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On the Cubic Equation with Four Unknowns $x^3 + y^3 = 24zw^2$

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ABSTRACT

The cubic equation $x^3 + y^3 = 24zw^2$ is analyzed for its non – zero distinct integer solutions. Three different patterns of non-zero distinct integer solutions to the equation under consideration are obtained. A few interesting relation between the solutions and special numbers are exhibited.

***Keywords* -** Integral solutions, nasty number, Ternary Cubic

I. INTRODUCTION

The cubic equation offers an unlimited field for research because of their variety [1-2]. For an extensive review of various problems, one may refer [3-10]. This communication concerns with yet another interesting ternary quadratic equation $x^3 + y^3 = 24zw^2$ representing a homogenous cone for determining its infinitely may non-zero integral solutions. Also a few interesting relations among the solutions have been presented.

II. METHOD OF ANALYSIS

The cubic Diophantine equation with four unknowns to be solved for getting non – zero integral solution is

$x^3 + y^3 = 24zw^2$ ------------ *(1)*

On substituting the linear transformations

$x = u + v, y = u - v, z = u$ ---------*(2)*

in (1) leads to

$$(u+v)^3 + (u-v)^3 = 24uw^2$$

$$(u^3 + v^3 + 3uv(u+v)) + (u^3 - v^3 - 3uv(u-v))$$
$$= 24uw^2$$

$$u^3 + v^3 + 3u^2v + 3uv^2 + u^3 - v^3 - 3u^2v + 3uv^2$$
$$= 24uw^2$$

$$2u^3 + 2(3uv^2) = 24uw^2$$

$$2u^3 + 6uv^2 = 24uw^2$$

$$2u(u^2 + 3v^2) = 2u(12w^2)$$

$u^2 + 3v^2 = 12w^2$ ---------------(3)

We obtain four different patterns of integral solutions to (1) through solving (3) which are illustrated as follows :

Pattern 1:

Assume , $w = a^2 + 3b^2$ -----------(4)

Write '12' as

$12 = (3+i\sqrt{3})(3-i\sqrt{3})$ ---------(5)

Using (4) and (5) in (3) and employing factorization,

$u^2+3v^2=12w^2$

$u^2 + 3v^2 = 12w^2$

$(u+i\sqrt{3}v)(u-i\sqrt{3}v) = (3+i\sqrt{3})(3-i\sqrt{3})(a^2+3b^2)^2$
$= (3+i\sqrt{3})(3-i\sqrt{3})(a+i\sqrt{3}b)^2(a-i\sqrt{3}b)^2$

Which is equivalent to the system of equations

$(u+i\sqrt{3}v) = (3+i\sqrt{3})(a+i\sqrt{3}b)^2$ ---------(6a)

$(u-i\sqrt{3}v) = (3-i\sqrt{3})(a-i\sqrt{3}b)^2$ ---------(6b)

Equating the real and imaginary parts either in (6a) or (6b), we have

$(u+i\sqrt{3}v) = (3+i\sqrt{3})(a+i\sqrt{3}b)^2$
$= (3+i\sqrt{3})(a^2-3b^2+2ai\sqrt{3}b)$
$= 3a^2-9b^2+6ai\sqrt{3}b+a^2i\sqrt{3}-3i\sqrt{3}b^2-6ab$

$u = 3a^2-9b^2-6ab$ -------------(7)

$v = a^2-3b^2\ \_\ 6ab$ -------------(8)

On substituting (7) and (8) in (2), we get

$x = u+v$

$= 4a^2-12b^2$

$y = u-v$

$= 2a^2-6b^2-12ab$

$z = u$

$= 3a^2-9b^2-6ab$

The non – zero distinct integral solutions of (1), are

$x = x(a,b) = 4a^2-12b^2$

$y = y(a,b) = 2a^2-6b^2-12ab$

$z = z(a,b) = 3a^2-9b^2-6ab$

$w = w(a,b) = a^2+3b^2$

***Properties*:**

(i) $x(a,b)-y(a,b) \equiv 0 \pmod 2$

(ii) $x(a,b)-z(a,b)-t_{4,a} \equiv 0 \pmod 3$

(iii) $x(a,b)+2w(a,b)$ is a nasty number

(iv) $y(a,b)-2z(a,b) \equiv 0 \pmod 4$

(v) $z(a,b)-3w(a,b)$ is a nasty number

Pattern 2:

Write '12' as

$12 = \frac{(6+2i\sqrt{3})(6-2i\sqrt{3})}{4}$ ---- *(9)*

Using (4) and (9) in

$u^2+3v^2=12w^2$

$(u^2+3v^2)\frac{= (6+2i\sqrt{3})(6-2i\sqrt{3})(a^2+3b^2)^2}{4}$

$(u^2+3v^2)\frac{= (6+2i\sqrt{3})(6-2i\sqrt{3})(a+i\sqrt{3}b)^2(a-i\sqrt{3}b)^2}{4}$
$(u+i\sqrt{3}v)(u-i\sqrt{3}v)\frac{= (6+2i\sqrt{3})(6-2i\sqrt{3})(a+i\sqrt{3}b)^2(a-i\sqrt{3}b)^2}{4}$

Which is equivalent to the system of equations

$$(u+i\sqrt{3}v)\frac{=(6+2i\sqrt{3})(a+i\sqrt{3}b)^2}{2}$$ ------(10a)

$$(u-i\sqrt{3}v)\frac{=(6-2i\sqrt{3})(a-i\sqrt{3}b)^2}{2}$$ ------(10b)

Equating the real and imaginary parts either in (10a) or (10b), we have

$$(u+i\sqrt{3}v)\frac{=(6+2i\sqrt{3})(a+i\sqrt{3}b)^2}{2}$$

$$\frac{=1(6+2i\sqrt{3})(a^2-3b^2+2ai\sqrt{3}b)}{2}$$

Equating real and Imaginary parts, we have

$$u=\frac{1}{2}(6a^2-18b^2-12ab)$$ --------(11)

$$v=\frac{1}{2}(2a^2-6b^2+12ab)$$ ----------(12)

On substituting (11) and (12) in (2), we get

$x=u+v$

$=4a^2-12b^2$

$y=u-v$

$=2a^2-6b^2-12ab$

$z=u$

$=3a^2-9b^2-6ab$

The corresponding integer solutions are given by

$x=x(a,b)=4a^2-12b^2$

$y=y(a,b)=2a^2-6b^2-12ab$

$z=z(a,b)=3a^2-9b^2-6ab$

$w=w(a,b)=a^2+3b^2$

Now choose a and b suitably so that the solutions are in integers.

In particular, the choice a = 2A and

b = 2B leads to the integer solutions to (1)3are given by

$x=x(a,b)=16a^2-48b^2$

$y=y(a,b)=8A^2-24B^2-96ab$

$z=z(a,b)=12A^2-36B^2-24AB$

$w=w(a,b)=4A^2+12B^2$

Properties

(i) $x(a,b)-2y(a,b)$ is a nasty number

(ii) $y(a,b)-2z(a,b)\equiv 0(\mathrm{mod}\,4)$

(iii) $z(a,b)-3w(a,b)\equiv 0(\mathrm{mod}\,6)$

(iv) $3x(a,b)-4z(a,b)$ is a nasty number

(v) $y(a,b)-2w(a,1)\equiv 0(\mathrm{mod}\,3)$

Pattern 3:

Equation (3) can also be written as

$u^2+3v^2=12w^2*1$-------*(13)*

write '1' as

$$1=\frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4}$$ ------ *(14)*

Using*(4)* ,*(9)* and *(14)* in

$u^2+3v^2=12w^2*1$

$$u^2+3v^2=\frac{(6+2i\sqrt{3})(6-2i\sqrt{3})(1+i\sqrt{3})(1-i\sqrt{3})(a^2+3b^2)^2}{4}$$
$$(u+i\sqrt{3}v)(u-i\sqrt{3}v)=\frac{(6+2i\sqrt{3})(6-2i\sqrt{3})(1+i\sqrt{3})(1-i\sqrt{3})(a^2+3b^2)^2}{4}$$

Which is equivalent to the system of equations

$$(u+i\sqrt{3}v)=\frac{(6+2i\sqrt{3})(1+i\sqrt{3})(a+i\sqrt{3})}{2}$$
$$(u-i\sqrt{3}v)=\frac{(6-2i\sqrt{3})(1-i\sqrt{3})(a-i\sqrt{3})}{2}$$

Equating the real and imaginary parts,

$$u=\frac{1}{4}(-48ab)=-12ab \text{ ----- } (15)$$

$$v=\frac{1}{4}(8a^2-24b^2)=2a^2-6b^2 \text{ ----- } (16)$$

On substituting (15) and (16) in (2), we get

$$x=u+v = 2a^2-6b^2-12ab$$

$$y=u-v = -2a^2+6b^2-12ab$$

$$z=u=-12ab$$

The corresponding integer solutions are given by

$$x=x(a,b)=2a^2-6b^2-12ab$$

$$y=y(a,b)=-2a^2+6b^2-12ab$$

$$z=z(a,b)=-12ab$$

$$w=w(a,b)=a^2+3b^2$$

To find integral solutions, choose a = 2A and b = 2B leads to the integer solutions to (1) are given by

$$x=8A^2-24B^2-48AB$$

$$y=-8A^2+24B^2-48AB$$

$$z=-48AB$$

$$w=-4A^2+12B^2$$

Properties

(i) $x(a,b)+y(a,b)$ is a nasty number

(ii) $y(a,1)-z(a,1)-t_{18,a}-7a$ is a nasty number

(iii) $z(a,b)+w(a,b)\equiv 0(\bmod 4)$

(iv) $x(a,b)-z(a,b)\equiv 0(\bmod 8)$

(v) $x(a,1)-z(a,1)-8a^2$ is a nasty number

Pattern 4:

Equation (3) can be rewritten as

$$u^2-9w^2 = 3(w^2-v^2)$$

which is written in the form of ratio as,

$$\frac{u+3v}{w-v}=\frac{3(w+v)}{u-3v}=\frac{a}{b}$$

which is equivalent to the system of equations,

$$bu+3wb-aw+av=0$$
$$bu+av+(3b-a)w=0$$

and

$$\frac{3(w+v)}{u-3w}=\frac{a}{b}$$

which is equivalent to the system of equations,

$$au-3aw-3bw-3bv=0$$
$$au-3bv-(3a+3b)w=0$$

Applying the method of cross multiplication we have,

$u = -3a^2 + 9b^2 - 6ab$

$v = -a^2 + 3b^2 + 6ab$

$w = a^2 + 3b^2$

Substituting the values of 'u' and 'v' we get the non – zero distinct integral solutions to be

$x = x(a,b) = -4a^2 + 12b^2$

$y = y(a,b) = -2a^2 + 6b^2 - 12ab$

$z = z(a,b) = -3a^2 + 9b^2 - 6ab$

$w = w(a,b) = a^2 + 3b^2$

Properties

(i) $x(a,b) - y(a,b) \equiv 0 (\mathrm{mod}\, 2)$

(ii) $x(a,b) - z(a,b) - t_{4,a} \equiv 0 (\mathrm{mod}\, 3)$

(iii) $x(a,b) + 2w(a,b)$ *is a nasty number*

(iv) $y(a,b) - 2z(a,b) \equiv 0 (\mathrm{mod}\, 4)$

(v) $z(a,b) - 3w(a,b)$ is a nasty number

III. CONCLUSION

To conclude, one may search for other patterns of solutions to the equation under consideration.

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