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On Z_{α}-Open Sets and Decompositions of Continuity

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ABSTRACT
In this paper, we introduce and study the notion of Z_{α}-open sets and some properties of this class of sets are investigated. Also, we introduce the class of A^*L-sets via Z_{α}-open sets. Further, by using these sets, a new decompositions of continuous functions are presented.
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I. INTRODUCTION

J. Tong [21] introduced the notion of B-set and B-continuity in topological spaces. The concept of A^*-sets, DS -set, A^*-continuity, DS-continuity introduced by E. Ekici [4, 8] and used them to obtain a new decomposition of continuity. Noiri et.al [17] introduced the notion of η-set and η-continuity in topological spaces. The main purpose of this paper is to obtain a new decompositions of continuous functions. We introduce and study the notion of Z_{α}-open sets and A^*L-sets. The relationships among of Z_{α}-open sets, A^*L-sets and the related sets are investigated. By using these notions, we obtain a new decompositions of continuous functions. Also, some characterizations of these notions are presented.

II.PRELIMINARIES

A subset A of a topological space (X, τ) is called regular open (resp. regular closed) [20] if A = int(cl(A)) (resp. A= cl(int(A))). The δ-interior [22] of a subset A of X is the union of all regular open sets of X contained in A and is denoted by δ-int(A). A subset A of a space X is called δ-open [22] if it is the union of regular open sets. The complement of a δ-open set is called δ-closed. Alternatively, a set A of (X, τ) is called δ-closed [22] if A = δ-cl(A), where δ-cl(A) = $\{x\in X: A \cap int(cl(U)) \neq \varphi, U\in\tau \text{ and } x\in U\}$. Throughout this paper (X, τ) and (Y, σ) (simply, X and Y) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned . For a subset A of a space (X, τ), cl(A), int(A) and X \ A denote the closure of A, the interior of A and the complement of A respectively. A space X is called submaximal [3] if every dense subset of X is open. A space (X, τ) is called extremally disconnected (briefly. E. D.) [19] if the closure of every open set of X is open. A subset A of a space X is called δ-dense [6] if and only if δ-cl(A) = X. A subset A of a space X is called a-open [4] (resp. α-open [16],preopen [13],δ-semiopen [18], semiopen [12], Z-open [11], b-open [1] or γ-open [10] or sp-open [3], e-open [5]) if A ⊆ int(cl(δ-int(A))) (resp. A⊆ int(cl(int(A))), A⊆int(cl(A)), A⊆cl(δ-int(A)), A⊆ cl(int(A)), A ⊆ cl(δ-int(A)) ∪ int(cl(A))), A ⊆ int(cl(A)) ∪ cl(int(A)), A ⊆ cl(δ-int(A)) ∪ int(δ-cl(A))). The complement of a-open (resp. α-open, preopen, δ-semiopen, semiopen) sets is called a-closed [4] (resp. α-closed [16], pre-closed [13], δ-semi-closed [18], semi-closed [2]). The intersection of all a-closed (resp. α-closed, pre-closed, δ-semi-closed, semi-closed) sets containing A is called the a-closure (resp. α-closure, pre-closure, δ-semi-closure, semi-closure) of A and is denoted by a-cl(A) (resp. α-cl(A), pcl(A), δ-scl(A), scl(A)). The union of all a-open (resp. α-open, preopen, δ-semiopen, semiopen) sets contained in A is called the a-interior (resp. α-interior, pre-interior, δ-semi-interior, semi-interior) of A and is denoted by a-int(A) (resp. α-int(A), pint(A), δ-sint(A), sint(A)). The family of all δ-open (resp. a-open, α-open, preopen, δ-semiopen, semiopen) is denoted by δO(X) (resp. aO(X), αO(X), PO(X), δSO(X), SO(X)).

Lemma 2.1. Let A, B be two subset of (X, τ). Then the following are hold:
(1) α-cl(A) = A ∪ cl(int(cl(A))) and α-int (A) = A ∩ int(cl(int(A))) [1],
(2) δ-scl(A) = A ∪ int(δ-cl(A)) and δ-sint(A) = A ∩ cl(δ-int(A)) [17],
(3) pcl(A) = A ∪ cl(int(A)) and pint (A) = A ∩ int(cl(A)) [1].

Definition 2.1. A subset A of a space (X, τ) is called:
(1) a A^*-set [4] if A = U ∩ V, where U is open and V is a-closed,
(2) a DS-set [8] if A = U ∩ V, where U is open and V is δ-semi-closed,
(3) a B-set [21] if A = U ∩ V, where U is open and V is semi-closed,
(4) a η-set [17] if A = U ∩ V, where U is open and

V is α-closed,
(5) a δ^*-set [7] if δ-int(A) is δ-closed.

III. Z_A-OPEN SETS

Definition 3.1. A subset A of a topological space (X, τ) is called
(1) Z_α-open if $A \subseteq int(cl(int(A))) \cup cl(\delta\text{-}int(A))$,
(2) Z_α-closed if $cl(int(cl(A))) \cap int(\delta\text{-}cl(A)) \subseteq A$.
The family of all Z_α-open (resp. Z_α-closed) subsets of a space (X, τ) will be as always denoted by $Z_\alpha O(X)$ (resp. $Z_\alpha C(X)$).

Remark 2.1. The following diagram holds for a subset of a space X:

$$\begin{tikzcd}
 & & \textit{semiopen} \arrow[r] & \gamma\textit{-open} \\
\textit{open} \arrow[r] & \alpha\textit{-open} \arrow[r] & Z_\alpha\textit{-open} \arrow[u] \arrow[r] & Z\textit{-open} \arrow[u] \arrow[d] \\
\delta\textit{-open} \arrow[u] \arrow[r] & \text{a-}\textit{open} \arrow[u] \arrow[r] & \delta\textit{-semiopen} \arrow[u] \arrow[r] & e\textit{-open}
\end{tikzcd}$$

The converse of the above implications need not necessary be true as shown by [1, 3, 4, 5, 10, 11, 16, 18] and the following examples.

Example 3.1. Let X = {a, b, c, d, e} with topology τ = {φ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d}, {a, b, c, d}, X}. Then:
(1) A subset {a, b, e} of X is Z_α-open but it is not δ-semiopen and it is not α-open,
(2) A subset {b, e} of X is semiopen but it is not Z_α-open.
Example 3.2. Let X = {a, b, c, d, e} with topology τ = {φ,{a, b},{c, d}, {a, b, c, d}, X}. Then the subset {a, b, c} is a Z-open set but it is not Z_α-open.
Theorem 3.1. Let (X, τ) be a topological space. Then a Z_α-open set A of X is α-open if one of the following conditions are hold:
(1) (X, τ) is E.D.,
(2) A is δ^*-set of X,
(3) X \ A is δ-dense of X.
Proof. (1) Since, $A \in Z_\alpha O(X)$ and X is E.D., then
$A \subseteq int(cl(int(A))) \cup cl(\delta\text{-}int(A)) \subseteq int(cl(int(A))) \cup int(cl(\delta\text{-}int(A))) = int(cl(int(A)))$ and therefore $A \in \alpha O(X, \tau)$,
(2) Let A be a δ^*-set and Z_α-open. Then $A \subseteq int(cl(int(A))) \cup cl(\delta\text{-}int(A)) = int(cl(int(A))) \cup \delta\text{-}int(A) = int(cl(int(A)))$. Therefore A is α-open,
(3) Let $A \in Z_\alpha O(X)$ and X \ A be a δ-dense set of X. Then δ-int(A)= φ and hence $A \subseteq int(cl(int(A)))$. Therefore A is α-open.
Lemma 3.1. Let (X, τ) be a topological space. Then the following statements are hold.
(1) The union of arbitrary Z_α-open sets is Z_α-open,
(2) The intersection of arbitrary Z_α-closed sets is Z_α-closed.
Remark 3.2. By the following example we show that the intersection of any two Z_α-open sets is not Z_α-open.
Example 3.3. Let X = {a, b, c} with topology τ = {φ, {a}, {b}, {a, b}, X}. Then A = {a, c} and B ={b, c} are Z_α-open sets. But $A \cap B$ = {c} is not Z_α-open.
Definition 3.2. Let (X, τ) be a topological space. Then :
(1) The union of all Z_α-open sets of X contained in A is called the Z_α-interior of A and is denoted by Z_α-int(A),
(2) The intersection of all Z_α-closed sets of X containing A is called the Z_α-closure of A and is denoted by Z_α-cl(A).
Theorem 3.2. Let A be subset of a topological space (X, τ). Then the following are statements are equivalent:
(1) A is Z_α-open set,
(2) A = Z_α-int(A),
(3) $A = \alpha\text{-}int(A) \cup \delta\text{-}sint(A)$.
Proof. (1) $\leftrightarrow$(2). Obvious,
(1) $\rightarrow$(2). Let A be a Z_α-open set. Then $A \subseteq int(cl(int(A))) \cup cl(\delta\text{-}int(A))$. BY Lemma 2.1, $\alpha\text{-}int(A) \cup \delta\text{-}sint(A) = (A \cap int(cl(int(A)))) \cup (A \cap cl(\delta\text{-}int(A))) = A \cap (int(cl(int(A))) \cup cl(\delta\text{-}int(A))) = A$,
(2) $\rightarrow$(1). Let $A = \alpha\text{-}int(A) \cup \delta\text{-}sint(A)$. Then by Lemma 2.1, we have
$A = (A \cap int(cl(int(A)))) \cup (A \cap cl(\delta\text{-}int(A)))$
$\subseteq int(cl(int(A))) \cup cl(\delta\text{-}int(A))$. Therefore A is Z_α-open set.
Theorem 3.2. Let A be subset of a topological space (X, τ). Then the following are statements are equivalent:
(1) A is a Z_α-closed,
(2) A = Z_α-cl(A),
(3) $A = \alpha\text{-}cl(A) \cap \delta\text{-}scl(A)$.
Proof. It is clear.

IV. A^*L-SETS

Definition 4.1. A subset A of a space (X, τ) is said to be an A^*L-set if there exist an open set U and an Z_α-closed set V such that $A = U \cap V$.
The family of A^*L-sets of X is denoted by $A^*L(X)$.
Remark 4.1. (1) The following diagram holds for a subset A of a space X,

$$\begin{tikzcd}
\eta\text{-set} \arrow[r] & A^*L\text{-set} \arrow[r] & B\text{-set} \\
A^*\text{-set} \arrow[u] \arrow[r] & DS\text{-set} \arrow[u] &
\end{tikzcd}$$

(2) Every open set and every Z_α-closed set is A^*L-set,
(3) None of the above implications is reversible as shown by [4, 7, 16] and the following examples.
Example 4.1. Let X ={a, b, c, d, e} with topology τ = {φ, {a}, {c}, {a, b},{a, c}, {a, b ,c}, X}. Then the set {b, c, e} is an B-set but it is not an A^*L-set . Also, the set {b, e} it is an A^*L-set but it is not DS-set and it is not open. Further, the set {a} is A^*L-set but not Z_α-closed.
Example 4.2. Let X = {a, b, c, d} with topology τ =

$\{\varphi,\{a\},\{b\},\{a, b\}, X\}$. Then the set $\{b, c\}$ is an A^*L-set but not an η-set.

Theorem 4.1. Let A be a subset of a space (X, τ). Then $A \in A^*L(X)$ if and only if $A = U \cap Z_\alpha\text{-cl}(A)$, for some open set U.

Proof. Let $A \in A^*L(X)$. Then $A = U \cap V$, where U is open and V is Z_α-closed. Since $A \subseteq V$, then $Z_\alpha\text{-cl}(A) \subseteq Z_\alpha\text{-cl}(V) = V$. Thus $U \cap Z_\alpha\text{-cl}(A) \subseteq U \cap V = A \subseteq U \cap Z_\alpha\text{-cl}(A)$. Therefore, $A = U \cap Z_\alpha\text{-cl}(A)$.

Conversely. Since $A = U \cap Z_\alpha\text{-cl}(A)$, for some open set U and $Z_\alpha\text{-cl}(A)$ is Z_α-closed, then by Definition 4.1, A is A^*L-set.

Lemma 4.1 [12]**.** Let A be a subset of a space(X,τ). Then, A is semi-closed if and only if $\text{int}(A) = \text{int}(\text{cl}(A))$.

Theorem 4.2. Let X be a topological space and $A \subseteq X$. If $A \in A^*L(X)$, then $\text{pint}(A) = \text{int}(A)$.

Proof. Let $A \in A^*L(X)$. Then, $A = U \cap V$, where U is open and V is Z_α-closed. Since V is Z_α-closed, then V is semi-closed. Hence by Lemmas 2.1, 4.1, we have $\text{pint}(A) = A \cap \text{int}(\text{cl}(A)) \subseteq U \cap \text{int}(\text{cl}(V)) = U \cap \text{int}(V) = \text{int}(A)$. Thus, $\text{pint}(A) = \text{int}(A)$.

Theorem 4.3. Let A be a subset of a space(X, τ).Then the following are equivalent:
(1) A is open,
(2) A is α-open and A^*L-set,
(3) A is preopen and A^*L-set.

Proof. (1) $\to$(2) and (2) $\to$(3) Obvious,
(3) $\to$(1). Let A be a preopen set and A^*L-set. Then by Theorem 4.2, we have $\text{pint}(A) = \text{int}(A)$. But, A is preopen, then $A = \text{pint}(A) = \text{int}(A)$. Thus A is open.

Theorem 4.4. For an extremally disconnected space X. The following are equivalent:
(1) A is open,
(2) A is Z_α-open and A^*L-set,
(3) A is preopen and A^*L-set.

Proof. It follows directly from Theorems 3.1, 4.3.

Theorem 4.5. Let (X, τ) be a topological space. Then the following are equivalent:
(1) X is submaximal,
(2) Every dense subset of X is an A^*L-set.

Proof. (1) $\to$(2). Let X be a submaximal space. Then every dense subset of X is an open sets, so is an A^*L-set.
(2) $\to$(1). It is known that every dense set is preopen. Also, by hypothesis, every dense is A^*L-set. So, by Theorem 4. 3, it is open. Therefore, X is submaximal.

Theorem 4.6. Let X be a topological space. Then the following are equivalent:
(1) X is indiscrete,
(2) The A^*L-set of X are only trivial ones.

Proof. (1) $\to$(2). Let A be an A^*L-set of X. Then there exists an open set U and an Z_α-closed set V such that $A = U \cap V$. If $A \neq \varphi$, then $U \neq \varphi$. We obtain $U = X$ and $A = V$. Hence $X = Z_\alpha\text{-cl}(A) \subseteq A$ and $A = X$,
(2) $\to$(1). Every open set is an A^*L-set. So, open sets in X are only the trivial ones. Hence, X is indiscrete.

V. DECOMPOSITIONS OF CONTINUOUS FUNCTIONS

Definition 5.1. A function $f:(X, \tau)\to(Y, \sigma)$ is said to be Z_α-continuous if $f^{-1}(V)$ is Z_α-open in X, for every $V\in \sigma$.

Definition 4.2. A function $f:(X, \tau)\to(Y, \sigma)$ is called super-continuous [15] (resp. a-continuous [4], α-continuous [14], pre-continuous [13], δ-semi-continuous [9], semi-continuous [12],γ-continuous [10], e-continuous [5], Z-continuous [11]) if $f^{-1}(V)$ is δ-open (resp. a-open, α-open, preopen, δ-semiopen, semiopen, γ-open, e-open, Z-open) of X, for each $V\in \sigma$.

Remark 5.1. Let $f: (X, \tau) \to(Y, \sigma)$ be a function. Then The following diagram is hold:

$$\begin{tikzcd}
 & & \text{semi-continuous} \arrow[r] & \gamma\text{-continuous} \\
\text{continuous} \arrow[r] & \alpha\text{-continuous} \arrow[r] & Z_\alpha\text{-continuous} \arrow[r] \arrow[u] & Z\text{-continuous} \arrow[u] \arrow[d] \\
\text{super-continuous} \arrow[r] \arrow[u] & \text{a-continuous} \arrow[r] \arrow[u] & \delta\text{-semi-continuous} \arrow[r] \arrow[u] & \text{e-continuous}
\end{tikzcd}$$

The implications of the above diagram are not reversible as shown by [4, 9, 10, 11, 15] and the following examples.

Example 5.1. Let $X = \{a, b, c, d\}$ with topology $\tau = \{\varphi, \{a\},\{c\},\{a, c\}, \{a, b\}, \{a, b ,c\}, X\}$. Then:
(1) the function $f :(X, \tau) \to (X, \tau)$ which defined by $f(a) = a$, $f(b) = d$ and $f(c) = c$, $f(d) = b$ is semi-continuous but it is not Z_α-continuous,
(2) the function $f :(X, \tau) \to (X, \tau)$ which defined by, $f(a) = a$, $f(b) = b$, and $f(c) = f(d) = c$ is Z_α-continuous but it is not δ-semi-continuous.

Example 5.2. In Example 3.2, the function $f :(X, \tau) \to (X, \tau)$ which defined by, $f(a) = c$, $f(b) = f(c) = d$ and $f(d) = f(e) = e$ is Z-continuous but it is not Z_α-continuous. Also, the function $f :(X, \tau) \to (X, \tau)$ which defined by, $f(a) = a$, $f(b) = b$, $f(c) = c$ and $f(d) = f(e) = d$, is Z_α-continuous but it is not α-continuous.

Definition 5.3. A function $f:(X, \tau) \to (Y, \sigma)$ is said to be A^*L-continuous if $f^{-1}(V)$ is an A^*L-set of X, for every $V\in \sigma$.

Definition 5.4. A function $f:(X, \tau) \to (Y, \sigma)$ is called B-continuous [21] (resp. η-continuous [17], DS-continuous [8]) if $f^{-1}(V)$ is a B-set (resp. η-set, DS-set) in X, for each $V\in \sigma$.

Remark 5.2. (1) Let $f: X \to Y$ be a function. Then the following implications are hold:

η-continuous $\to$ A^*L-continuous $\to$ B-continuous
↑ ↑
A^*-continuous $\to$ DS-continuous

(2) Every continuous is A^*L-continuous.
(3) These implications are not reversible as shown by [4, 8]and the following examples.

Example 5.3. Let $X = \{a, b, c, d, e\} = Y$ with topology $\tau = \{\varphi,\{a\},\{c\}, \{a, c\},\{a, b\},\{a, b, c\}, X\}$

and σ = {φ,{c},{d, e}, {c, d, e}, Y}. Then:
(1) the function f :(X, τ)→(Y, σ) which defined by, f(a) = a, f(b) = d, f(c) = e, f(d)= b and f(e) = c is B-continuous but it is not A^*L-continuous,
(2) the function f :(X, τ)→(Y, σ) which defined by, f(a) = f(c) = a, f(b) = d, f(d) = b and f(e) = e is A^*L-continuous but it is not DS-continuous and it is not continuous.

Example 5.4. In Example 4.2, the function f :(X, τ)→(X, τ) which defined by f(a)= f(d) = d and f(b) = f(c) = b is A^*L-continuous but it is not η-continuous.

Theorem 5.1. The following are equivalent for a function f: X → Y:
(1) f is continuous,
(2) f is α-continuous and A^*L-continuous,
(3) f is pre-continuous and A^*L-continuous.

Proof. It is an immediate consequence of Theorem 4.3.

Theorem 5.2. Let X be an extremely disconnected space and f: X → Y be a function. Then following are equivalent:
(1) f is continuous,
(2) f is Z_α-continuous and A^*L-continuous,
(3) f is α-continuous and A^*L-continuous,
(4) f is pre-continuous and A^*L-continuous.

Proof. It is an immediate consequence of Theorem 4.4.

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