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Existence the Solution for Fractional order Nonlinear Functional Integro-Differential Equation in Banach Space

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ABSTRACT
In this paper we prove existence the solution for fractional order nonlinear functional integro-differential equation. A hybrid fixed point theorem for the three operators are used for proving the main result.
Keywords: Banach Space, Fixed point theorem, Nonlinear functional integro-differential equation.

I. INTRODUCTION

Fractional Calculus is the field of Mathematical Analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. The concept of fractional calculus can be considered as a generalization of ordinary differentiation and integration to arbitrary order. However great efforts must be done before the ordinary derivatives could be truly interpreted as a special case of fractional derivatives .For more details; we refer the book by Miller and Ross [1].Fractional differential equations arise in the mathematical modeling of system and processes occurring in many engineering and scientific disciplines such as physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer theology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, etc [2-7]. For some recent developments on the topic, see [8].The theory of differential and integral equations of fractional order has recently received a lot of attention and new constitutes a significant of nonlinear analysis. The class of equation involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Numerous research papers have appeared devoted for hybrid differential and integral equations [9-12] .Fixed point theory constitutes an important and the core part of the subject of nonlinear functional analysis and is useful for proving the existence theorems for nonlinear differential and integral equations.

In this paper we study the existence result is obtained for initial value problem fractional order nonlinear functional integro-differential equation by using a hybrid fixed point theorem for three operators in Banach algebras due to B.C. Dhage [13].

We consider the following initial value problem fractional order nonlinear integro-differential equation (FNFIDE):

$$\left.\begin{aligned} \frac{d^{\xi}}{dt^{\xi}}\left\{\frac{x(t)-\sum_{i=1}^{n} I^{\beta_i} q_i\big(t,x(t)\big)}{f\Big(t,x\big(\mu_1(t)\big),x\big(\mu_2(t)\big)\Big)}\right\} \\ = g\Big(t,s,x\big(\gamma_1(s)\big),x\big(\gamma_2(s)\big)\Big) \\ x(0)=0 \end{aligned}\right\} \qquad (1.1)$$

for all $t \in \mathbb{J} = [0,\mathbb{T}], \mathbb{T} > 0$.

Where $\frac{d^{\xi}}{dt^{\xi}}$ denotes the Riemann-Liouville fractional derivative of order $\xi \in (0,1)$, I^{β_i} is the Riemann-Liouville fractional integral of order $\beta_i > 0, i = 1,2,3 \dots\dots.. n$, $f: \mathbb{J}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}\backslash\{0\}, g: \mathbb{J}\times\mathbb{J}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $q_i: \mathbb{J}\times\mathbb{R} \to \mathbb{R}$.

II. PRELIMINARIES

In this section, we introduce some notations and definitions of fractional calculus [8] and present preliminary results needed in our proofs later.

Definition 2.1[14]: Let $\mathbb{X}$ be a Banach space. A mapping $\mathbb{A}: \mathbb{X} \to \mathbb{X}$ is called Lipschitz if there is a constant $\alpha > 0$ such that,

$$\|\mathbb{A}x - \mathbb{A}y\| \le \alpha\|x - y\| \text{ for all } x, y \in \mathbb{X}.$$

If $\alpha < 1$, then $\mathbb{A}$ is called a contraction on $\mathbb{X}$ with the contraction constant α.

Definition 2.2[14]: An operator $\mathbb{Q}$ on a Banach space $\mathbb{X}$ into itself is called compact if for any bounded subset S of $\mathbb{X}$, $\mathbb{Q}(S)$ is relatively compact subset of $\mathbb{X}$. If $\mathbb{Q}$ is continuous and compact, then it is called completely continuous on $\mathbb{X}$.

Definition 2.3[14]: Let $\mathbb{X}$ be a Banach space with the norm $\|\cdot\|$ and let $\mathbb{Q}: \mathbb{X} \to \mathbb{X}$, be an operator (in general nonlinear). Then $\mathbb{Q}$ is called

i) Compact if $\mathbb{Q}(\mathbb{X})$ is relatively compact subset of $\mathbb{X}$.
ii) Totally compact if $\mathbb{Q}(S)$ is totally bounded subset of $\mathbb{X}$ for any bounded subset S of $\mathbb{X}$.
iii) Completely continuous if it is continuous and totally bounded operator on $\mathbb{X}$.

Definition2.4 [14]**:** Let $f \in \mathcal{L}^1[c,d]$and $\alpha > 0$. The Riemann – Liouville fractional derivative of order ξ of real function f is defined as

$$\frac{d^\xi}{dt^\xi} f(t) = \frac{1}{\Gamma(1-\xi)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\xi} ds, 0 < \xi < 1$$

Such that, $\frac{d^{-\xi}}{dt^{-\xi}} f(t) = I^\xi f(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{f(s)}{(t-s)^{1-\xi}} ds$ respectively.

Let $\mathcal{L}^1[c,d]$ be the space of all real and lebesgue integrable functions on the interval $[c,d]$. The space $\mathcal{L}^1[c,d]$ is equipped with the standard norm.

Definition 2.5(14]): Let $f \in \mathcal{L}^1[c,d]$ and $\xi > 0$ be a fixed number.The Riemann-Liouville fractional integral of order $\xi \in (0,1)$ of the function $f \in \mathcal{L}^1[c,d]$ is defined by the formula:

$$I^\xi f(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{f(s)}{(t-s)^{1-\xi}} ds, \quad t \in [0, \mathbb{T}]$$

Where $\Gamma(\xi)$ denote the Euler gamma function. The Riemann-Liouville fractional derivative operator of order ξ defined by

$$\frac{d^\xi}{dt^\xi} = \frac{d}{dt} \circ I^{1-\xi}$$

It may be shown that the fractional integral operator I^ξ transforms the space $\mathcal{L}^1[c,d]$ into itself and has some other properties. (See [17]).

Definition 2.6 :(Dominated convergence theorem): Let $\{f_n\}$ be a sequence of measurable functions on a measurable set A of X. If for all t $\in A$, $|f_n(t)| \le f(t)$ for some measurable function f on A with $\int_A f du < \infty$ and $f_n(t) \to f(t) \forall t \in A$, then$\int_A f_n du \to \int_A f du$

Theorem 2.6[14]: (Arzela-Ascoli Theorem) If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, then it has a convergent subsequence.

Corollary 2.7[14]: A metric space X is compact iff every sequence in X has a convergent subsequence.

Lemma 2.8[4]: *Let* $p > 0$ *and* $x \in \mathcal{C}(0,\mathbb{T}) \cap \mathcal{L}(0,\mathbb{T})$ then we have

$$I^p \frac{d^p}{dt^p} x(t) = x(t) - \sum_{j=1}^{n} \frac{(I^{n-p}x)^{(n-j)}(0)}{\Gamma(p-j+1)} t^{p-j}$$

where $n-1 < p < n$.

Let $\mathbb{X} = \mathcal{C}(\mathbb{J}, \mathbb{R})$ be a space of continuous real valued functions defined on $\mathbb{J} = [0, \mathbb{T}]$.Define a norm $\|\cdot\|$ and a multiplication in $\mathbb{X}$ by

$$\|x\| = sup_{t\in\mathbb{J}} |x(t)| \text{ and } (xy)(t) = (t)y(t), \quad \forall\, t \in \mathbb{J}$$

Clearly $\mathbb{X}$ is Banach Algebra with respect to above supremum norm and multiplication in it.

III. MAIN RESULT

In this section we consider the fractional order functional integro-differential equation (1.1).The following hybrid fixed point theorem for three operators in Banach algebras $\mathbb{X}$, due to B.C.Dhage [13] will be used to prove existence the solution for given equation(1.1)

Theorem 3.1[12]: Let S be a non empty, convex, closed and bounded subset of the Banach space $\mathbb{X}$ and let $\mathbb{A}, \mathbb{C}: \mathbb{X} \to \mathbb{X}$ and $\mathbb{B}: S \to \mathbb{X}$ are two operators satisfying:

a) $\mathbb{A}$ and $\mathbb{C}$ are Lipschitzian with lipschitz constants ζ, η respectively.
b) $\mathbb{B}$ is completely continuous, and
c) $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}x \in S$ for all $y \in S$
d) $\zeta M + \eta < 1$ where $M = \|\mathbb{B}(s)\| = sup\{\|\mathbb{B}x\| : x \in S\}$

Then the operator equation $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}x$ has a solution in S.

Lemma 3.1: Suppose that $\xi \in (0,1)$ and the function $f, g, q_i, i = 1,2,3, \dots n$ satisfying FNFIDE (1.1). Then x is the solution of the FNFIDE (1.1) if and only if it is the solution of integral equation

$$x(t) = \frac{f\Big(t, x(\mu_1(t)), x(\mu_2(t))\Big)}{\Gamma(\xi)} \times \int_0^t \frac{g\Big(t, s, x\Big(\gamma_1(t), x(\gamma_2(t))\Big)\Big)}{(t-s)^{1-\xi}} ds + \sum_{i=1}^{n} I^{\beta_i} q_i(t, x(t)), t \in \mathbb{J}, \xi \in (0,1) \quad (3.1)$$

Proof: Applying the Riemann-liouville fractional integral of order ξ to both sides of (1.1) and using lemma (2.8), we have

$$I^{\xi}\frac{d^{\xi}}{dt^{\xi}}\left\{\frac{x(t)-\sum_{i=1}^{n}I^{\beta_i}q_i(t,x(t))}{f\left(t,x(\mu_1(t)),x(\mu_2(t))\right)}\right\}_0^t$$
$$=I^{\xi}g\left(t,s,x(\gamma_1(s)),x(\gamma_2(s))\right)$$
$$\Rightarrow\left\{\frac{x(t)-\sum_{i=1}^{n}I^{\beta_i}q_i(t,x(t))}{f\left(t,x(\mu_1(t)),x(\mu_2(t))\right)}\right\}_0^t$$
$$=I^{\xi}g\left(t,s,x(\gamma_1(s)),x(\gamma_2(s))\right)$$
$$\frac{x(t)-\sum_{i=1}^{n}I^{\beta_i}q_i(t,x(t))}{f\left(t,x(\mu_1(t)),x(\mu_2(t))\right)}-\frac{t^{\xi-1}}{\Gamma(\xi-1+1)}I^{1-\xi}$$
$$\times\left[\frac{x(t)-\sum_{i=1}^{n}I^{\beta_i}q_i(t,x(t))}{f\left(t,x(\mu_1(t)),x(\mu_2(t))\right)}\right]_{t=0}$$
$$=\frac{1}{\Gamma(\xi)}\int_0^t\frac{g\left(t,s,x\left(\gamma_1(s),x(\gamma_2(s))\right)\right)}{(t-s)^{1-\xi}}ds$$
$$\therefore\frac{x(t)-\sum_{i=1}^{n}I^{\beta_i}q_i(t,x(t))}{f\left(t,x(\mu_1(t)),x(\mu_2(t))\right)}$$
$$-\frac{t^{\xi-1}}{\Gamma(\xi)}I^{1-\xi}\left[\frac{x(t)-\sum_{i=1}^{n}I^{\beta_i}q_i(t,x(t))}{f\left(t,x(\mu_1(t)),x(\mu_2(t))\right)}\right]_{t=0}$$
$$=\frac{1}{\Gamma(\xi)}\int_0^t\frac{g\left(t,s,x\left(\gamma_1(s),x(\gamma_2(s))\right)\right)}{(t-s)^{1-\xi}}ds$$
Since $x(0)=0$, $q_i(0,0)=0, f(0,0,0)\neq 0$
It follows that
$$x(t)=\frac{f\left(t,x(\mu_1(t)),x(\mu_2(t))\right)}{\Gamma(\xi)}\times$$
$$\int_0^t\frac{g\left(t,s,x\left(\gamma_1(t),x(\gamma_2(t))\right)\right)}{(t-s)^{1-\xi}}ds$$
$$+\sum_{i=1}^{n}I^{\beta_i}q_i(t,x(t)),t\in\mathbb{J}$$
Conversely differentiate (3.1) of order ξ with respect to t , we get,
$$\frac{d^{\xi}}{dt^{\xi}}\left\{\frac{x(t)-\sum_{i=1}^{n}I^{\beta_i}q_i(t,x(t))}{f\left(t,x(\mu_1(t)),x(\mu_2(t))\right)}\right\}$$
$$=\frac{d^{\xi}}{dt^{\xi}}\frac{1}{\Gamma(\xi)}\int_0^t\frac{g\left(t,s,x\left(\gamma_1(t),x(\gamma_2(t))\right)\right)}{(t-s)^{1-\xi}}ds$$
$$\Rightarrow\frac{d^{\xi}}{dt^{\xi}}\left\{\frac{x(t)-\sum_{i=1}^{n}I^{\beta_i}q_i(t,x(t))}{f\left(t,x(\mu_1(t)),x(\mu_2(t))\right)}\right\}$$
$$=\frac{d^{\xi}}{dt^{\xi}}I^{\xi}g\left(t,s,x(\gamma_1(t)),x(\gamma_2(t))\right)$$
$$\therefore\frac{d^{\xi}}{dt^{\xi}}\left\{\frac{x(t)-\sum_{i=1}^{n}I^{\beta_i}q_i(t,x(t))}{f\left(t,x(\mu_1(t)),x(\mu_2(t))\right)}\right\}$$
$$=g\left(t,s,x(\gamma_1(t)),x(\gamma_2(t))\right)\text{ for all }t\in\mathbb{J}.$$
Now by applying lemma (3.1) we study the existence of solution for the (FNFIDE) (1.1) under the following general assumptions:

($\mathcal{H}$1) The function $q_i:\mathbb{J}\times\mathbb{R}\to\mathbb{R}, i=1,2,3\ldots\ldots..n$, with $q_i(0,0,)=0, i=1,2,3\ldots\ldots..n$ are continuous and there exist positive functions λ_i, $i=1,2,3\ldots\ldots..n$ with bound $\|\lambda_i\|$ such that
$$\left|q_i(t,x(t))-q_i(t,y(t))\right|$$
$$\leq\lambda_i(t)|x(t)-y(t)| \tag{3.2}$$
$\forall\, t\in\mathbb{J}$ and $x,y\in\mathbb{R}$.

($\mathcal{H}$2) The function $f:\mathbb{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$ is continuous and bounded with bound $\mathbb{F}=sup_{(t,x_1,x_2)}|f(t,x_1,x_2)|$ there exist a bounded function $\alpha:\mathbb{J}\to\mathbb{R}$ with bound $\|\alpha\|$ such that
$$\left|\begin{matrix}f\left(t,x_1(\mu_1(t)),x_2(\mu_2(t))\right)\\-f\left(t,y_1(\mu_1(t)),y_2(\mu_2(t))\right)\end{matrix}\right|$$
$$\leq\alpha(t)max\{|x_1-y_1|,|x_2-y_2|\} \tag{3.3}$$
for all $t\in\mathbb{J}$ and $x,y\in\mathbb{R}$

($\mathcal{H}$3) The function $g:\mathbb{J}\times\mathbb{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfy Caratheodory conditions with the function $h(t,s):\mathbb{J}\times\mathbb{J}\to\mathbb{J}$ such that
$$g\left(t,s,x(\gamma_1(t)),x(\gamma_2(t))\right)\leq h(t,s) \tag{3.4}$$
$\forall t,s\in\mathbb{J}$ and $x\in\mathbb{R}$.

($\mathcal{H}$4) The function $v:\mathbb{J}\to\mathbb{J}$ defined by the function $v(t)=\int_0^t\frac{h(t,s)}{(t-s)^{1-\xi}}ds$ (3.5)
is bounded on $\mathbb{J}$.

Remark 3.1: Note that with the hypothesis ($\mathcal{H}1$ –$\mathcal{H}4$) holds then there exist a constant $\mathcal{K}_1>0$ such that $\mathcal{K}_1=sup_{t\geq 0}\frac{v(t)}{\Gamma(\xi)}$

Theorem 3.2: Assume that hypothesis ($\mathcal{H}1 - \mathcal{H}3$) holds then FNFIDE (1.1). Further if $\|\alpha\|\mathcal{K}_1 + \|\beta\| < 1$ then FNFIDE (1.1) has a solution in the space $\mathcal{AC}(\mathbb{J}, \mathbb{R})$.

Proof: Set $\mathbb{X} = \mathcal{AC}(\mathbb{J}, \mathbb{R})$ and define a subset S of $\mathbb{X}$ as $S = \{x \in \mathbb{X} : \|x\| \le r\}$.
where r satisfies the inequality

$$\mathbb{F}\mathcal{K}_1 + \|\lambda_i\| \sum_{i=1}^{n} \frac{\mathbb{T}^{\beta_i}}{\Gamma(\beta_i + 1)} \le r \tag{3.6}$$

Clearly S be a non empty, convex, closed and bounded subset of the Banach space $\mathbb{X}$. By lemma (3.1), problem (1.1) is equivalent to (3.1).

Now we define three operators $\mathbb{A} : \mathbb{X} \to \mathbb{X}$ and $\mathbb{B} : S \to \mathbb{X}$ and $\mathbb{C} : \mathbb{X} \to \mathbb{X}$ by

$$\mathbb{A}x(t) = f\left(t, x(\mu_1(t)), x(\mu_2(t))\right) \tag{3.7}$$

$$\mathbb{B}x(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{g\left(t, s, x\left(\gamma_1(t), x(\gamma_2(t))\right)\right)}{(t-s)^{1-\xi}} ds \tag{3.8}$$

$$\mathbb{C}x(t) = \sum_{i=1}^{n} I^{\beta_i} q_i\left(t, x(t)\right) \tag{3.9}$$

i.e. $\mathbb{C}x(t) = \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{\beta_i - 1}}{\Gamma(\beta_i)} q_i\left(s, x(s)\right) ds,$
$\forall\, t \in \mathbb{J}$ (3.10)

We shall show that, the operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ satisfy all the conditions of lemma (3.1) This will be achieved in the following series of steps.

Step I: First show that $\mathbb{A}$ and $\mathbb{C}$ are lipschitzian on $\mathbb{X}$.
Let $x, y \in \mathbb{X}$, then by ($\mathcal{H}1$) for $t \in \mathbb{J}$ we have,

$$|\mathbb{A}x(t) - \mathbb{A}y(t)| = \left| \begin{array}{l} f\left(t, x(\mu_1(t)), x(\mu_2(t))\right) \\ -f\left(t, y(\mu_1(t)), y(\mu_2(t))\right) \end{array} \right|$$

$$\le \alpha(t) max\{|x_1 - y_1|, |x_2 - y_2|\}$$

$$\le \|\alpha\| |x(t) - y(t)|$$

Taking the supremum over t, we obtain

$$\|\mathbb{A}x - \mathbb{A}y\| \le \|\alpha\| \|x - y\|$$

for all $x, y \in \mathbb{R}$.
Therefore $\mathbb{A}$ is lipschitzian with lipschitz constant $\|\alpha\|$.
Analogously, for any $x, y \in \mathbb{R}$, we have,

$$|\mathbb{C}x(t) - \mathbb{C}y(t)| = \left| \begin{array}{l} \sum_{i=1}^{n} I^{\beta_i} q_i\left(t, x(t)\right) \\ -\sum_{i=1}^{n} I^{\beta_i} q_i\left(t, y(t)\right) \end{array} \right|$$

$$\le \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{\beta_i - 1}}{\Gamma(\beta_i)} \lambda_i(s) |x(s) - y(s)| ds$$

$$\le \|x - y\| \sum_{i=1}^{n} \|\lambda_i\| \frac{\mathbb{T}^{\beta_i}}{\Gamma(\beta_i + 1)}$$

This means that,

$$\|\mathbb{C}x - \mathbb{C}y\| \le \sum_{i=1}^{n} \|\lambda_i\| \frac{\mathbb{T}^{\beta_i}}{\Gamma(\beta_i + 1)} \|x - y\|$$

$$\therefore\ \|\mathbb{C}x - \mathbb{C}y\| \le \|\beta\| \|x - y\|$$

Thus $\mathbb{C}$ is lipschitz on $\mathbb{X}$ with lipschitz constant constant $\|\beta\|$.

Step II: To show the operator $\mathbb{B}$ is completely continuous on $\mathbb{X}$. Let $\{x_n\}$ be a sequence in S converging to a point x. Then by lebesgue dominated convergence theorem for all $t \in \mathbb{J}$, we obtain

$$\lim_{n\to\infty} \mathbb{B}x_n(t) =$$

$$\lim_{n\to\infty} \left\{ \begin{array}{c} \frac{1}{\Gamma(\xi)} \times \\ \int_0^t \frac{g\left(t, s, x_n\left(\gamma_1(s), x_n(\gamma_2(s))\right)\right)}{(t-s)^{1-\xi}} ds \end{array} \right\}$$

$$= \frac{1}{\Gamma(\xi)} \int_0^t \frac{g\left(t, s, x\left(\gamma_1(s), x(\gamma_2(s))\right)\right)}{(t-s)^{1-\xi}} ds$$

$$= \mathbb{B}x(t), \forall\, t \in \mathbb{J}$$

Implies that, $\mathbb{B}$ is continuous on S.
Next we will prove that the set $\mathbb{B}(S)$ is uniformly bounded in S. for any $x \in S$,
we have $|\mathbb{B}x(t)|$

$$= \left| \frac{1}{\Gamma(\xi)} \int_0^t \frac{g\left(t, s, x\left(\gamma_1(s), x(\gamma_2(s))\right)\right)}{(t-s)^{1-\xi}} ds \right|$$

$$\le \frac{1}{\Gamma(\xi)} \int_0^t \left| \frac{g\left(t, s, x\left(\gamma_1(s), x(\gamma_2(s))\right)\right)}{(t-s)^{1-\xi}} \right| |ds|$$

$$\le \frac{1}{\Gamma(\xi)} \int_0^t \frac{h(t,s)}{(t-s)^{1-\xi}} ds \le \frac{v(t)}{\Gamma(\xi)}$$

Taking supremum over t, we obtain

$\|\mathbb{B}x\| \leq \frac{v(t)}{\Gamma(\xi)} = \mathcal{K}_1, \forall\, t \in \mathbb{J}$

Therefore $\|\mathbb{B}\| \leq \mathcal{K}_1$, which shows that $\mathbb{B}$ is uniformly bounded on S.

Now we will show that $\mathbb{B}$(S) is equicontinuous set in $\mathbb{X}$. Let $t_1, t_2 \in \mathbb{J}$ with $t_1 > t_1$ and $x \in S$, then we have

$$|\mathbb{B}x(t_1) - \mathbb{B}x(t_2)| = \left| \frac{1}{\Gamma(\xi)} \int_0^{t_1} \frac{g\left(t_1, s, x\left(\gamma_1(t_1), s(\gamma_2(t_1))\right)\right)}{(t_1 - s)^{1-\xi}} ds - \frac{1}{\Gamma(\xi)} \int_0^{t_2} \frac{g\left(t_2, s, x\left(\gamma_1(t_2), s(\gamma_2(t_2))\right)\right)}{(t_2 - s)^{1-\xi}} ds \right|$$

$$\leq \frac{1}{\Gamma(\xi)} \left| \int_0^{t_1} h(t_1, s) ds - \int_0^{t_2} h(t_2, s)\, ds \right|$$

$$\leq \frac{1}{\Gamma(\xi)} |v(t_1) - v(t_2)|$$

$$\to 0 \; as \; t_1 \to t_2$$

Therefore by Arzela- Ascoli theorem that $\mathbb{B}$ is completely continuous operator on S.

Step III: The hypothesis (c) of lemma (3.1) is satisfies.

Let $x \in \mathbb{X}$ and $y \in S$ be arbitrary elements such that $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}x$ then we have

$$|x(t)| = |\mathbb{A}x(t)\mathbb{B}x(t) + \mathbb{C}x(t)|$$

$$\leq |\mathbb{A}x(t)||\mathbb{B}x(t)| + |\mathbb{C}x(t)|$$

$$\leq \left| f\left(t, x(\mu_1(t)), x(\mu_2(t))\right) \right| \times \left| \frac{1}{\Gamma(\xi)} \int_0^t \frac{g\left(t, s, x\left(\gamma_1(s), x(\gamma_2(s))\right)\right)}{(t-s)^{1-\xi}} ds \right| + \left| \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{\beta_i - 1}}{\Gamma(\beta_i)} q_i(s, x(s)) ds \right|$$

$$\leq \left| f\left(t, x(\mu_1(t)), x(\mu_2(t))\right) \right| \times \frac{1}{\Gamma(\xi)} \int_0^t \frac{\left| g\left(t, s, x\left(\gamma_1(s), x(\gamma_2(s))\right)\right) \right|}{(t-s)^{1-\xi}} ds + \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{\beta_i - 1}}{\Gamma(\beta_i)} |q_i(s, x(s))| ds$$

$$\leq \mathbb{F} \frac{1}{\Gamma(\xi)} \int_0^t (t-s)^{\xi - 1} h(t,s)\, ds + \sum_{i=1}^{n} \int_0^t \frac{(t-s)^{\beta_i - 1} \|\lambda_i\|}{\Gamma(\beta_i)}$$

$$\leq \mathbb{F} \frac{v(t)}{\Gamma(\xi)} + \|\lambda_i\| \sum_{i=1}^{n} \frac{\mathbb{T}^{\beta_i}}{\Gamma(\beta_i + 1)}$$

which leads to

$$\|x\| \leq \mathbb{F}\mathcal{K}_1 + \|\lambda_i\| \sum_{i=1}^{n} \frac{\mathbb{T}^{\beta_i}}{\Gamma(\beta_i + 1)} \leq r$$

Therefore $x \in S$.

Step IV: Finally we show that $\zeta M + \eta < 1$ that is condition (d) of lemma (3.1) holds.

Since

$$M = \|\mathbb{B}(S)\| = sup_{x \in S}\{sup_{t \in \mathbb{J}} |\mathbb{B}x(t)|\}$$

$$= sup_{x \in S} \left\{ sup_{t \in \mathbb{J}} \left| \frac{1}{\Gamma(\xi)} \times \int_0^t \frac{g\left(t, s, x\left(\gamma_1(s), x(\gamma_2(s))\right)\right)}{(t-s)^{1-\xi}} ds \right| \right\}$$

$$\leq sup_{x \in S} \left\{ \frac{1}{\Gamma(\xi)} \times \int_0^t \frac{\left| g\left(t, s, x\left(\gamma_1(s), x(\gamma_2(s))\right)\right) \right|}{(t-s)^{1-\xi}} ds \right\}$$

$$\leq sup_{x \in S} \left\{ \frac{1}{\Gamma(\xi)} \times \int_0^t \frac{h(t,s)}{(t-s)^{1-\xi}} ds \right\}$$

$$\leq sup_{x \in S} \left\{ \frac{v(t)}{\Gamma(\xi)} \right\} = \mathcal{K}_1$$

and therefore $\zeta M + \eta$, we have

$(\|\alpha\|\mathcal{K}_1 + \|\beta\|) < 1$,

Where$\zeta = \|\alpha\|$ and $\eta = \|\beta\|$

Thus all the conditions of lemma (3.1) are satisfied and hence the operator equation $x = \mathbb{A}x\mathbb{B}y + \mathbb{C}x$ has a solution in S.

In consequence problem (1.1) has a solution on $\mathbb{J}$. This completes the proof.

IV. CONCLUSION

In this paper we have studied the existence of solution for initial value problem for fractional order functional nonlinear integro-differential equation. The result has been obtained by using hybrid fixed point theorem for three operators in Banach space due to Dhage [1].

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