_x, \nu_y), \qquad (11d)$$

if {x or $y < \mathrm{M}$ and simultaneously $\min(x, y) > \overline{\max(x, y)}$}.

$$\nu_{\min(x,y)} = \nu_{\max(x,y)} = \min(\nu_x, \nu_y) = \max(\nu_x, \nu_y), \qquad (11e)$$

if {x or $y < \mathrm{M}$ and simultaneously $\min(x, y) = \overline{\max(x, y)}$}, where the line above means the opposite value of fuzzy gradation (vide supra). If several relations are valid at the same time, then we must choose a relation with the smallest permissible value of the right-hand side, ceteris paribus. Here is an example of calculation in which the results obtained by the numerical method. Let $x =$ VL, $y = \mathrm{H}$, then $\nu_x =$ VL-L, $\nu_y =$ M-H, $\nu_{xy} =$ VL, $\nu_x\nu_y =$ VL, $\nu_{x[+]y} =$ M-H, $\nu_x\ [+]\ \nu_y =$ H, $\nu_{x+y} =$ L-M, $\nu_x + \nu_y =$ H-VH, $\nu_{\min(x,y)} =$ VL-L, $\nu_{\max(x,y)} =$ M-H, $\min(\nu_x, \nu_y) =$ VL-L, $\max(\nu_x, \nu_y) =$ M-H, $\min(x, y) =$ VL, $\max(x, y) =$ H, $\overline{\max(x, y)} = \mathrm{L}$. It is seen that the obtained general relations are correct. For ν_{xy} we have the case (8a); for $\nu_{x[+]y}$ we have the case (9c); for ν_{x+y} we have the case (10b); for $\nu_{\min(x,y)}$ we have the case (11c); for $\nu_{\max(x,y)}$ we have the case (11b). When we use expression (6) for the index of fuzziness it is easy to see that for the operation of multiplication the relation (8a) is shifted towards smaller values, because the right part is reduced stronger than the left. Equation (8a) will be correct, if {x and $y < \mathrm{M}$}. Ratios (8b) – (8d) do not change. For the operation of bounded summation the ratio (9a) will be always true, as the right side increases because of the decrease of value $\nu_x\nu_y$ in expression

v_x [+] $v_y = v_x + v_y - v_x v_y$. Relation (9b) will be always true. Relations (9c), (9d) will be true under the same conditions. For the operation of summation ratios (10a) – (10c) remain correct under the same conditions.

IV. 4. FUZZY MODELS OF OPTIMAL CHOICE

We apply our results to the problem of optimal choice. The problem is formulated as follows. There are many possible solutions (alternatives) $X = \{x_1,..., x_m\}$. Each alternative is evaluated by a set of criteria $\{K_1,..., K_n\}$. We also know the weights (importance) of the criteria $\{a_1,..., a_n\}$. The values of the criteria and weights are represented in the form of fuzzy gradations. It is required to determine the fitness of alternatives for purpose and to choose the best solution. To solve the formulated problem, it is expedient to use the method of threshold criteria and the distance method. The first method allows us to obtain a lower estimate, and the second gives an upper estimate and allows us to determine the so-called indirect costs. We also consider two methods occupying an intermediate position: convolution by the worst criterion, at which the risk of a selection error due to the model is the smallest and additive convolution corresponding to the averaging strategy. To apply the threshold criteria method, the threshold values of the criteria should be known. We define them directly from the initial data (see table 7). Then the value of the general criterion for an arbitrary alternative *x* is given by the following expression that does not depend on the weight of criteria

$$K(x) = \min_j K_j(x) / \min_x K_j(x), \qquad (12)$$

and the best solution is

$$x^* = \arg\max_{x \in X} K(x). \qquad (13)$$

For the application of the distance method, an "ideal" solution should be known. We define it, as above, directly from the experimental data (see table 7). As a measure of distance, we use the Hamming function and Euclid's function, corresponding to the strategy of the mean and the mean square, respectively, as well as the functions of the greatest and smallest difference, corresponding to the limiting strategies. In the case of the Hamming function, we have for the distance of an arbitrary alternative *x* to an ideal solution

$$d(x) = \sum_j a_j \left| K_j(x) - \max_{x \in X} K_j(x) \right| \qquad (14)$$

In the case of Euclid's function, the analogous expression has the form

$$d(x) = \left(\sum_j a_j^2 \left| K_j(x) - \max_{x \in X} K_j(x) \right|^2 \right)^{1/2} \qquad (15)$$

In the case of a function of the greatest difference, we have the expression

$$d(x) = \max_j a_j \left| K_j(x) - \max_{x \in X} K_j(x) \right| \qquad (16)$$

In the case of a function of the smallest difference, we can write

$$d(x) = \min_j a_j \left| K_j(x) - \max_{x \in X} K_j(x) \right| \qquad (17)$$

The best solution in all cases is defined as the closest to the ideal

$$x^* = \arg\min_{x \in X} d(x). \qquad (18)$$

In the case of the convolution by the worst criterion the value of the general criterion for an arbitrary alternative *x* is given by the following expression

$$K(x) = \min_j a_j K_j(x) \qquad (19)$$

In the case of the additive convolution we have the relation

$$K(x) = \sum_j a_j K_j(x) \qquad (20)$$

The best solution in both cases is defined from (13). Note that when using a representation in the form of fuzzy gradations, it is not necessary to fulfill the normalization condition for the weights of the criteria, it is only necessary that the results of the calculations do not go beyond the scale, which is ensured by multiplying by a small gradation (see below). It should be borne in mind that the estimates obtained are not absolute, but relative, since they are satisfied in the scale of order and allow any monotonic transformation at which the result remains within the scale. Let's consider a concrete example. Let *X* – the set of objects, for example, projects of technical system, consisting of five variants of solutions (alternatives), each of which is evaluated according to five criteria, where K_1 is a functional criterion, K_2 is economic, K_3 is ergonomic, K_4 is ecological, and K_5 is social. Generalized criteria were used to simplify the calculations. The initial data are given in table 7. The degree of certainty (reliability) of the data is assumed to be equal to VH, so that condition (5) is satisfied. It is easy to see that alternatives form the Pareto set, so none of them can be excluded.

Table 7
The initial data for the example

Alternatives, x_i	Criteria, K_j				
	K_1	K_2	K_3	K_4	K_5
x_1	H	M	M	L	VH
x_2	VH	H	M	L	H
x_3	M-H	M	L	M	M-H
x_4	M	M	M	M	M
x_5	M	L	VH	H	M
"Threshold" solution	M	L	L	L	M
"Ideal" solution	VH	H	VH	H	VH

Note. All values are given in the direct scale. Hereinafter, a short line is used to denote an intermediate gradation, and long line means a subtraction operation.

When using the threshold criteria method, calculations according to (12) give $K(x_1)$ = min{H/M, M/L, M/L, L/L, VL/M} = 1(one unit). Similarly, we have $K(x_2)$ = 1(one unit), $K(x_3)$ = 1(one unit), $K(x_4)$ = 1(two units), $K(x_5)$ = 1(three units). It follows from (13) that all solutions are equivalent. If we take into account the number of units, i.e. coincidences with the threshold values, then solutions can be distinguished. The most preferable solutions are x_1, x_2, x_3, which correspond to the smallest number of units (one unit). The ranking according to the "degree of admissibility" has the form {x_1, x_2, x_3}, x_4, x_5. We apply the distance method using the Hamming function, Euclid's function and functions of the greatest and smallest difference. Suppose that the importance of the criteria is the same $\boldsymbol{a_1 = a_2 = ... = a_5 =}$ **M**. In this case, the weight of the criteria may not be taken into account; it can be used to ensure that the result remains within the scale. In the case of the *Hamming function*, calculations from equation (14) using tables 2 and 3 give for alternative x_1: $d(x_1)$ = M·(VH – H) + M·(H – M) + M·(VH – M) + M·(H – L) + M·(VH – VH) = M-H. Similarly for other alternatives, we have $d(x_2)$ = M, $d(x_3)$ = H-VH, $d(x_4)$ = H-VH, $d(x_5)$ = M-H. The best solution in accordance with (18) is x_2. The ranking by degree of proximity to the ideal solution has the form x_2, {x_1, x_5}, {x_3, x_4}. The solutions in braces are equivalent. In the case of the *Euclid's function*, the results are almost the same, but the solutions are less distinct. We have from (15): $d^2(x_1) = d^2(x_2) = d^2(x_4) = d^2(x_5)$ = VL, $d^2(x_3)$ = VL-L. The ranking has the form{x_1, x_2, x_4, x_5}, x_3. In the case of *function of the greatest difference*, calculations from equation (16) using tables 2, 3 and 6 give for alternative x_1: $d(x_1)$ = max{M·(VH – H); M·(H – M); M·(VH – M); M·(H – L); M·(VH – VH)}= VL-L. Similarly for other alternatives, we obtain $d(x_2) = d(x_4) = d(x_5)$ = V-L, $d(x_3)$ = L. So, the solutions 1, 2, 4 and 5 are equivalent. The ranking has the form{x_1, x_2, x_4, x_5}, x_3. In the case of *function of the smallest difference*, calculations from equation (17) using tables 2, 3 and 5 give for alternative x_1: $d(x_1)$ = min{M·(VH – H); M·(H – M); M·(VH – M); M·(H – L); M·(VH – VH)}= 0(one zero). Similarly $d(x_2) = d(x_5)$ = 0(two zeros), $d(x_3) = d(x_4)$ =VL. The best solutions in accordance with (18) are x_2 and x_5. The ranking has the form{x_2, x_5}, x_1, {x_3, x_4}. Note that with our initial data, the result for the function of the smallest difference does not depend on the weight of the criteria (see below). In the case of the *convolution by the worst criterion*, calculations from equation (19) using tables 2 and 5 give for alternative x_1: $K(x_1)$ = min{M·H; M·M; M·M; M·L; M·VH} = VL-L. Similarly $K(x_2) = K(x_3) = K(x_5)$ = V-L, $K(x_4)$ = L (with rounding). So, the best solution in accordance with (13) is x_4; the solutions 1, 2, 3 and 5 are equivalent. The ranking has the form x_4, {x_1, x_2, x_3, x_5}. In the case of the *additive convolution*, calculations from equation (20) using tables 2 and 3 give for alternative x_1: $K(x_1)$ = M·H + M·M + M·M + M·L + M·VH = L-M. Similarly $K(x_3) = K(x_4) = K(x_5)$ = L-M, $K(x_2)$ = M. So, the best solution in accordance with (13) is x_2; the solutions 1, 3, 4 and 5 are equivalent. The ranking has the form x_2, {x_1, x_3, x_4, x_5}. Let's explore how the result depends on changing the importance of the criteria. Suppose that the importance of criteria increases from K_1 to K_5, and the importance of the criteria K_1 and K_2 is approximately the same. Let $\boldsymbol{a_1 = a_2 =}$ **L**, $\boldsymbol{a_3 =}$ **L-M**, $\boldsymbol{a_4 =}$ **M**, $\boldsymbol{a_5 =}$ **M-H**. In the case of the *Hamming function*, calculations from equation (14) using tables 2 and 3 give for alternative x_1: $d(x_1)$ = L·(VH – H) + L·(H – M) + (L-M)·(VH – M) + M·(H – L) + (M-H)·(VH – VH) = M. Similarly we have $d(x_2)$ = M, $d(x_3)$ = H, $d(x_4)$ = H, $d(x_5)$ = L-M. It follows from (18) that the best solution is x_5. The ranking by degree of proximity to the ideal solution has the form x_5, {x_1, x_2,}, {x_3, x_4}. For the *Euclid's function*, the results are similar. We have from (15): $d^2(x_1) = d^2(x_2) = d^2(x_3) = d^2(x_4) = d^2(x_5)$ = VL, so all solutions are equivalent; the ranking has the form {x_1, x_2, x_3, x_4, x_5}. In the case of *function of the greatest difference*, calculations from equation (16)

using tables 2, 3 and 6 give $d(x_1) = d(x_2) = d(x_3) = d(x_4) = d(x_5)$ = VL-L. So, all solutions are equivalent. The ranking has the form $\{x_1, x_2, x_3, x_4, x_5\}$. In the case of *function of the smallest difference*, calculations from equation (17) using tables 2, 3 and 5 give the same result as above with an equal weight of the criteria: $d(x_1) = 0$(one zero), $d(x_2) = d(x_5) = 0$(two zeros), $d(x_3) = d(x_4)$ =VL. The best solutions in accordance with (18) are x_2 and x_5. The ranking has the form$\{x_2, x_5\}$, x_1, $\{x_3, x_4\}$. In the case of the *convolution by the worst criterion*, calculations from equation (19) using tables 2 and 5 give: $K(x_1) = K(x_2) = K(x_3) = K(x_4)$ = VL-L, $K(x_5)$ = VL (with rounding). So, the solutions 1, 2, 3 and 4 are equivalent. The ranking has the form $\{x_1, x_2, x_3, x_4\}$, x_5. In the case of the *additive convolution*, calculations from equation (20) using tables 2 and 3 give: $K(x_1) = K(x_2) = K(x_5)$ = L-M, $K(x_3) = K(x_4)$ = L. In the calculation, each term of the sum was multiplied by a gradation L so that the results remained within the scale. The solutions x_1, x_2, x_5 are equivalent. The ranking has the form $\{x_1, x_2, x_5\}$, $\{x_3, x_4\}$. We will change the priorities, assuming that the importance of the criteria decreases from K_1 to K_5, and the importance of the criteria K_4 and K_5 is approximately the same. Assume that $\boldsymbol{a_1}$ **= M-H,** $\boldsymbol{a_2}$ **= M,** $\boldsymbol{a_3}$ **= L-M,** $\boldsymbol{a_4} = \boldsymbol{a_5}$ **= L**. In the case of the *Hamming function*, the calculations from (14) using tables 2 and 3 give $d(x_1)$ = (L-H)·(VH – H) + M·(H – M) +(L-M)·(VH – M) + L·(H – L) + L·(VH – VH) = M. Similarly we have for other alternatives $d(x_2)$ = L-M, $d(x_3) = d(x_4)$ = H, $d(x_5)$ = M. The best solution in accordance with (18) is x_2. The ranking has the form x_2, $\{x_1, x_5\}$, $\{x_3, x_4\}$. For *Euclid's function*, we have from (15): $d^2(x_1) = d^2(x_3) = d^2(x_4) = d^2(x_5)$ = VL, $d^2(x_2)$ = VVL. So, the best solution in accordance with (18) is x_2; the ranking has the form x_2, $\{x_1, x_3, x_4, x_5\}$. In the case of *function of the greatest difference*, we have from equation (16):
$d(x_1) = d(x_2) = d(x_3) = d(x_4) = d(x_5)$ = VL-L. So, all solutions are equivalent. The ranking has the form $\{x_1, x_2, x_3, x_4, x_5\}$. In the case of *function of the smallest difference*, the results do not change (see above). In the case of the *convolution by the worst criterion*, we obtain from (19) using tables 2 and 5: $K(x_1) = K(x_2) = K(x_3)$ = VL, $K(x_4) = K(x_5)$ = VL-L. The best solutions in accordance with (13) are x_4 and x_5. The ranking has the form $\{x_4, x_5\}$, $\{x_1, x_2, x_3\}$. In the case of the *additive convolution*, we obtain from (20) using tables 2 and 3: $K(x_1)$ = (M-H)·H + M·M + (L-M)·M + L·L + L·VH = L-M. In the calculation, each term of the sum was multiplied by a gradation L so that the results remained within the scale. Similarly for other alternatives, taking into account the same factor L, we obtain $K(x_2)$ = L-M, $K(x_3)$ = L, $K(x_4)$ = L, $K(x_5)$ = L-M (the last result is obtained with rounding towards a larger gradation). The best alternatives from (13) are $\{x_1, x_2, x_5\}$. The ranking has the form $\{x_1, x_2, x_5\}$, $\{x_3, x_4\}$. A summary of the results is given in table 8. So, in our example, the most justified is the application of the distance method (the distance of the smallest difference). The calculations show that the most "intensive" is the variant x_2 (economic components prevail), and the most "gentle"(humane) is x_5 (ergonomic and ecological components prevail).

Table 8
The summary of results

Method (model)	The best solution	Ranking	Weight (importance) of criteria
Threshold criteria method	$\{x_1, x_2, x_3\}$	$\{x_1, x_2, x_3\}$, x_4, x_5	arbitrary
Hamming function	x_2	x_2, $\{x_1, x_5\}$, $\{x_3, x_4\}$	equal
	x_5	x_5, $\{x_1, x_2\}$, $\{x_3, x_4\}$	$a_1 = a_2$ = L, a_3 = L-M, a_4 = M, a_5 = M-H
	x_2	x_2, $\{x_1, x_5\}$, $\{x_3, x_4\}$	a_1 = M-H, a_2 = M, a_3 = L-M, $a_4 = a_5$ = L.
Euclid's function	$\{x_1, x_2, x_4, x_5\}$	$\{x_1, x_2, x_4, x_5\}$, x_3	equal
	$\{x_1, x_2, x_3, x_4, x_5\}$	$\{x_1, x_2, x_3, x_4, x_5\}$	$a_1 = a_2$ = L, a_3 = L-M, a_4 = M, a_5 = M-H
	x_2	x_2, $\{x_1, x_3, x_4, x_5\}$	a_1 = M-H, a_2 = M, a_3 = L-M, $a_4 = a_5$ = L.
Function of the greatest difference	$\{x_1, x_2, x_4, x_5\}$	$\{x_1, x_2, x_4, x_5\}$, x_3	equal
	$\{x_1, x_2, x_3, x_4, x_5\}$	$\{x_1, x_2, x_3, x_4, x_5\}$	$a_1 = a_2$ = L, a_3 = L-M, a_4 = M, a_5 = M-H
	$\{x_1, x_2, x_3, x_4, x_5\}$	$\{x_1, x_2, x_3, x_4, x_5\}$	a_1 = M-H, a_2 = M, a_3 = L-M, $a_4 = a_5$ = L.
Function of the	$\{x_2, x_5\}$	$\{x_2, x_5\}$, x_1, $\{x_3, x_4\}$	equal

smallest difference	$\{x_2, x_5\}$	$\{x_2, x_5\}, x_1, \{x_3, x_4\}$	$a_1 = a_2$ = L, a_3 = L-M, a_4 = M, a_5 = M-H
	$\{x_2, x_5\}$	$\{x_2, x_5\}, x_1, \{x_3, x_4\}$	a_1 = M-H, a_2 = M, a_3 = L-M, $a_4 = a_5$ = L.
Convolution by the worst criterion	x_4	$x_4, \{x_1, x_2, x_3, x_5\}$	equal
	$\{x_1, x_2, x_3, x_4\}$	$\{x_1, x_2, x_3, x_4\}, x_5$	$a_1 = a_2$ = L, a_3 = L-M, a_4 = M, a_5 = M-H
	$\{x_4, x_5\}$	$\{x_4, x_5\}, \{x_1, x_2, x_3\}$	a_1 = M-H, a_2 = M, a_3 = L-M, $a_4 = a_5$ = L.
Additive convolution	x_2	$x_2, \{x_1, x_3, x_4, x_5\}$	equal
	$\{x_1, x_2, x_5\}$	$\{x_1, x_2, x_5\}, \{x_3, x_4\}$	$a_1 = a_2$ = L, a_3 = L-M, a_4 = M, a_5 = M-H
	$\{x_1, x_2, x_5\}$	$\{x_1, x_2, x_5\}, \{x_3, x_4\}$	a_1 = M-H, a_2 = M, a_3 = L-M, $a_4 = a_5$ = L.

V. CONCLUSION

Thus, the selection of a preferred solution depends on the importance of criteria (priorities) defined by external purpose, as well as, the type of the model, which is determined by the initial data and preferences of the person (subject) making the decisions. The representation of data in the form of fuzzy gradations eliminates the problem of standardization of criteria and their importance arising in decision-making on many criteria in connection with the transition from the physical scale to the scale of order for each criterion. Approximation of calculations and estimates using fuzzy arithmetic allows us to smooth or eliminate inconsistency and errors of the initial data, which increases the reliability of decision making and allows us to reduce the risk associated with the inadequacy of the optimization model.

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