Applications and Properties of Unique Coloring of Graphs

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ABSTRACT
This paper studies the concepts of origin of uniquely colorable graphs, general results about unique vertex colorings, assorted results about uniquely colorable graphs, complexity results for unique coloring. Mathematics Subject Classification 2000: 05CXX, 05C15, 05C20, 37E25.

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I. ORIGIN OF UNIQUE COLORING
The origin of unique coloring appears to have been, perhaps surprisingly, in the field of psychology. There the problem of a signed graph was introduced, together with a coloring of signed graphs, to model a problem in that field [4]. A signed graph is a signed pair (G, t), where G is an undirected graph and t is a function: E(G) → {−1, 0, 1}. These signed graphs are used in psychology to model the idea of clusterings. From there the idea of colorings and unique colorings a signed graph, as well as a 1967 paper of Gleason and Cartwright [31], established conditions for a signed graph to have a unique coloring. In addition, a 1968 paper of Cartwright and Harary [32] defined a partition of the vertices of a graph into color classes. This paper of Cartwright and Harary, as well as a 1967 paper of Gleason and Cartwright [31], established conditions for a signed graph to have a unique coloring, and introduced the notion of a unique coloring of a signed graph. To wit, a signed graph S is uniquely colorable if there is exactly one partition of S into color classes. Both papers gave fairly simple criteria for a signed graph to be uniquely colorable. In addition, introduced the notion of unique coloring of a “normal” (unsigned) graph G, which is the topic of interest in this thesis.

As usual the set \( V(G) \) defines a partition of the vertices of \( G \) into color classes. This paper of Cartwright and Harary, as well as a 1967 paper of Gleason and Cartwright [31], established conditions for a signed graph to have a unique coloring, and introduced the notion of a unique coloring of a signed graph. To wit, a signed graph S is uniquely colorable if there is exactly one partition of S into color classes. Both papers gave fairly simple criteria for a signed graph to be uniquely colorable. In addition, introduced the notion of unique coloring of a “normal” (unsigned) graph G, which is the topic of interest in this thesis.

Under the usual notion of a coloring \( c \) of a graph \( G \) being a function from the set of vertices to a set of integers (colors) having the property that adjacent vertices receive a different assignment under \( c \), Cartwright and Harary defined a graph \( G \) to be uniquely colorable if either \( G \) is complete or \( G \) has a unique partition of the vertices of \( G \) into \( t \) color classes. In this same paper, they showed that if \( G \) has a unique coloring with say \( t \) colors, then, in fact \( t = \chi(G) \), where \( \chi(G) \) is the chromatic number of \( G \), that is, the smallest positive integer \( s \) for which there is a coloring of \( G \) using exactly \( s \) colors.

1.2 General Results about Unique Vertex Colorings
Necessary Conditions for a Graph to be uniquely Colorable
To warm up our understandings of unique coloring we mention some easy necessary consequences of a graph being uniquely vertex-colorable. The first is that the number of colors used in a unique coloring is unique and equals the chromatic number of \( G \).

Proposition 1.2.1. (Cartwright, Harary) If \( G \) has a unique coloring with \( t \) colors then \( t = \chi(G) \).

Proof: We may assume that \( G \) is not the complete graph on \( (G) \) vertices. Clearly \( \chi(G) \leq t \leq (V(G)) \), since a unique coloring is also a proper coloring. If \( t > \chi(G) \) then \( (V(G)) > (G) \) and for any \( \chi(G) \)-coloring \( c \) of \( G \), pick a set of vertices \( \{x_1, x_2, \ldots, x_t\} \) having the property that \( c(x_i) = i \). There are at least \( t \) vertices in \( G \) other than \( \{x_1, x_2, \ldots, x_t\} \) and these can be assigned colors from \( (v(G)) - 1 \). (For \( (G) + 2 \) vertices in \( G \) other than \( \{x_1, x_2, \ldots, x_t\} \) to get two distinct \( t \)-colorings of \( G \).

By this proposition, we may say unambiguously that \( G \) is uniquely vertex-colorable, and mean that \( G \) is uniquely vertex-(\( \chi(G) \))-colorable.

Let \( G \) be a graph and let \( c : V(G) \rightarrow \{1, 2, \ldots, k\} \) be a unique vertex-\( k \)-coloring of \( G \). For \( i, j \in \{1, 2, \ldots, k\} \), define \( G_{ij} \) to be the subgraph of \( G \) induced by the vertices which \( c \) assigns the colors \( i \) or \( j \). A very useful necessary condition for \( G \) to be uniquely vertex-\( k \)-colorable was noticed by Harary et. al. in the following theorem which appears in [30].
Theorem 1.2.2. (Harary, Hedetniemi, Robinson, 1969) If \(c:V(G) \rightarrow \{1, 2, \ldots, k\}\) is a unique vertex-k-coloring of \(G\), then for all \(i \neq j, i, j \in \{1, 2, \ldots, k\}\), the graph \(G_{ij}\) is connected.

Proof: If some \(G_{ij}\) had two or more components, then by interchanging the colors \(i\) and \(j\) in exactly one of these components, we would arrive at a valid coloring different than \(c\).

Corollary 1.2.3. Let \(c\) be a unique vertex-k-coloring of \(G\), let \(c\) be a color different from \(c(x)\). By Theorem 1.2.2, \(G_{(x)c}\) is a connected graph, and in particular, \(x\) is not an isolated vertex in \(G_{(x)c}\) because Proposition 1.2.1 insures that some vertex receives the color \(i\). Since there are \(k - 1\) other colors besides \(c(x)\), the minimum degree of \(G\) must be at least \(k - 1\). This completes the proof of Corollary 1.2.3.

Corollary 1.2.4. (Harary et al.) If \(G\) is a uniquely vertex-k-colorable, then \(G\) has at least \((k - 1)\) edges.

Proof: Let \(V(G)\) be the set of vertices colored. Theorem 1.2.2 insures that for \(1 \leq i < j \leq k\), the graph \(G_{ij}\) with vertex set \(V_{ij}\) is connected. Thus \(|E(G_{ij})| \geq |V_{ij}| + |V_{ij}| - 1\). Summing this inequality over all pairs \(i\), we have that \(|E(G)|\) which is the desired result.

Corollary 1.2.5. (Geller, Chartrand) If \(G\) is a uniquely vertex-4-colorable simple planar graph, then any drawing of the graph \(G\) is a triangulation. Moreover, for \(i \neq j\) and \(i, j \in \{1; 2; 3; 4\}\), each subgraph \(G_{ij}\) is a tree.

Proof: By Euler's formula \(|E(G)| = 3|V(G)| - 6\), and from Corollary 1.2.4, \(|E(G)| = 3|V(G)| - 6\), \(|E(G)| = 3|V(G)| - 6\). This implies that any drawing of \(G\) must be a triangulation. It also implies that equality holds throughout in the proof of the Corollary 1.2.4, so . Since \(G_{ij}\) is connected, it follows that \(G_{ij}\) is a tree. This completes the proof of the corollary.

II. ASSORTED RESULTS ABOUT UNIQUELY COLORABLE GRAPHS

A function \(V(G) \rightarrow V(G')\) is said to be a homomorphism of the graph \(G\) into the graph \(G'\) if it preserves adjacency of vertices, that is, if \((x, y)\) \(E(G)\) implies \((x); (y)) \(E(G')\). If it is true that for every pair of vertices \(x', y'\), \(x'\) is adjacent to \(y'\) in \(G'\) if and only if there is a pair \(x, y\) of adjacent vertices in \(G\) such that \((x) = x'\) and \((y) = y'\), then it is said to be a homomorphism of \(G\) onto \(G'\), and \(G'\) is said to be a homomorphism image of \(G\). The following propositions appear in [32].

Proposition 2.1.1. If \(G\) is uniquely vertex-k-colorable and \(H\) is a homomorphism image of \(G\) such that \((H) = k\), then \(H\) is uniquely vertex-k-colorable.

Proposition 2.1.2. If \(G\) is uniquely vertex-k-colorable then \(G\) is \((k - 1)\)-connected.

Proof: Let \(A\) be a set with \(|A| \leq k - 2\), let \(c\) be a unique vertex-k-coloring of \(G\), and let \(x, y \in V(G)\). If \(x\) and \(y\) are two distinct colors \(i\) and \(j\) such that no vertex of \(A\) has a vertex colored \(i\) or \(j\). Therefore \(c\). By Corollary 1.2.3, there are vertices \(u_i\) and \(u_j\) such that \(x\) is adjacent to \(u_i\) and \(y\) is adjacent to \(u_j\). Since \(G_{ij}\) is connected, there is a path \(P\) in \(G_{ij}\) joining \(u_i\) to \(u_j\) and thus there is a path in \(G\) joining \(x\) and \(y\). Thus, \(G\) is connected. This completes the proof of the proposition.

III. COMPLEXITY RESULTS FOR UNIQUE COLORING

The following proposition is obvious.

Proposition 3.1.1. A graph is uniquely vertex-1-colorable if and only if it consists of isolated vertices. A graph is uniquely vertex-2-colorable if and only if it is a connected bipartite graph.

Beyond this there is not much hope of finding a "good" characterization of arbitrary uniquely vertex-k-colorable graphs when \(k > 2\) because of the following complexity results contained in or implied by the work of Dailey in 1981 [33]

Theorem 3.1.2. The following decision problems are NP-Complete:

1. Given a graph \(G\) and a vertex-k-coloring \(c\) of \(G\), is there a vertex-k-coloring \(c'\) of \(G\) that is not equivalent to \(c'\)?
2. Given an integer \(k\) and a graph \(G\), does \(G\) have either 0 or at least 2 vertex-k-colorings?

The result of Dailey probably doom's any possibility of a polynomial time algorithm for problems 1) or 2) above. In [31], the authors pose the question of whether there is a polynomial time algorithm for deciding whether a given planar graph is uniquely vertex-3-colorable. This problem is still open as far as this author knows.
IV. A SUFFICIENT CONDITION FOR DETERMINING UNIQUE VERTEX-K-COLORABILITY

The following sufficient condition for a graph to be uniquely vertex-k-colorable was given by Bollobas in [14].

**Theorem 4.1.1.** Let k be an integer greater than one, let G be a vertex-k-colorable graph on n vertices, and let \( \Delta(G) \) denote the minimum degree of G. If then G is uniquely vertex-k-colorable. Moreover, if G has a vertex-k-coloring in which \( G_{ij} \) is connected for every \( 1 \leq i < j \leq k \), and, then G is uniquely vertex-k-colorable. These results are best possible.

This was generalized by Dmitriev according to a review of [15]. As we can see, this condition will apply only to very dense graphs.

**REFERENCES**


