RESEARCH ARTICLE OPEN ACCESS

Applications and Properties of Unique Coloring of Graphs

A. Sri Krishna Chaitanya[1] Dr. R. Sivaprasad[2]
[1]A. Sri Krishna Chaitanya, Research Scholar, Department of Mathematics, Sri Krishnadevaraya University, Anantapur Dist., A.P.
[2]Dr. R. Sivaprasad, Professor, Department of Mathematics, Sri Krishnadevaraya University, Anantapur Dist., A.P.

ABSTRACT
This paper studies the concepts of origin of uniquely colorable graphs, general results about unique vertex colorings, assorted results about uniquely colorable graphs, complexity results for unique coloring
Mathematics Subject Classification 2000: 05CXX, 05C15, 05C20, 37E25.
Keywords: complete graph. bipartite graph, k-chromatic graph, adjacent vertices, maximal planar graphs.

I. ORIGIN OF UNIQUE COLORING

The origin of unique coloring appears to have been, perhaps surprisingly, in the field of psychology. There the problem of a signed graph was introduced, together with a coloring of signed graphs, to model a problem in that field [4]. A signed graph*S* is a ordered pair (*G;*), where *G* is an undirected graph and is a function: *E(G)*→**{-1,1}**. These signed graphs are used in psychology to model the idea of clusterings. From there the idea of colorings and unique colorings a signed graph, closely related to the normal notion of coloring a graph arose in a 1968 paper of Cartwright and Harary [31]A coloring cof a signed graph is a function from the vertex set of *G* to{1; 2,….*k*} having the property that if *x* and *y* are two adjacentvertices in *G*, then

1) *If ({x;y}) = 1 then c(x) = c(y).*
2) *If ({x;y}) = 1 then c(x) ≠ c(y).*

As usual the set $\{c^{-1}(\{i\}) : i\ \{1; 2,\ldots.k\}\}$ defines a partition of the vertices of*S* into color classes. This paper of Cartwright and Harary, as well as a 1967 paper of Gleason and Cartwright [31], established conditions for a signed graph to have a coloring, and introduced the notion of a unique coloring of a signed graph. To wit, a signed graph *S* is uniquely colorable if there is exactly one partition of *S* into color classes. Both papers gave fairly simple criterion for a signed graph to be uniquely colorable. In addition introduced the notion of unique coloring of a “normal" (unsigned) graph *G,* which is the topic of interest in this thesis.

Under the usual notion of a coloring *c* of a graph *G* being a function from the set of vertices to a set of integers (colors) having the property that adjacent vertices receive a different assignment under *c*, Cartwright and Harary defined a graph *G* to be uniquely colorable if either *G* is complete or *G* has a unique partition of the vertices of *G* into *t* <|*V (G)*| color classes. In this same paper, they showed that if *G* has a unique coloring with say *t* colors, then, in fact *t = (G),* where *(G)* is the chromatic number of *G*, that is, the smallest positive integer *s* for which there is a coloring of *G* using exactly *s*colors.

1.2 General Results about Unique Vertex Colorings

Necessary Conditions for a Graph to be uniquely Colorable

To warm up our understandings of unique coloring we mention some easy necessary consequences of a graph being uniquely vertex-colorable. The first is that the number of colors used in a unique coloring is unique and equals the chromatic number of *G*.

Proposition1.2.1. *(Cartwright, Harary) IfGhas a unique coloring with t colors then t = (G).*

Proof: We may assume that *G*is not the complete graph on *(G)* vertices. Clearly$(G) \leq t \leq |V(G)|$, since a unique coloring is also a proper coloring. If $t > (G)$ then $|V(G)| > (G)$ and for any *(G)*-coloring *c* of *G*, pick a set of vertices $\{x_1;x_2,\ldots.,\}$ having the property that $c(x_i) = i$. There are at least $t - x\ (G)$ vertices in *G* other than $\{x_1,\ldots,\}$ and these can be assigned colors from $\{(G) + 1; (G) + 2,\ldots,t\}$, to get two distinct *t*-colorings of *G*.

By this proposition, we may say unambiguously that *G* is uniquely vertex colorable, and mean that *G* is uniquely vertex- *(G)*-colorable.

Let *G* be a graph and let $c : V(G) \rightarrow \{1, 2, \ldots, k\}$ be a unique vertex-*k*-coloring of *G*. For *i*, *j* $\{1, 2, \ldots, k\}$, define $G_{i;j}$ to be the subgraph of *G* induced by the vertices which *c* assigns the colors *i* or *j*. A very useful necessary condition for *G* tobe uniquely vertex-*k*-colorable was noticed by Harary et. al. in the following theorem which appears in [30].

Theorem 1.2.2. *(Harary, Hedetniemi, Robinson, 1969) Ifc:V(G) →{1,2,…,k} is a unique vertex-k-coloring of G, then for all i ≠ j, i, j {1, 2, …, k}, the graph $G_{i;j}$ is connected.*

Proof: If some$G_{i;j}$had two or more components, then by interchanging the colors*i*and *j* in exactly one of these components, we would arrive at a valid coloring different than *c*.

Corollary*1.2.3.* *Letcbe a unique vertex-k-coloring ofG, letxbe a vertex inV(G)and let i {1,….,k}. If i ≠ c(x) then there is a vertex yV (G) such that x is adjacent to y and c(y) = i. In particular, every vertex of G has degree at least k1.*

Proof: Let*uV(G),* let*c*be a unique vertex-*k*-coloring of*G*and let*i*be a colordifferent from *c(x).* By Theorem 1.2.2., $G_{i;c(x)}$ is a connected graph, and in particular, *x* is not an isolated vertex in $G_{i;c(x)}$ because Proposition 1.2.1 insures that some vertex receives the color *i*. Since there are *k1* other colors besides *c(x)*, the minimum degree of *G* must be at least *k1*. This completes the proof of Corollary 1.2.3.

Corollary 1.2.4.*(Harary et al.) IfGis a uniquely vertex-k-colorable, thenGhas at least (kedges.*

Proof: Let$V_i$be the set of vertices colored*i*. Theorem 1.2.2 insures that for *1<u>≤</u>i < j<u>≤</u>k*, the graph $G_{i;j}$ with vertex set V_iV_j is connected. Thus |$E(G_{i;j})$| <u>≥</u> |V_i|+|V_j|-*1*. Summing this inequality over all pairs*i≠j*, we have that
|*E(G)*|)which isthe desired result.

Corollary1.2.5.*(Geller, Chartrand) IfGis a uniquely vertex-4-colorable simpleplanar graph, then any drawing of the graph G is a triangulation. Moreover, for i ≠ j and i, j{1; 2; 3; 4}, each subgraph $G_{i;j}$ is a tree.*

Proof: By Euler's formula|*E(G)*| <u>≤</u>*3*|*V(G)*|*6*, and from Corollary 1.2.4,|*E(G)*|<u>≥</u>*3*|*V (G)*|*6*, so |*E(G)*| = *3*|*V (G)*|*6*. This implies that any drawing of *G* must be a triangulation. It also implies that equality holds throughout in the proof of the Corollary 1.2.4, so . Since $G_{i,j}$ is connected, it follows that $G_{i,j}$ is a tree. This completes the proof of the corollary.

II. ASSORTED RESULTS ABOUT UNIQUELY COLORABLE GRAPHS

A function : $V(G) \to V(G')$ is said to be a homomorphism of the graph *G* into the graph G' if it preserves adjacency of vertices, that is, if *{x,y} E(G)* implies *{(x);(y)}E(*G'*)*. If it is true that for every pair of vertices x', y', x' isadjacent to y' in *G'* if and only if there is a pair *x, y* of adjacent vertices in *G* such that *(x) =* x' *and (y) =* y', then is said to be a homomorphism of *G* onto G', and G' is said to be a *homomorphism image* of *G*. The following propositions appear in [32].

Proposition 2.1.1. *IfGis uniquely vertex-k-colorable andHis a homomorphism image of G such that (H) = k, then H is uniquely vertex-k-colorable .*

Proposition 2.1.2. *IfGis uniquely vertex-k-colorable thenGis(k-1)-connected.*

Proof: Let*A*be a set with|*A*| <u>≤</u>*k-2*, let*c*be a unique vertex-*k*-coloring of*G*,and let *x, y V(G) A*. There are two distinct colors *i, j {1,….,k}* such that novertex of *A* has a vertex colored *i* or *j* by *c*. Therefore, . By Corollary 1.2.3, there are vertices u_x and u_y such that *x* is adjacent to u_x, *y* is adjacent to u_y and $c(u_x) = c(u_y) = i$. Since $G_{i;j}$ is connected, there is a path *P* in $G_{i,j}$ joining $u_x$to u_y and thus there is a path in *GA* joining *x* and *y*. Thus, *G A* is connected. This completes the proof of the proposition.

III. COMPLEXITY RESULTS FOR UNIQUE COLORING

The following proposition is obvious.

Proposition 3.1.1. *A graph is uniquely vertex-1-colorable if and only if it consists ofisolated vertices. A graph is uniquely vertex-2-colorable if and only if it is a connected bipartite graph.*
Beyond this there is not much hope of finding a “good" characterization of arbitrary uniquely vertex-*k*-colorable graphs when *k<u>≥</u>3* because of the following complexity results contained in or implied by the work of Dailey in 1981 [33]

Theorem 3.1.2. *The following decision problems are NP-Complete:*

1. *Given a graph G and a vertex-k-coloring c of G, is there a vertex-k-coloring* c' *of G that is not equivalent to c?*
2. *Given an integer k and a graph G, does G have either 0 or at least 2 vertex-k-colorings?*

The result of Dailey probably dooms any possibility of a polynomial time algorithm for problems 1) or 2) above. In [31], the authors pose the question of whether there is a polynomial time algorithm for deciding whether a given planar graph is uniquely vertex-3-colorable. This problem is still open as far as this author knows.

IV. A SUFFICIENT CONDITION FOR DETERMINING UNIQUE VERTEX-K-COLORABILITY

The following sufficient condition for a graph to be uniquely vertex-k-colorable was given by Bollobas in [14].

Theorem 4.1.1. *Letkbe an integer greater than one, letGbe a vertex-k-colorablegraph on n vertices, and let (G) denote the minimum degree of G. If then G is uniquely vertex-k-colorable . Moreover, if G has a vertex-k-coloring in which* $G_{i;j}$ *is connected for every* $1 \leq i < j \leq k$*, and, then G is uniquely vertex-k-colorable. These results are best possible.*

This was generalized by Dmitriev according to a review of [15]. As we can see, this condition will apply only to very dense graphs.

REFERENCES

[1]. Appel, K. and W. Haken(1977). Every planar map is four colorable. Part I. Discharging Illinois Journal of Mathematics21, 429-490.
[2]. Bodlaender, H.L. 1991. “On the complexity of some coloring games”, Lecture Notes in Computer Science. No. 484, pp. 30-40.
[3]. Bondy, J.A. and Murty, U.S.R., 1985. Graph Theory with Applications, (Elsevier Science Publishing Co., Inc.).
[4]. F. Harary. Conditional colorability in graphs. Appears in Graphs and Applications: Proceedings of the First Colorado Symposium on Graph Theory. John Wiley and Sons Inc., New York, 1985.
[5]. F. Harary. Graph Theory. Addison-Wesley, Massachusetts, 1969.
[6]. F. Harary and K. Fraughnaugh (Jones). Conditional colorability ii:bipartite variations. Congr. Numer.,50:205-218, 1985.
[7]. F. Harary and K. Fraughnaugh (Jones). Degree conditional bipartition numbers in graphs. Congr. Numer., 55:39-50, 1986.