On $Z^\alpha$-Open Sets and Decompositions of Continuity

A. M. Mubarki 1, M. A. Al-Juhani 2, M. M. Al-Rshudi 3, M. O. AL-Jabri 4

1 Department of Mathematics, Faculty of Science, Taif University, 2,3,4 Department of Mathematics, Faculty of Science, Taibah University Saudi Arabia

ABSTRACT

In this paper, we introduce and study the notion of $Z^\alpha$-open sets and some properties of this class of sets are investigated. Also, we introduce the class of $A^\L$-sets via $Z^\alpha$-open sets. Further, by using these sets, a new decompositions of continuous functions are presented.

(2000) AMS Subject Classifications: 54D10; 54C05; 54C08.

Keywords: $Z^\alpha$-open sets , $A^\L$-set, $Z_\alpha$-continuity and decomposition of continuity.

1. INTRODUCTION

J. Tong [21] introduced the notion of B-set and B-continuity in topological spaces. The concept of A'-sets, DS-set, A'-continuity, DS-continuity introduced by E. Ekici [4, 8] and used them to obtain a new decomposition of continuity. Noiri et al. [17] introduced the notion of $\eta$-set and $\eta$-continuity in topological spaces. The main purpose of this paper is to obtain a new decompositions of continuous functions. We introduce and study the notion of $Z^\alpha$-open sets and $A^\L$-sets. The relationships among $Z^\alpha$-open sets, $A^\L$-sets and the related sets are investigated. By using these notions, we obtain a new decompositions of continuous functions. Also, some characterizations of these notions are presented.

II.PRELIMINARIES

A subset $A$ of a topological space $(X, \tau)$ is called regular open (resp. regular closed) [20] if $A = \text{int}(\text{cl}(A))$ (resp. $A = \text{cl}(\text{int}(A))$). The $\delta$-interior [22] of a subset $A$ of $X$ is the union of all regular open sets of $X$ contained in $A$ and is denoted by $\text{int}(A)$. A subset $A$ of a space $X$ is called $\delta$-open [22] if it is the union of regular open sets. The complement of a $\delta$-open set is called $\delta$-closed. Alternatively, a set $A$ of $(X, \tau)$ is called $\delta$-closed [22] if $A = \text{cl}(\text{int}(A))$, where $\text{cl}(A) = \{x \in X: A \cap \text{int}(U) \neq \emptyset, U \in \tau \text{ and } x \in U\}$. Throughout this paper $(X, \tau)$ and $(Y, \sigma)$ (simply, $X$ and $Y$) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of a space $(X, \tau)$, $\text{cl}(A)$, $\text{int}(A)$ and $X \setminus A$ denote the closure of $A$, the interior of $A$ and the complement of $A$ respectively. A space $X$ is called submaximal [3] if every dense subset of $X$ is open. A space $(X, \tau)$ is called extremally disconnected (briefly. E. D.) [19] if the closure of every open set of $X$ is open. A subset $A$ of a space $X$ is called $\delta$-dense [6] if and only if $\text{cl}(A) = X$. A subset $A$ of a space $X$ is called $\alpha$-open [4], preopen [13], $\delta$-semiopen [18], semiopen [12], Z-open [11], $b$-open [1] or $\gamma$-open [10] or sp-open [3], e-open [5]) if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$ (resp. $\text{cl}(\text{int}(\text{cl}(A)))$, $\text{A}\subset \text{int}(\text{cl}(A))$, $\text{A}\subseteq \text{cl}(\text{int}(A))$, $\text{A}\subset \text{cl}(\delta(\text{int}(A)))$ $\cup \text{cl}(\text{int}(A)))$, $\text{A}\subset \text{cl}(\text{int}(A))$ $\cup \text{cl}(\text{int}(A))$, $\text{A}\subseteq \text{cl}(\delta(\text{int}(A)))$ $\cup \text{int}(\text{cl}(A)))$. The complement of a-open (resp. $\alpha$-open, preopen, $\delta$-semiopen, semiopen) sets contained a called $\alpha$-closed [4] (resp. $\alpha$-closed [16], preclosed [13], $\delta$-semi-closed [18], semi-closed [2]). The intersection of all $\alpha$-closed (resp. $\alpha$-closed, preclosed, $\delta$-semi-closed, semi-closed) sets containing $A$ is called the $\alpha$-closure (resp. $\alpha$-closure, preclosure, $\delta$-semi-closure, semi-closure) of $A$ and is denoted by $\text{cl}(A)$ (resp. $\text{cl}(A)$, $\text{cl}(A)$, $\text{cl}(A)$, $\text{cl}(A)$). The union of all a-open (resp. $\alpha$-open, preopen, $\delta$-semiopen, semiopen) sets contained in $A$ is called the $\alpha$-interior (resp. $\alpha$-interior, pre-interior, $\delta$-semi-interior, semi-interior) of $A$ and is denoted by $\text{int}(A)$ (resp. $\text{int}(A)$, $\text{int}(A)$, $\text{int}(A)$, $\text{int}(A)$). The family of all $\delta$-open (resp. $\alpha$-open, $\alpha$-open, preopen, $\delta$-semiopen, semiopen) is denoted by $\delta(X)$ (resp. $\delta(X)$, $\delta(X)$, $\delta(X)$, $\delta(X)$).

Lemma 2.1. Let $A, B$ be two subset of $(X, \tau)$. Then the following are hold: (1) $\text{cl}(A) = A \cup \text{cl}(\text{int}(A)))$ and $\text{int}(A) = A \cap \text{int}(\text{cl}(A)))$ [1], (2) $\delta(\text{cl}(A)) = A \cup \text{int}(\text{cl}(A)))$ and $\delta(\text{int}(A)) = A \cap \text{int}(\text{cl}(A)))$ [17], (3) $\text{p}(A) = A \cup \text{cl}(\text{int}(A)))$ and $\text{p}(A) = A \cap \text{cl}(\text{int}(A)))$ [1].

Definition 2.1. A subset $A$ of a space $(X, \tau)$ is called: (1) a $A^\L$-set [4] if $A = A \cup V$, where $U$ is open and $V$ is $\alpha$-closed, (2) a DS-set [8] if $A = A \cup V$, where $U$ is open and $V$ is $\delta$-semi-closed, (3) a B-set [21] if $A = A \cup V$, where $U$ is open and $V$ is semi-closed, (4) a $\eta$-set [17] if $A = A \cup V$, where $U$ is open and
V is $\alpha$-closed,
(5) a $\delta$-set [7] if $\delta$-int(A) is $\delta$-closed.

III. $Z_\alpha$-OPEN SETS

Definition 3.1. A subset A of a topological space $(X, \tau)$ is called

(1) $Z_\alpha$-open if $A \subseteq \text{int}(\text{cl}(\text{int}(A))) \cup \text{cl}(\delta\text{-int}(A))$,
(2) $Z_\alpha$-closed if $\text{cl}(\text{int}(A)) \cap \text{int}(\delta\text{-int}(A)) \subseteq A$.

The family of all $Z_\alpha$-open (resp. $Z_\alpha$-closed) subsets of a space $(X, \tau)$ will be always denoted by $Z_\alpha O(X)$ (resp. $Z_\alpha C(X)$).

Remark 2.1. The following diagram holds for a subset of a space $X$:

$$
\text{semiopen} \rightarrow \gamma\text{-open} \uparrow \uparrow \uparrow \uparrow
\text{open} \rightarrow \alpha\text{-open} \rightarrow Z_\alpha\text{-open} \rightarrow Z\text{-open} \uparrow \uparrow \uparrow \uparrow
\delta\text{-open} \rightarrow \beta\text{-open} \rightarrow \delta\text{-semiopen} \rightarrow \varepsilon\text{-open}
$$

The converse of the above implications need not necessary be true as shown by [1, 3, 4, 5, 10, 11, 16, 18] and the following examples.

Example 3.1. Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}, \{a, b, c, d, e\}\}$. Then:
(1) A subset $\{a, b, e\}$ of $X$ is $Z_\alpha$-open but it is not $\delta$-semiopen and it is not $\alpha$-open,
(2) A subset $\{b, e\}$ of $X$ is semiopen but it is not $Z_\alpha$-open.

Example 3.2. Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c, d\}, X\}$. Then the subset $\{a, b, c\}$ is a $Z_\alpha$-open set but it is not $Z_\alpha$-open.

Theorem 3.1. Let $(X, \tau)$ be a topological space. Then a $Z_\alpha$-open set $A$ of $X$ is $\alpha$-open if one of the following conditions are hold:

(1) $(X, \tau)$ is E.D.,
(2) $A$ is $\delta$-set of $X$,
(3) $X \setminus A$ is $\delta$-dense of $X$.

Proof. (1) Since, $A \in Z_\alpha O(X)$ and $X$ is E.D., then $A \subseteq \text{int}(\text{cl}(\text{int}(A))) \cup \text{cl}(\delta\text{-int}(A)) \subseteq \text{int}(\text{cl}(\text{int}(A))) \cup \text{cl}(\delta\text{-int}(A))$ and therefore $A \in \alpha O(X, \tau)$.

(2) Let $A$ be a $\delta$-set and $Z_\alpha$-open. Then $A \subseteq \text{int}(\text{cl}(\text{int}(A))) \cup \text{cl}(\delta\text{-int}(A)) = \text{int}(\text{cl}(\text{int}(A)))$ and therefore $A \in \alpha O(X, \tau)$. Then $\delta\text{-int}(A) = \emptyset$ and hence $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$. Therefore $A$ is $\alpha$-open.

(3) Let $A \in Z_\alpha O(X)$ and $X \setminus A$ be a $\delta$-dense set of $X$. Then $\delta\text{-int}(A) = \emptyset$ and hence $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$. Therefore $A$ is $\alpha$-open.

Lemma 3.1. Let $(X, \tau)$ be a topological space. Then the following statements are hold.

(1) The union of arbitrary $Z_\alpha$-open sets is $Z_\alpha$-open,
(2) The intersection of arbitrary $Z_\alpha$-closed sets is $Z_\alpha$-closed.

Remark 3.2. By the following example we show that the intersection of any two $Z_\alpha$-open sets is not $Z_\alpha$-open.

Example 3.3. Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}, X\}$. Then $A = \{a, c\}$ and $B = \{b, c\}$ are $Z_\alpha$-open sets. But $A \cap B = \{c\}$ is not $Z_\alpha$-open.

Definition 3.2. Let $(X, \tau)$ be a topological space. Then:

(1) The union of all $Z_\alpha$-open sets of $X$ contained in $A$ is called the $Z_\alpha$-interior of $A$ and is denoted by $Z_\alpha \text{-int}(A)$,
(2) The intersection of all $Z_\alpha$-closed sets of $X$ containing $A$ is called the $Z_\alpha$-closure of $A$ and is denoted by $Z_\alpha \text{-cl}(A)$.

Theorem 3.2. Let $A$ be a subset of a topological space $(X, \tau)$. Then the following are statements are equivalent:

(1) $A$ is $Z_\alpha$-open set,
(2) $A = Z_\alpha \text{-int}(A)$,
(3) $A = \alpha\text{-int}(A) \cup \delta\text{-int}(A)$.

Proof. (1) $\iff$ (2). Let $A$ be a $Z_\alpha$-open set. Then $A \subseteq \text{int}(\text{cl}(\text{int}(A))) \cup \text{cl}(\delta\text{-int}(A))$. By Lemma 2.1, $\alpha\text{-int}(A) \cup \delta\text{-int}(A) = (A \cap \text{int}(\text{cl}(\text{int}(A)))) \cup (A \cap \text{cl}(\delta\text{-int}(A))) = A \cap (\text{int}(\text{cl}(\text{int}(A))) \cup \text{cl}(\delta\text{-int}(A))) = A$.

(2) $\iff$ (1). Let $A = \alpha\text{-int}(A) \cup \delta\text{-int}(A)$. Then by Lemma 2.1, we have $A = \text{int}(\text{cl}(\text{int}(A))) \cup (A \cap \text{cl}(\delta\text{-int}(A))) \subseteq \text{int}(\text{cl}(\text{int}(A))) \cup \text{cl}(\delta\text{-int}(A))$. Therefore $A$ is $Z_\alpha$-open set.

Theorem 3.2. Let $A$ be a subset of a topological space $(X, \tau)$. Then the following are statements are equivalent:

(1) $A$ is a $Z_\alpha$-closed,
(2) $A = Z_\alpha \text{-cl}(A)$,
(3) $A = \alpha\text{-cl}(A) \cap \delta\text{-cl}(A)$.

Proof. It is clear.

IV. $A^\ast L$-SETS

Definition 4.1. A subset $A$ of a space $(X, \tau)$ is said to be an $A^\ast L$-set if there exist an open set $U$ and an $Z_{\alpha}$-closed set $V$ such that $A = U \cap V$.

The family of $A^\ast L$-sets of $X$ is denoted by $A^\ast L(X)$.

Remark 4.1. (1) The following diagram holds for a subset $A$ of a space $X$:

$$
\eta\text{-set} \rightarrow A^\ast L\text{-set} \rightarrow B\text{-set} \uparrow \uparrow \uparrow \uparrow
A^\ast\text{-set} \rightarrow DS\text{-set}
$$

(2) Every open set and every $Z_{\alpha}$-closed set is $A^\ast L$-set,
(3) None of the above implications is reversible as shown by [4, 7, 16] and the following examples.

Example 4.1. Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$. Then the set $\{b, c, e\}$ is a $B$-set but it is not an $A^\ast L$-set. Also, the set $\{b, e\}$ it is an $A^\ast L$-set but it is not $DS$-set and it is not open. Further, the set $\{a\}$ is an $A^\ast L$-set but not $Z_{\alpha}$-closed.

Example 4.2. Let $X = \{a, b, c, d\}$ with topology $\tau =...
\{\varphi,\{a\},\{b\},\{a,b\}, X\}. Then the set \{b, c\} is an \(A\ L\)-set but not an \(\eta\)-set.

**Theorem 4.1.** Let \(A\) be a subset of a space \((X, \tau)\). Then \(A \in A' L(X)\) if and only if \(A = U \cap Z_c(\text{cl}(A)),\) for some open set \(U\).

**Proof.** Let \(A \in A' L(X)\). Then \(A = U \cap V\), where \(U\) is open and \(V\) is \(Z_c\)-closed. Since \(A \subseteq V\), then \(Z_c(\text{cl}(A)) \subseteq Z_c(\text{cl}(V)) = V\). Thus \(U \cap Z_c(\text{cl}(A)) \subseteq U \cap V = A \subseteq U \cap Z_c(\text{cl}(A))\). Therefore, \(A = U \cap Z_c(\text{cl}(A))\).

Conversely, since \(A = U \cap Z_c(\text{cl}(A)),\) for some open set \(U\) and \(Z_c(\text{cl}(A))\) is \(Z_c\)-closed, then by Definition 4.1, \(A\) is \(A' L\)-set.

**Lemma 4.1** [12]. Let \(A\) be a subset of a space \((X, \tau)\). Then, \(A\) is semi-closed if and only if \(\text{int}(\text{cl}(A)) = \text{int}(A)\).

**Theorem 4.2.** Let \(X\) be a topological space and \(A \subseteq X\). If \(A \in A' L(X)\), then \(\text{pint}(A) = \text{int}(A)\).

**Proof.** Let \(A \in A' L(X)\). Then, \(A = U \cap V\), where \(U\) is open and \(V\) is \(Z_c\)-closed. Since \(V\) is \(Z_c\)-closed, then \(V\) is semi-closed. Hence by Lemmas 2.1, 4.1, we have \(\text{pint}(A) = A \cap \text{int}(\text{cl}(A)) \subseteq U \cap \text{int}(\text{cl}(V)) = U \cap \text{int}(V) = \text{int}(A)\). Thus, \(\text{pint}(A) = \text{int}(A)\).

**Theorem 4.3.** Let \(A\) be a subset of a space \((X, \tau)\). Then the following are equivalent:

1. \(A\) is open,
2. \(A\) is an \(\alpha\)-open and \(A'\ L\)-set,
3. \(A\) is a preopen and \(A'\ L\)-set.

**Proof.** (1) \(\rightarrow\) (2) and (2) \(\rightarrow\) (3) Obvious. (3) \(\rightarrow\) (1). Let \(A\) be a preopen set and \(A'\ L\)-set. Then \(A\) is preopen and \(A'\ L\)-set.

**Theorem 4.4.** For an extremally disconnected space \(X\), the following are equivalent:

1. \(A\) is open,
2. \(A\) is \(Z_c\)-open and \(A'\ L\)-set,
3. \(A\) is preopen and \(A'\ L\)-set.

**Proof.** It follows directly from Theorems 3.1, 4.3.

**Theorem 4.5.** Let \((X, \tau)\) be a topological space. Then the following are equivalent:

1. \(X\) is submaximal,
2. Every dense subset of \(X\) is an \(A'\ L\)-set.

**Proof.** (1) \(\rightarrow\) (2). Let \(X\) be a submaximal space. Then every dense subset of \(X\) is an open set, so is an \(A'\ L\)-set.

(2) \(\rightarrow\) (1). It is known that every dense set is preopen. Also, by hypothesis, every dense is an \(A'\ L\)-set. So, by Theorem 4.3, it is open. Therefore, \(X\) is submaximal.

**Theorem 4.6.** Let \((X, \tau)\) be a topological space. Then the following are equivalent:

1. \(X\) is indiscrte,
2. Every \(A'\ L\)-set of \(X\) is only trivial ones.

**Proof.** (1) \(\rightarrow\) (2). Let \(A\) be an \(A'\ L\)-set of \(X\). Then there exists an open set \(U\) and an \(Z_c\)-closed set \(V\) such that \(A = U \cap V\). If \(A \neq \emptyset\), then \(U \neq \emptyset\). We obtain \(U = X\) and \(A = V\). Hence \(X = Z_c(\text{cl}(A)) \subseteq A\) and \(A = X\).

(2) \(\rightarrow\) (1). Every open set is an \(A'\ L\)-set. So, open sets in \(X\) are only the trivial ones. Hence, \(X\) is indiscrte.

**V. DECOMPOSITIONS OF CONTINUOUS FUNCTIONS**

**Definition 5.1.** A function \(f:(X, \tau) \rightarrow (Y, \sigma)\) is said to be \(Z_c\)-continuous if \(\overline{f^{-1}(V)} \subseteq \overline{g^{-1}(V)}\) is \(Z_c\)-open in \(X\), for every \(V \in \sigma\).

**Definition 4.2.** A function \(f:(X, \tau) \rightarrow (Y, \sigma)\) is called super-continuous [15] (resp. \(a\)-continuous [4], \(\alpha\)-continuous [14], pre-continuous [13], \(\delta\)-semi-continuous [9], semi-continuous [12], \(\gamma\)-continuous [10], \(\epsilon\)-continuous [5], \(Z\)-continuous [11]) if \(\overline{f^{-1}(V)}\) is \(\delta\)-open (resp. \(a\)-open, \(\alpha\)-open, \(\delta\)-open, \(\epsilon\)-open, \(Z\)-open) of \(X\), for each \(V \in \sigma\).

**Remark 5.1.** Let \(f:(X, \tau) \rightarrow (Y, \sigma)\) be a function. Then the following diagram is hold:

The implications of the above diagram are not reversible as shown by [4, 9, 10, 11, 15] and the following examples.

**Example 5.1.** Let \(X = \{a, b, c, d\}\) with topology \(\tau = \{\varphi, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, X\}\). Then:

1. The function \(f:(X, \tau) \rightarrow (X, \tau)\) which defined by \(f(a) = a, f(b) = d\) and \(f(c) = c, f(d) = b\) is semi-continuous but it is not \(Z_c\)-continuous,
2. The function \(f:(X, \tau) \rightarrow (X, \tau)\) which defined by, \(f(a) = a, f(b) = b, f(c) = c, f(d) = d\) is \(Z_c\)-continuous but it is not \(\alpha\)-continuous.

**Example 5.2.** In Example 3.2, the function \(f:(X, \tau) \rightarrow (X, \tau)\) which defined by, \(f(a) = c, f(b) = f(c) = d\) and \(f(d) = f(e) = e\) is \(Z\)-continuous but it is not \(Z_c\)-continuous. Also, the function \(f:(X, \tau) \rightarrow (X, \tau)\) which defined by, \(f(a) = a, f(b) = b, f(c) = c, f(d) = f(e) = d\) is \(Z_c\)-continuous but it is not \(a\)-continuous.

**Definition 5.3.** A function \(f:(X, \tau) \rightarrow (Y, \sigma)\) is said to be \(A'\ L\)-continuous if \(\overline{f^{-1}(V)}\) is an \(A'\ L\)-set of \(X\), for every \(V \in \sigma\).

**Definition 5.4.** A function \(f:(X, \tau) \rightarrow (Y, \sigma)\) is called \(B\)-continuous [21] (resp. \(\eta\)-continuous [17], \(DS\)-continuous [8]) if \(\overline{f^{-1}(V)}\) is a \(B\)-set (resp. \(\eta\)-set, \(DS\)-set) in \(X\), for each \(V \in \sigma\).

**Remark 5.2.** (1) Let \(f:X \rightarrow Y\) be a function. Then the following implications are hold:

\(\eta\)-continuous \(\rightarrow A'\ L\)-continuous \(\rightarrow B\)-continuous

(2) Every continuous is \(A'\ L\)-continuous.

(3) These implications are not reversible as shown by [4, 8] and the following examples.

**Example 5.3.** Let \(X = \{a, b, c, d, e\}\) with topology \(\tau = \{\varphi, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{a, b, c\}, X\}\).
and $\sigma = \{a, c\}, \{d, e\}, \{c, d, e\}, Y\) Then:

1. The function $f : (X, \tau) \rightarrow (Y, \sigma)$ which defined by,
   $f(a) = a, f(b) = d, f(c) = c, f(d) = b$ and $f(e) = c$ is
   continuous and it is not $A$-$L$-continuous but it is not $\eta$-continuous.

2. The function $f : (X, \tau) \rightarrow (Y, \sigma)$ which defined by,
   $f(a) = f(c) = a, f(b) = d, f(d) = b$ and $f(e) = e$ is
   $A$-$L$-continuous but it is not $D$-continuous and it is not continuous.

Example 5.4. In Example 4.2, the function $f : (X, \tau) \rightarrow (X, \tau)$ which defined by $f(a) = f(d) = d$ and $f(b) = f(c) = b$ is $A$-$L$-continuous but it is not $\eta$-continuous.

Theorem 5.1. The following are equivalent for a function $f : X \rightarrow Y$:

1. $f$ is continuous,
2. $f$ is $\alpha$-continuous and $A$-$L$-continuous,
3. $f$ is precontinuous and $A$-$L$-continuous.

Proof. It is an immediate consequence of Theorem 4.3.

Theorem 5.2. Let $X$ be an extremely disconnected space and $f : X \rightarrow Y$ be a function. Then following are equivalent:

1. $f$ is continuous,
2. $f$ is $Z_\sigma$-continuous and $A$-$L$-continuous,
3. $f$ is $\alpha$-continuous and $A$-$L$-continuous,
4. $f$ is precontinuous and $A$-$L$-continuous.

Proof. It is an immediate consequence of Theorem 4.4.

REFERENCES


