RESEARCH ARTICLE OPEN ACCESS

Evaluation of Vibrational Behavior for A System With Two-Degree-of-Freedom Undamped.

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ABSTRACT
Analysis of the vibrational behavior of a system is extremely important, both for the evaluation of operating conditions, as performance and safety reason. The studies on vibration concentrate their efforts on understanding the natural phenomena and the development of mathematical theories to describe the vibration of physical systems. The purpose of this study is to evaluate an undamped system with two-degrees-of-freedom and demonstrate by comparing the results obtained in the experimental, numerical and analytical modeling the characteristics that describe a structure in terms of its natural characteristics. The experiment was conducted in PUC-MG where the data were acquired to determine the natural frequency of the system. We also developed an experimental test bed for vibrations studies for graduate and undergraduate students. In analytical modeling were represented all the important aspects of the system. In order, to obtain the mathematical equations is used MATLAB to solve the equations that describe the characteristics of system behavior. For the simulation and numerical solution of the system, we use a computational tool ABAQUS. The comparison between the results obtained in the experiment and the numerical was considered satisfactory using the exact solutions. This study demonstrates that calculation of the adopted conditions on a system with two-degrees-of-freedom can be applied to complex systems with many degrees of freedom and proved to be an excellent learning tool for determining the modal analysis of a system. One of the goals is to use the developed platform to be used as a didactical experiment system for vibration and modal analysis classes at PUC Minas. The idea is to give the students an opportunity to test, play, calculate and confirm the results in vibration and modal analysis in a low-cost platform.
Keywords-Modal Analysis,Vibration of two-degrees-of-freedom system, Didactical experiment.

I. INTRODUCTION

The properties of vibration for the engineering devices are often limiting factors in performance. Where the natural frequency of vibration of a machine or structure coincide with the frequency of external excitation, the system resonance occurs that results in excessive deflections and failures. It is therefore very important to analyze, evaluate and control the vibration behavior of any set or system to full operating conditions, performance and safety.In general, a vibration system includes a means to store potential energy (spring or elasticity), a means to store kinetic energy (mass or inertia) and a means of gradual loss of energy. The vibration of a system involves the transfer of alternating its potential energy to kinetic energy and kinetic energy to potential energy.

Vibrations can be classified in several ways. They are defined as free vibration or forced by the condition of external forces acting on the system. If no energy is lost or dissipated by friction or other resistance during oscillation, we have an undamped vibration. However, if any energy is lost is called damped vibration. The behavior of the vibrating system can be linear or nonlinear behavior is a function of spring, mass and damper. We have yet to define the degrees of freedom which is given by the minimum number of independent coordinates required to completely determine the positions of all parts of the system.The analysis of vibrating system usually involves the mathematical modeling that have the purpose to represent all important aspects of the system to obtain the mathematical equations that govern the behavior of the system, solution of these equations by standard methods and interpretation of results.

This paper aims to present the vibrational behavior for a system with two-degrees-of-freedom undamped through the data obtained in the experiment conducted at the Laboratory of the Department's Graduate Program in Mechanical Engineering, PUC-MG and comparison of numerical and exact solutions. We will present the preparation of the experiment for data acquisition of the natural frequency of the system, beyond the description of analytical modeling to obtain the mathematical equations that describe the behavior of the system and solution the characteristic equations. It will be also presented the numerical modeling using theABAQUS software package for simulation of the system evaluation.

System With Two-Degree-of-Freedom Undamped

Systems that require two independent coordinates to describe its motion are called systems with two-degree-of-freedom. The condition of undamped vibration is given by any energy to be lost or dissipated by friction during oscillation of the set. The study of vibration for the system starts with the definition of the equations of motion. Adopting that the system vibrates in a specific direction in the plane with a mass *m* when supported on two springs of stiffness k_1 and k_2, as shown in Figure 1, the displacement of the system at any instant can specified by a linear coordinate *x(t)* indicates that the displacement center of gravity (*CG*) of the mass and an angular coordinate *ɵ(t)* that triggers the rotation of mass m relative to its *CG*.

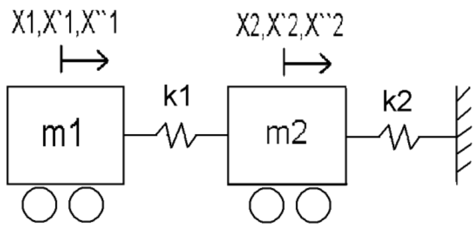

Fig. 1 –Proposed system: 2 masses and 2 springs.

Initially, the solution to this system, we define the free-body diagram as show in Figure 2.

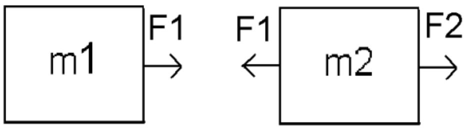

Fig. 2. Free-body diagram

From the free-body diagram and based on the equation that describes a system of multiple degrees of freedom, we have,

$$[M]\ddot{x}(t) + [c]\dot{x}(t) + [k]x(t) = F(t) (1)$$

and taken on that all forces are zero and that the damping is neglected (c_1=c_2=0), it is possible to obtain the equations of each of the masses:

$$m_1\ddot{X}_1 + K_1(X_1 - X_2) = 0, \qquad (2)$$

$$m_2\ddot{X}_2 + K_2X_2 + K_1(X_2 - X_1) = 0 \ (3)$$

The Equations (2) and (3) can be written in matrix form as

$$m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}; K = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 + K_2 \end{bmatrix}; (4)$$

where $[m]$ and $[k]$ are called the matrices of mass and stiffness of the system. The vectors of displacement and force, *x(t)* and *F(t)* are given by

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}; \quad F(t) = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix} (5)$$

For the analysis of the vibration system shown in Figure 1, substituting the values of *m* and *k* in Equation (1), we have:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 + K_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} (6)$$

The Equation of motion (4), or simultaneous homogeneous algebraic equations with unknowns x_1 and x_2 are reduced to:

$$\begin{cases} m_2\ddot{X}_2 - K_1X_1 + (K_1 + K_2)X_2 = 0 \\ m_1\ddot{X}_1 + K_1X_1 - K_1X_2 = 0 \end{cases} (7)$$

Based on a solution to harmonic motion is given by the solution of elementary differential equations, we assume that the solution of *x(t)* is of the form:

$$x(t) = \{x\}e^{jwt} \Rightarrow \ddot{x}(t) = -w^2\{x\}e^{jwt}, (8)$$

and substituting the Equation (8) to the vector equation of motion, we have:

$$-w^2[m]\{X\}e^{jwt} + [K]\{X\}e^{jwt} = 0 (9)$$

Applying the condition of singularity for the coefficient of Equation (9), we have a nontrivial solution of x_1 and x_2 and the determinant of coefficients must be zero;

$$\det([K] - w^2[m]) = 0, (10)$$

and substituting the values of the matrices of *m* and *k* in Equation (10), we obtain

$$\det \begin{bmatrix} K_{11} - w^2m_1 & K_{12} \\ K_{21} & K_{22} - w^2m_2 \end{bmatrix} = 0, (11)$$

and solving the determinant of Equation (11), we have:

$$m_1m_2w^4 - (K_{11}m_2 + K_{22}m_1)w^2 + K_{11}K_{22} - K_{21}K_{12} = 0 (12)$$

The Equation (12) is called the characteristic equation because the solution of this equation we have the frequencies or characteristics values of the system. The roost of this equation is given by

$$W^{2`}, W^{2``} = \frac{(K_{11}m_2 + K_{22}m_1) \pm}{2m_1m_2} \frac{\pm\sqrt{(K_{11}m_2+K_{22}m_1)^2 - 4m_1m_2(K_{11}K_{22} - K_{21}K_{12})}}{2m_1m_2}$$

or

$$W^{2`}, W^{2``} = \frac{(K_1m_2 + (K_1+K_2)m_1)}{2m_1m_2} \frac{\pm\sqrt{(K_1m_2+(K_1+K_2)m_1)^2 - 4m_1m_2(K_1(K_1+K_2) - K_1{}^2)}}{2m_1m_2} (13)$$

The parameters $W^{2\prime}$ and $W^{2\prime\prime}$ are called the frequencies of the system since the values of physical parameters m_1, m_2, k_1 and k_2 are known.
The modeling for system with multi-degrees-of-freedom is complex, requires the description of dynamic behavior for several independent variables, but the approach for this system can also be described by Equation (1) that governing the motion of free vibration systems with two-degree-of-freedom, since include all variables of the system.

II. DESCRIPTION OF THE EXPERIMENT

The experiment consists in evaluating the system with two-degree-of-freedom undamped to obtain the natural frequency of the system through excitation pulse. The installations for the experiment that was conducted in the Dynamics Laboratory of PUC-MG are shown in Figure 3. This is a low-cost test bed developed to demonstrate vibration effects for graduate and undergraduate students. The focus is to allow the students to apply analytical, numerical and experimental concepts to understand vibration phenomenon in engineering systems. Based on this platform we want to simulate engineering systems like car suspension and others.

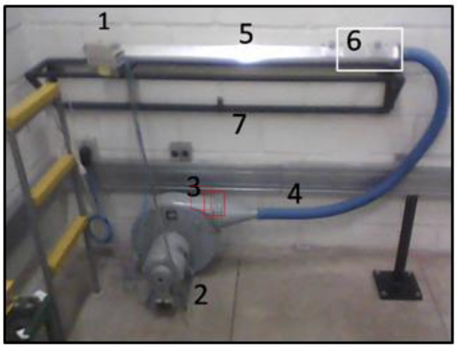

Fig. 3. Schematic of experiment setup

The goal in this part of the project was to create a low-cost installation that could be easily reproduced but that still allowed several experiments to be made, this was achieved as show bellow:

1. Frequency inversor: Of the shelf equipment from WEG, model CFW080040S2024PSZ. This model operates within 200-240VAC at 8.8A and 50/60Hz at the input, it provides three outputs that can be adjusted from 0 to the input voltage at 4A and 0 to 300Hz, only one output was used.
2. A.C. Motor: An of the shelf equipment from Compton Parkinson. This motor has 3/4kW and can achieve a maximum spin of 3450RPM. Its voltage its stated at 220/254V and 2.1A at 60Hz.
3. Blowing Equipment: An of the shelf produced by DISCUS, this equipment is of the B28 77/1 model.
4. Flexible Tube: We use 1.5 meters of tube with a diameter of 55mm.
5. The test section: To this installation the laboratory made L shaped cantilever beam out of aluminum, the test section has a length of 1180mm, each leg of the L measure 50mm and it's 3.5mm thick. In each leg 79 columns of roles at three parallel rows were made, each role with 0.5mm in diameter, the roles were equally spaced at 1.5mm of each other. In this experimental only 22 columns were left open the others were closed with tape. The bottom of the test section is a 60mm of diameter PVC tube cut in half and fixed to the cantilever with silicone. One side of the cantilever was fixed to the flexible tube and the other was closed with glass, everything was isolated with silicone.
6. The test body: We used two aluminum plates shaped in "L" with the same shape as the cantilever but with a length of only 90mm. In each plate were fixated to screws and a 25mm platform, the screws were to fixate the springs used in the test and the platform is the place to put the accelerometers. Even since only two bodies and two springs were used in the test this installation allows to include more bodies and different kinds of springs to make more tests. This system is shown in Figure 4, where is emphasized the point of attachment of the accelerometer at plate.

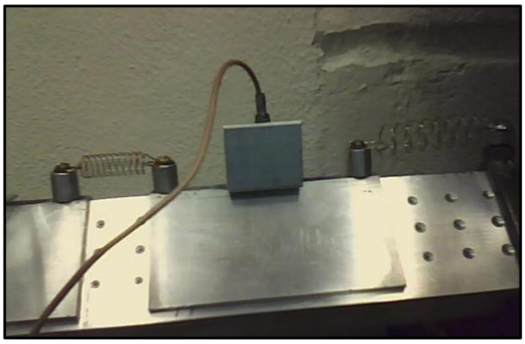

Fig. 4. Position of the accelerometer.

7. Support: The support was made in the laboratory out of steel tubes it measured 1600mm in length and its base have 240mm.The two accelerometers used in the experiment are brand Kistler8632C10 model, type uniaxial, cubic configuration with low impedance and high sensitivity.This installation was created, in order, to eliminate the friction between the metal plates and the cantilever beam, the pressurized air that pass through the small holes in the cantilever beam allowed us to

create a thin layer of air between the components. This also allow for experiments with differences in the friction value by changes of the air flow generated by a variation in the spin of the motor. This changes were not used in this test and the flow was the same for all times, avoiding friction in all tests for this paper.

Measurements were taken of the physical quantities of all components of the mass-spring system used in the process of experimental vibration analysis. Measurements were performed in the laboratory of Physics, PUC Minas, which used a balance of Toledo Scout SP 2020 Brazil's capacity 200g, resolution of 0.01g of capacity testing division of 0.01g, 10290 493 series, product load tested in accordance with the ordinance 236/94 metrology.

To check the displacement of position of the springs were used for this process standard base, which is in the physics laboratory, with a fixed resolution of 1 mm and weight standards of 20 g each. Pendulum-type base fixed with scale patterns and weights 20g, observing minimize parallax error and process stability. Fixation was performed for the corner points of the masses classified as *m1* and *m2*, the masses of all the springs (manufactured in spring steel 1070) in having the same compression characteristics and repulsion, seeking to obtain a value of K (elastic constant) the same as befitted the best way to process and the mass involved in the process being studied experimentally in the laboratory of dynamic analysis.

The plates were classified as *m1* and *m2*, and before the measurements for the system we have that the masses are 91.14g and 92.40g, respectively. We have calculated the uncertainty for the measurements of the masses is ± 0.03g. For the total values of each mass is the mass of the accelerometer included. Steel springs have the same characteristics of compression and tension. These springs are defined by *k1* and *k2* with elastic constant 70.58N/m and 75.46N/m, respectively. The excitation system was manually at the end of the cantilever beam of mass m_2 causing the system to stay tensioned. Thus, initiating the oscillatory motion and process data acquisition using MATLAB software for processing signals from the accelerometer and obtain the values of the accelerations in time domain and frequency.

The equipment for conditioning the signals from the accelerometer was set for a gain of 2 and a cutoff frequency equal to 1K. We use a green board installed in 6251 model PCI slot of your computer to convert the analog signal to digital. To obtain the numerical processing in MATLAB using the computer code described in Table 1.

Table 1. MATLAB code for collection data

```
hw = daqhwinfo('nidaq');
// Hardware definition for data aquisition
hw.InstalledBoardIds;
// addressing a PCI6251 National Instruments board.
hw.BoardNames;
ai = analoginput('nidaq','Dev1');
addchannel(ai, [0 1]);
set(ai,'InputType','Differential');

% Sample rate
ai.SampleRate = 10000;

% Number of acquired values
ai.SamplesPerTrigger = 250000;
// acquisition parameters
delay = (ai.SamplesPerTrigger/ai.SampleRate)+1;
start(ai);
wait(ai,delay);
data = getdata(ai);
dt=inv(ai.SampleRate)
tf=delay-1
t=0:dt:(tf-dt);
t=t';
% Next power of 2 from length of y
NFFT = 2^nextpow2(ai.SamplesPerTrigger);
// buffer length
Y = fft(data,NFFT)/ai.SamplesPerTrigger;
// FFT computation
f = ai.SampleRate/2*linspace(0,1,NFFT/2);
// autoscale adjust
```

The acquisitions were made considering a sampling rate equal to 10,000 and the number of acquired values equal to 25,000 during a period of 25 seconds. It was found the behavior of the records to validate only those who showed a symmetrical sinusoidal oscillation for the signs of tension.

III. RESULTS OF THE EXPERIMENT

In the analysis of the vibrational behavior of the system, we found that the natural frequency in the first vibrational mode is 2.899Hz and in the second vibrational mode is 7.286Hz, as showed the results in Figure5.

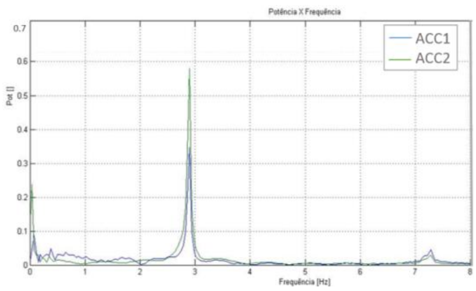

Fig. 5. Results for the values of the natural frequency.

Accordingto the results obtained in the experiment, we have that the signals obtained for the accelerations in time domain were satisfactory and did not surpassed the limits set for the signal conditioner for accelerometers and the results in time domain. It showed a symmetrical sinusoidal oscillation in the whole study period, indicating that the voltage signals generated excitation system showed no faults in their acquisition. If saturation occurs, new data acquisition should be done.

IV. ANALYTICAL SOLUTION

Using the equations presented in Chapter II of this paper for a system of two-degree-of-freedom undamped and considering the parameters of components adopted in the experiment, we can analytically determine the natural frequency for the system.
The system parameters are:

$$m_1 = 91.14g\,, m_2 = 92.40g\,,$$
$$K_1 = 70.58\frac{N}{m}, K_2 = 75.46\frac{N}{m}$$

Using these parameters and according toEquation (13) previously deducted to determine the natural frequency, we find the following results:

$$W^{2`}, W^{2``} = \frac{19.8317 \pm \sqrt{(19.8317)^2 - 179.407)}}{2 * 0.0084}$$
$$W^{2`} = 309.141 => w^{2`} = \sqrt{W^{2`}}$$
$$\lambda_1 = 17.58\frac{rad}{s} = 2.798Hz$$

$$W^{2``} = 2045.792 => w^{2``} = \sqrt{W^{2``}}$$
$$\lambda_2 = 45.23\frac{rad}{s} = 7.199Hz$$

Therefore, we have the analytical calculation of the first mode natural frequency of the system that is 2.798Hz.

We can now determine the characteristic equation that governing the system using the method of solution of Eigenvalue. This method allows the solution of vibration problems for systems with multi-degree-of-freedom. Considering the Equation (10), we have the roost for λ_1 and λ_2 by following equations:

$$[[K] - \lambda_v[m]]\{v_v\} = 0$$

solving λ_1 and λ_2, we obtain:

$$\begin{bmatrix} 70.58 - 309.14 * 0.09 & -70.58 \\ -70.58 & 146.04 - 309.14 * 0.09 \end{bmatrix}\begin{Bmatrix} v_{11} \\ v_{21} \end{Bmatrix} = 0$$
$$\rightarrow v_{11} = 1.665v_{21}$$

$$\begin{bmatrix} 70.58 - 2045.79 * 0.09 & -70.58 \\ -70.58 & 146.04 - 2045.7 * 0.09 \end{bmatrix}\begin{Bmatrix} v_{12} \\ v_{22} \end{Bmatrix} = 0$$

$$\rightarrow v_{12} = -0.609v_{22}$$

since $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$, we have

$$v = \begin{bmatrix} 1.665 & -0.609 \\ 1 & 1 \end{bmatrix}$$

To find the new mass matrix we use the following equation

$$[M] = [V]^T[m][V](14)$$

where, we have a solving the matrix by

$$[M] = \begin{bmatrix} 1.665 & 1 \\ -0.609 & 1 \end{bmatrix}\begin{bmatrix} 0.09114 & 0 \\ 0 & 0.09240 \end{bmatrix}\begin{bmatrix} 1.665 & -0.609 \\ 1 & 1 \end{bmatrix}$$
$$[M] = \begin{bmatrix} 0.345 & 0 \\ 0 & 0.126 \end{bmatrix} kg$$

The same condition is considered for the stiffness matrix, where we have defined the equation by

$$[K] = [V]^T[k][V] \qquad (15)$$

where, we have a solving the matrix by

$$[K] = \begin{bmatrix} 1.665 & 1 \\ -0.609 & 1 \end{bmatrix}\begin{bmatrix} 70.58 & 0 \\ 0 & 75.46 \end{bmatrix}\begin{bmatrix} 1.665 & -0.609 \\ 1 & 1 \end{bmatrix}$$
$$[K] = \begin{bmatrix} 271.12 & 0 \\ 0 & 101.64 \end{bmatrix} N/m$$

With the new definitions of mass and stiffness matrices, we find the independent equations by:

$$[M]\{\ddot{\eta}\} + [K]\{\eta\} = \{N\}(16)$$

obtaining the solution

$$\begin{bmatrix} 0.345 & 0 \\ 0 & 0.126 \end{bmatrix}\begin{Bmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \end{Bmatrix} + \begin{bmatrix} 271.12 & 0 \\ 0 & 101.64 \end{bmatrix}\begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{Bmatrix} n_1(t) \\ n_2(t) \end{Bmatrix}$$
$$\rightarrow \begin{cases} 0.345\ddot{\eta}_1(t) + 271.12\eta_1(t) = n_1(t) \\ 0.126\ddot{\eta}_2(t) + 101.64\eta_2(t) = n_2(t) \end{cases}$$

Using the MATLAB software to plot the solutions of the characteristic equation and calculate the natural frequency of the system. The code used for these tests is presented in table 2.

Table 2. MATLAB code for solution the natural frequency.

```
% System parameters
m1 = 0.09114;
m2 = 0.09240;
k1 = 70.58;
k2 = 75.46;
% System modeling
% M*ddX + K*X = 0
M = [m1 0;0 m2];
K = [k1 -k1;-k1 k1+k2];
Mt = [inv(sqrt(m1)) 0; 0 inv(sqrt(m2))];
Kt = Mt*K*Mt;
lbd = sym ('lbd');
lbd_I = sym([lbd 0;0 lbd]);
equation = Kt - lbd_I
determinant = det(equation)
r = roots (determinant)
lbd1 = sqrt(r(2));
lbd2 = sqrt(r(1));
```

```
% Calculating the frequency natural
f1 = lbd1/2/pi
f2 = lbd2/2/pi
```

V. NUMERICAL MODELING

The computational analysis widely used in engineering for evaluation of the structural integrity and development of new projects. In addition to lower cost in comparison to experimental tests, the simulation has the benefit of agility in all phases of the project involved.

To assist this study we used the solver Abaqus 6.9-1® to simulate the system with two-degrees-of-freedom undamped. This software is a general-purpose finite element program to numerical solution for complex models.The software used for pre-processing and post processing was the Altair HyperWorks 10.0®, software package that offers an interactive environment that was used to create the finite element model,for definition of boundary conditions, diagnose and evaluation of results.We have in Figure 6 the finite element model where we seek the best way to discretize the real conditions of the experiment.

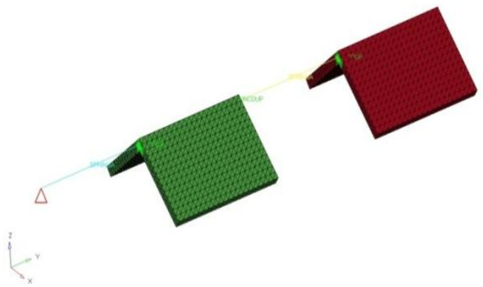

Figure 6. Finite Element Model

For the model of plates were used elements type *hexahedrical* in second order and the material set with elastic modulus of 205GPa, Poison of 0.30 and the density of each mass was adjusted according to the values obtained in measurements. The element size used was 5mm, which is sufficient to discretize the geometry of the plates.

As for the springs, we use the element type of *Springa* in which we define the direction and elastic constant in six degree-of-freedom, being in a direction defined values as measured data for springs and other restricted directions.Describing the actual condition of the experiment, in the lower plates were considered the restrictions of the displacements, being released in only one direction.The FEM model for the design and data processing used a machine with Microsoft Windows operating system with Core 2 Duo and 4GB of RAM. Because it is a simple model, the processing time is fast.

VI. NUMERICAL SOLUTION

The vibration modes for the evaluated system by numerical solution are show in the Figures 7 and 8.

Figure 7. Result of modal analysis: 1st mode of vibration in 2.789Hz

Figure 8. Result of modal analysis: 2nd mode of vibration in 7.222Hz

We verified the sensitivity of mesh to the model, however, because it is a simple model, the values showed no changes in the modal result.The software package offers good tools for configuring the models to be evaluated, so models with multiple degrees of freedom can be easily simulated and the results obtained from the natural frequency of the system.

VII. RESULTS

The Table 3 presents the results for all methods of measuring the natural frequency of the system with two degrees of freedom undamped evaluated in this study. Being shown the results of the experiment conducted in the laboratory, the exact solution and the results obtained from finite element model. Note that the results are very similar and demonstrate the correlation between the evaluated methods.The main goal was to validate the computational implementation in Matlab by comparing the results with a well-known numerical solver and also doing an experimental modal analysis. As a future works, we want to introduce a damper in order to introduce new effects in the system. We want to do this validation preparing laboratory experimental classes to present the application of physical concepts and experimental results to demonstrate for graduate and undergraduate students. Based on this platform we want to simulate dynamic systems like automotive suspension, machine components and others.

Table 3. Results for system with two degree-of-freedom undamped.

Resultsof Natural frequency	Firstvibration mode	Secondvibrationmode
Experimental Data	2.899Hz	7.286Hz
Analyticalsolution	2.798Hz	7.199Hz
Numericalsolution	2.789Hz	7.222Hz

VIII. CONCLUSION

The system with two-degree-of-freedom undamped evaluated in this paper is a low- cost method for making and provides an overview of aspects associated with the natural frequency analysis (modal analysis). In the experiment was possible to examine various influences of factors inherent to the process for assessing vibration behavior.

The results obtained with the analytical solution and numerical simulation are very satisfactory, because the natural frequency values obtained was with minimum variation in respect to the experiment, thus validating the proposed methodologies.

This study allowed a development platform to be used for teaching an experimental system to confirm the results in vibration and modal analysis, besides the application of a methodology for calculating direct and simple for systems with multi-degrees-of-freedom. Also, could be useful for analyzing any component that university received with this proposal, evaluate natural frequency.

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