RESEARCH ARTICLE OPEN ACCESS

Single Server Bulk Queue with Second Optional Service, Balking and Compulsory Vacation

S. Uma*, K. Punniyamoorthy**
(Department of Mathematics, D. G. Govt. Arts College for Women, Mayiladuthurai, Nagappattinam (Dt.), Tamil Nadu, India.)
***(Department of Mathematics, AnjalaiAmmal-Mahalingam Engineering College, Kovilvenni, Tiruvarur (Dt.), Tamil Nadu, India.)*

ABSTRACT
This paper is about the customers who arrive in bulk or group in a single server queueing system. The process takes place in Poisson distribution. The customers are provided with general services of two types in bulk of size M. The service is done on first come first serve basis in general distribution. Here the first service is essential and the second service is optional. Once the bulk gets serviced, the customer goes out of the system. The server compulsory goes to vacation when after completing the service. The vacation times are exponentially distributed. The server again goes to vacation or remain in the system if arrived bulk of customers is not enough to get the essential service. The server returns when arrived bulk is sufficient. The arriving batch balks during the period when the server is busy or when the server is on vacation or other constraints. This may result in the impatient behavior of the customers. We obtain the time dependent probability generating functions and from it the corresponding steady state results are derived. Also the average queue size and the system size and certain cases are explained.
Keywords: bulk arrival, bulk service, vacation, essential service, optional service, balking

I. INTRODUCTION

Among the enormous researchers, Chaudhry and Templeton [1], Cooper [2], and Medhi [3] have worked on bulk queues. Vacation queue has been surveyed extensively by Doshi [4]. Chang and Takine [5] have studied Stochastic Decompositions in the bulk queue with Generalized Vacations. Madan [6], Punniyamoorthy and Uma [8] have explained the queueing system with compulsory vacation. Thangaraj [7] has worked the M/G/1 Queue with Two Stage Heterogeneous Service Compulsory Server Vacation. Punniyamoorthy and Uma [8], Medhi [9], MonitaBaruah, Madan and TillalEldabi [10] have discussed the queueing system with second optional service. Artalejo and Herrero [11], Al-Seedy, EI-Sherbiny, Shehawy and Ammar [12], Wu, Brill, Hlynka and Wang [13] have worked about balking.

In this paper, we explain the single server queueing system with bulk arrival and bulk service with second optional service, balking and compulsory server vacation. The arrival is under Poisson distribution and the services are in general distribution. The vacation times are exponentially distributed. The bulk of customers are served under first come first served basis. The first service is essential and the second service is optional. Customers may arrive at the system but may leave without joining the system (balking). We assume that the system is subject to balking with different balking probabilities during the period.

These phenomena are observed in situations involving many real-time applications like web access, data communication networks, computer systems, waiting lines at clinic, automated teller machines, banking and so forth.

This paper is organized as follows: The mathematical model is explained in section 2. Definitions and Notations are briefed in section 3. Equations governing the system are given in section 4. The time dependent solutions have been derived in section 5 and corresponding steady state results have been calculated clearly in section 6. The average queue size and the system size are computed in section 7. Some particular cases are discussed in section 8.

II. THE MATHEMATICAL MODEL

We assume the following to describe the queueing model of our study:

(a) Customers (units) arrive at the system in batches of variable size in a compound Poisson process.
(b) Let $\lambda\pi_i dt\ (i = 1,2,3,\dots)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + dt]$, where $0 \le \pi_i \le 1$, $\sum_{i=1}^{\infty}\pi_i = 1$, $\lambda > 0$ is the mean arrival rate of batches.
(c) The service to customers in based on a first come first served basis (FCFS); they receive the

first essential service and may choose the second optional service if needed. The first essential service is required by all customers. As soon as first essential service is completed by a customer then he may choose second optional service with probability θ or leave the system with probability $1-\theta$.

(d) The service of customers (units) is rendered in batches of fixed size $M(\geq 1)$or $\min(n, M)$, where n is the number of customers in the queue.

(e) We assume that the random variable of service time $S_j (j = 1,2)$ of the j^{th} kind of service follows a general probability law with distribution function $G_j(s_j)$, $g_j(s_j)$ is the probability density function and $E(S_j^k)$ is the k^{th} moment $(k = 1,2,\ldots)$ of service time $j = 1,2$ (first essential service and the second optional service).

(f) Let $\mu_j(x)$ be the conditional probability of type j service completion during the period $(x, x+dx]$, given that elapsed service time is x, so that

$$\mu_j(x) = \frac{g_j(x)}{1-G_j(x)}, j = 1,2 \quad (1)$$

and therefore

$$g_j(s_j) = \mu_j(s_j)e^{-\int_0^{s_j}\mu_j(x)dx}, j = 1,2 \quad (2)$$

(g) After completion of continuous service to the batches of fixed size $M(\geq 1)$, the server will go for compulsory vacation.

(h) Server vacation starts after the completion of service to a batch. The duration of the vacation period is assumed to be exponential with mean vacation time $\frac{1}{\phi}$.

(i) On returning from vacation the server instantly starts the first essential service if there is a batch of size M or he waits idle in the system.

(j) We assume that $(1-a_1)(0 \leq a_1 \leq 1)$ is the probability that an arriving batch balks during the period when the server is busy (available on the system) and $(1-a_2)(0 \leq a_2 \leq 1)$ is the probability that an arriving batch balks during the period when the server is on vacation.

(k) Finally, it is assumed that the inter-arrival times of the customers, the service times of each kind of service and vacation times of the server, all these stochastic processes involved in the system are independent of each other.

III. DEFINITIONS AND NOTATIONS

We define:

$P_{n,j}(x,t)$: Probability that at time t, the server is active providing and there are n $(n \geq 0)$ customers in the queue, excluding a batch of M customers in type j service, $j = 1,2$ and the elapsed service time of this customer is x. Accordingly, $P_{n,j}(t) = \int_0^\infty P_{n,j}(x,t)dx$ denotes the probability that there are n customers in the queue excluding a batch of M customers in type j service, $j = 1,2$ irrespective of the elapsed service time x.

$V_n(t)$: Probability that at time t, there are n $(n \geq 0)$ customers in the queue and the server is on vacation.

$Q(t)$: Probability thatat time t, there are less than M customers in the system and the server is idle but available in the system.

IV. EQUATIONS GOVERNING THE SYSTEM

According to the Mathematical model mentioned above, the system has the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_{n,1}(x,t) + \frac{\partial}{\partial t}P_{n,1}(x,t) + (\lambda + \mu_1(x))P_{n,1}(x,t) = a_1\lambda\sum_{k=1}^{n}\pi_k P_{n-k,1}(x,t) + \lambda(1-a_1)P_{n,1}(x,t) \quad (3)$$

$$\frac{\partial}{\partial x}P_{0,1}(x,t) + \frac{\partial}{\partial t}P_{0,1}(x,t) + (\lambda + \mu_1(x))P_{0,1}(x,t) = \lambda(1-a_1)P_{0,1}(x,t) \quad (4)$$

$$\frac{\partial}{\partial x}P_{n,2}(x,t) + \frac{\partial}{\partial t}P_{n,2}(x,t) + (\lambda + \mu_2(x))P_{n,2}(x,t) = a_1\lambda\sum_{k=1}^{n}\pi_k P_{n-k,2}(x,t) + \lambda(1-a_1)P_{n,2}(x,t) \quad (5)$$

$$\frac{\partial}{\partial x}P_{0,2}(x,t) + \frac{\partial}{\partial t}P_{0,2}(x,t) + (\lambda + \mu_2(x))P_{0,2}(x,t) = \lambda(1-a_1)P_{0,2}(x,t) \quad (6)$$

$$\frac{d}{dt}V_n(t) + (\lambda+\phi)V_n(t) = \lambda a_2\sum_{k=1}^{n}\pi_k V_{n-k}(t) + \lambda(1-a_2)V_n(t) + (1-\theta)\int_0^\infty P_{n,1}(x,t)\,\mu_1(x)dx + \int_0^\infty P_{n,2}(x,t)\mu_2(x)dx \quad (7)$$

$$\frac{d}{dt}V_0(t) + (\lambda+\phi)V_0(t) = \lambda(1-a_2)V_0(t) + (1-\theta)\int_0^\infty P_{0,1}(x,t)\,\mu_1(x)dx + \int_0^\infty P_{0,2}(x,t)\mu_2(x)dx \quad (8)$$

$$\frac{d}{dt}Q(t) + \lambda Q(t) = \phi V_0(t) + \lambda(1-a_1)Q(t) \quad (9)$$

Equations (3) – (9) are to be solved subject to the following boundary conditions:

$$P_{n,1}(0,t) = \phi V_{n+M}(t) \quad (10)$$

$$P_{0,1}(0,t) = \phi\sum_{b=1}^{M}V_b(t) + \lambda a_1 Q(t) \quad (11)$$

$$P_{n,2}(0,t) = \theta\int_0^\infty P_{n,1}(x,t)\mu_1(x)dx \quad (12)$$

We assume that initially the server is available but idle because of less than M customers so that the initial conditions are

$$V_n(0) = 0;\ V_0(0) = 0;\ Q(0) = 1$$

$$P_{n,j}(0) = 0, \text{for } n = 0,1,2,\ldots \text{and } j = 1,2. \quad (13)$$

V. PROBABILITY GENERATING FUNCTION OF THE QUEUE SIZE: THE TIME DEPENDENT SOLUTION

We define the following probability generating functions:

$$\left.\begin{aligned} P_j(x,z,t) &= \sum_{n=0}^{\infty} P_{n,j}(x,t)z^n, \ j=1,2 \\ P_j(z,t) &= \sum_{n=0}^{\infty} P_{n,j}(t)z^n, \ j=1,2 \\ V(z,t) &= \sum_{n=0}^{\infty} V_n(t)z^n \\ \pi(z) &= \sum_{n=1}^{\infty} \pi_n z^n \end{aligned}\right\} \quad (14)$$

Define the Laplace-Stieltjes Transform of a function $f(t)$ as follows:

$$\bar{f}(s) = L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt \quad (15)$$

Taking Laplace Transform of equations (3) – (9) and using (14), we get,

$$\frac{\partial}{\partial x}\bar{P}_{n,1}(x,s) + \big(s+\lambda+\mu_1(x)\big)\bar{P}_{n,1}(x,s) = \lambda a_1 \sum_{k=1}^{n} \pi_k \bar{P}_{n-k,1}(x,s) + \lambda(1-a_1)\bar{P}_{n,1}(x,s) \quad (16)$$

$$\frac{\partial}{\partial x}\bar{P}_{0,1}(x,s) + \big(s+\lambda+\mu_1(x)\big)\bar{P}_{0,1}(x,s) = \lambda(1-a_1)\bar{P}_{0,1}(x,s) \quad (17)$$

$$\frac{\partial}{\partial x}\bar{P}_{n,2}(x,s) + \big(s+\lambda+\mu_2(x)\big)\bar{P}_{n,2}(x,s) = \lambda a_1 \sum_{k=1}^{n} \pi_k \bar{P}_{n-k,2}(x,s) + \lambda(1-a_1)\bar{P}_{n,2}(x,s) \quad (18)$$

$$\frac{\partial}{\partial x}\bar{P}_{0,2}(x,s) + \big(s+\lambda+\mu_2(x)\big)\bar{P}_{0,2}(x,s) = \lambda(1-a_1)\bar{P}_{0,2}(x,s) \quad (19)$$

$$(s+\lambda+\phi)\bar{V}_n(s) = \lambda a_2 \sum_{k=1}^{n} \pi_k \bar{V}_{n-k}(s) + \lambda(1-a_2)\bar{V}_n(s) + (1-\theta)\int_0^{\infty} \bar{P}_{n,1}(x,s)\,\mu_1(x)dx + \int_0^{\infty} \bar{P}_{n,2}(x,s)\mu_2(x)dx \quad (20)$$

$$(s+\lambda+\phi)\bar{V}_0(s) = \lambda(1-a_2)\bar{V}_0(s) + (1-\theta)\int_0^{\infty} \bar{P}_{0,1}(x,s)\,\mu_1(x)dx + \int_0^{\infty} \bar{P}_{0,2}(x,s)\mu_2(x)dx \quad (21)$$

$$(s+\lambda)\bar{Q}(s) = 1 + \phi\bar{V}_0(s) + \lambda(1-a_1)\,\bar{Q}(s) \quad (22)$$

$$\bar{P}_{n,1}(0,s) = \phi\bar{V}_{n+M}(s) \quad (23)$$

$$\bar{P}_{0,1}(0,s) = \phi\sum_{b=1}^{M} \bar{V}_b(s) + \lambda a_1 \bar{Q}(s) \quad (24)$$

$$\bar{P}_{n,2}(0,s) = \theta\int_0^{\infty} \bar{P}_{n,1}(x,s)\,\mu_1(x)dx \quad (25)$$

Multiplying the equation (16) by z^n and summing over n from 1to ∞, adding equation (17) and using the generating functions defined in (14), we obtain,

$$\frac{\partial}{\partial x}\bar{P}_1(x,z,s) + \{s+\lambda a_1\big(1-\pi(z)\big)+\mu_1(x)\}\bar{P}_1(x,z,s) = 0 \quad (26)$$

Performing similar operations on equations (18) – (21), we obtain,

$$\frac{\partial}{\partial x}\bar{P}_2(x,z,s) + \{R+\mu_2(x)\}\bar{P}_2(x,z,s) = 0 \quad (27)$$

$$\{\phi+R_1\}\bar{V}(z,s) = (1-\theta)\int_0^{\infty} \bar{P}_1(x,z,s)\,\mu_1(x)dx + \int_0^{\infty} \bar{P}_2(x,z,s)\mu_2(x)dx \quad (28)$$

Where $R = s+\lambda a_1\big(1-\pi(z)\big)$
and $R_1 = s+\lambda a_2\big(1-\pi(z)\big)$

Multiplying the equation (23) by z^{n+M} and summing over n from 1 to ∞ and adding, multiplying the equation (24) by z^M, and using the generating functions defined in (14) and using (22), we obtain,

$$\bar{P}_1(0,z,s) = [\lambda a_1 - (s+\lambda a_1)z^{-M}]\bar{Q}(s) + z^{-M} + \phi z^{-M}\bar{V}(z,s) + \phi\sum_{b=1}^{M-1}(1-z^{-M+b})\bar{V}_b(s) \quad (29)$$

Multiplying the equation (25) by z^n and summing over n from 0 to ∞ and using the generating functions defined in (14), we obtain,

$$\bar{P}_2(0,z,s) = \theta\int_0^{\infty} \bar{P}_1(x,z,s)\mu_1(x)dx \quad (30)$$

We now integrate equations (26) and (27) between the limits 0 and x and obtain,

$$\bar{P}_1(x,z,s) = \bar{P}_1(0,z,s)e^{-Rx-\int_0^x \mu_1(x)dx} \quad (31)$$

$$\bar{P}_2(x,z,s) = \bar{P}_2(0,z,s)e^{-Rx-\int_0^x \mu_2(x)dx} \quad (32)$$

Integrating equations (31) and (32) by parts, with respect to x, we get,

$$\bar{P}_1(z,s) = \bar{P}_1(0,z,s)\left[\frac{1-\bar{G}_1(R)}{R}\right] \quad (33)$$

$$\bar{P}_2(z,s) = \bar{P}_2(0,z,s)\left[\frac{1-\bar{G}_2(R)}{R}\right] \quad (34)$$

Where $\bar{G}_j(R) = \int_0^{\infty} e^{-Rx}dG_j(x)$, is the Laplace Transform of j^{th} type of service, $j = 1,2$.

Multiplying the equations (31) and (32) by $\mu_1(x)$ and $\mu_2(x)$, integrating by parts, with respect to x, we get,

$$\int_0^{\infty} \bar{P}_1(x,z,s)\mu_1(x)dx = \bar{P}_1(0,z,s)\bar{G}_1(R) \quad (35)$$

$$\int_0^{\infty} \bar{P}_2(x,z,s)\mu_2(x)dx = \bar{P}_2(0,z,s)\bar{G}_2(R) \quad (36)$$

Substituting (35) in (30) we get,

$$\bar{P}_2(0,z,s) = \theta\begin{bmatrix} \phi z^{-M}\bar{V}(z,s) + z^{-M} + \\ \phi\sum_{b=1}^{M-1}(1-z^{-M+b})\bar{V}_b(s) \\ +(\lambda a_1 - (s+\lambda a_1)z^{-M})\bar{Q}(s) \end{bmatrix}\bar{G}_1(R) \quad (37)$$

From (28), using (29), (35), (36) and (37), we obtain,

$$\bar{V}(z,s) = \left[\frac{\begin{Bmatrix} (1-\theta)\bar{G}_1(R) \\ +\theta\bar{G}_1(R)\bar{G}_2(R) \end{Bmatrix}\begin{Bmatrix} \phi\sum_{b=1}^{M-1}(z^M - z^b)\bar{V}_b(s)+1 \\ +\big(\lambda a_1 z^M - (s+\lambda a_1)\big)\bar{Q}(s) \end{Bmatrix}}{z^M\{R_1+\phi\} - \phi\{(1-\theta)\bar{G}_1(R) + \theta\bar{G}_1(R)\bar{G}_2(R)\}}\right] \quad (38)$$

Substituting (29) and (37) in (33) and (34) respectively, we get,

$$\bar{P}_1(z,s) = \begin{bmatrix} \phi z^{-M}\bar{V}(z,s) + z^{-M} \\ +\phi\sum_{b=1}^{M-1}(1-z^{-M+b})\bar{V}_b(s) \\ +(\lambda a_1 - (s+\lambda a_1)z^{-M})\bar{Q}(s) \end{bmatrix}\left[\frac{1-\bar{G}_1(R)}{R}\right] \quad (39)$$

$$\bar{P}_2(z,s) = \theta\begin{bmatrix} \phi z^{-M}\bar{V}(z,s) + z^{-M} \\ +\phi\sum_{b=1}^{M-1}(1-z^{-M+b})\bar{V}_b(s) \\ +(\lambda a_1 - (s+\lambda a_1)z^{-M})\bar{Q}(s) \end{bmatrix}\bar{G}_1(R)\left[\frac{1-\bar{G}_2(R)}{R}\right] \quad (40)$$

We note that there are M unknowns, $\bar{Q}(s)$and $\bar{V}_b(s), b = 1,2, \dots, M-1$ appearing in equation (38).

Now (38) gives the probability generating function of the service system with M unknowns. By Rouche's theorem of complex variables, it can be proved that $z^M\{R_1 + \phi\} - \phi\{(1-\theta)\bar{G}_1(R) +$ *θG1RG2R* has *M* zeroes inside the contour *z=1.* Since $\bar{V}(z,s)$ are analytic inside the unit circle $|z| = 1$, the numerator in the right hand side of equation (38) must vanish at these points, which gives rise to a set of M linear equations which are sufficient to determine Munknowns. Thus $\bar{V}(z,s)$, $\bar{P}_1(z,s)$ and $\bar{P}_2(z,s)$ can be completely determined.

VI. THE STEADY STATE RESULTS

To define the steady state probabilities and corresponding generating functions, we drop the argument t, and for that matter the argument s wherever it appears in the time-dependent analysis up to this point. Then the corresponding steady state results can be obtained by using the well-known Tauberian Property

$$\lim_{s\to 0} s\bar{f}(s) = \lim_{t\to\infty} f(t) \tag{41}$$

if the limit on the right exists.

Now (38), (39) and (40) we have,

$$V(z) = \frac{\begin{Bmatrix} (1-\theta)\bar{G}_1(f(z)) \\ +\theta\bar{G}_1(f(z))\bar{G}_2(f(z)) \end{Bmatrix}\{\phi U + \lambda a_1(z^M - 1)Q\}}{z^M\{f_1(z)+\phi\} - \phi\begin{Bmatrix} (1-\theta)\bar{G}_1(f(z)) \\ +\theta\bar{G}_1(f(z))\bar{G}_2(f(z)) \end{Bmatrix}} \tag{42}$$

$$P_1(z) = \begin{Bmatrix} \phi z^{-M}V(z) + \phi z^{-M}U \\ +\lambda a_1(1 - z^{-M})Q \end{Bmatrix}\left\{\frac{1-\bar{G}_1(f(z))}{f(z)}\right\} \tag{43}$$

$$P_2(z) = \theta\begin{Bmatrix} \phi z^{-M}V(z) + \phi z^{-M}U \\ +\lambda a_1(1 - z^{-M})Q \end{Bmatrix}\bar{G}_1(f(z))\left\{\frac{1-\bar{G}_2(f(z))}{f(z)}\right\} \tag{44}$$

Where $f(z) = \lambda a_1(1 - \pi(z))$;

$f_1(z) = \lambda a_2(1 - \pi(z))andU = \sum_{b=1}^{M-1}(z^M - z^b)V_b$

The M unknowns, Q and V_b,

$b = 1,2, \dots, M-1$can be determined as before.

Let $A_q(z)$ denote the probability generating function of the queue size irrespective of the state of the system.

i.e., $A_q(z) = P_1(z) + P_2(z) + V(z)$ (45)

In order to find Q, we use the normalization condition

$$A_q(1) + Q = 1 \tag{46}$$

Note that for $z = 1$, $A_q(1)$ is indeterminate of $\frac{0}{0}$ form.

Therefore, we apply L'Hôpital's Rule on (45), we get,

$$A_q(1) = \frac{\phi E_2^*[\phi U^* + \lambda a_1 MQ]}{M\phi - \lambda a_2 E(I) - \phi\lambda a_1 E(I)E_1^*} \tag{47}$$

Where $U^* = \sum_{b=1}^{M-1}(M-b)V_b$;

$E_1^* = E(S_1) + \theta E(S_2)$&$E_2^* = E(S_1) + \theta E(S_2) + \frac{1}{\phi}$

We used $\bar{G}_j(0) = 1,\ j = 1,2,\ \ \pi'(1) = E(I)$, where I denotes the number of customers in an arriving batch and therefore, $E(I)$ is the mean of the batch size of the arriving customers. Similarly $E(S_1)$and $E(S_2)$ are the mean service times of essential and optional services, respectively.

Therefore, adding Q to equation (47) and equating to 1 and simplifying we get,

$$Q = 1 - \frac{\phi E_2^*(M\lambda a_1 + \phi U^*)}{M\phi + \lambda a_1 M\phi E_2^* - \lambda E(I)[a_2 + a_1\phi E_1^*]} \tag{48}$$

Equation (48) gives the probability that the server is idle.

From equation (48) the utilization factor, ρ of the system is given by

$$\rho = \frac{\phi E_2^*(M\lambda a_1 + \phi U^*)}{M\phi + \lambda a_1 M\phi E_2^* - \lambda E(I)[a_2 + a_1\phi E_1^*]} \tag{49}$$

Where $\rho < 1$ is the stability condition under which the steady state exists.

VII. THE AVERAGE QUEUE SIZE AND THE SYSTEM SIZE

Let $L_q$denote the mean number of customers in the queue under the steady state.

Then $L_q = \frac{d}{dz}A_q(z)\big|_{z=1}$ (50)

Since the formula gives indeterminate form, then we write $A_q(z)$ as $A_q(z) = \frac{N(z)}{D(z)} + C(z)$

Where $N(z)$ and $D(z)$ are the numerator and denominator of the first term and $C(z)$is the second term of the right hand side of $A_q(z)$ respectively.

Then using L'Hôpital's Rule twice we obtain,

$$L_q = \frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} + \frac{d}{dz}C(z)\big|_{z=1} \tag{51}$$

We have,

$$N'(1) = \phi E_2^*(M\lambda a_1 Q + \phi U^*) \tag{52}$$

$$D'(1) = M\phi - \lambda a_2 E(I) - \phi\lambda a_1 E(I)E_1^* \tag{53}$$

$$N''(1) = \phi(M\lambda a_1 Q + \phi U^*)\begin{bmatrix} \lambda a_1 E(I)E_3^* \\ -2ME_1^* \end{bmatrix} + \phi E_2^*\left[2\lambda a_1 E(I)E_1^*\begin{pmatrix} M\lambda a_1 Q \\ +\phi U^* \end{pmatrix} + U^{**}\right] \tag{54}$$

$$D''(1) = \begin{bmatrix} M(M-1)\phi - \lambda a_2 E(I(I-1)) \\ -2M\lambda a_2 E(I) \end{bmatrix} - \phi\begin{bmatrix} \lambda^2 a_1^2 (E(I))^2 E_3^* \\ +\lambda a_1 E(I(I-1))E_1^* \end{bmatrix} \tag{55}$$

$$\frac{d}{dz}C(z)\big|_{z=1} = E_1^*(M\lambda a_1 Q + \phi U^*) \tag{56}$$

Where $E_3^* = E(S_1^2) + \theta E(S_2^2) + 2\theta E(S_1)E(S_2)$&

$$U^{**} = \phi\sum_{b=1}^{M-1}\begin{bmatrix} M(M-1) \\ -b(b-1) \end{bmatrix}V_b + \lambda a_1 M(M-1)\,Q$$

Where $E(I(I-1))$ is the second factorial moment of the batch size of the arriving customers. Similarly, $E(S_1^2)$and $E(S_2^2)$ are the second moments of the service times of essential and optional services, respectively Q has been obtained in (48).

Then if we substitute the equations (52) – (56) in the equation (51), we obtain L_q in a closed form.

Further, the average number of customers in the system can be found as $L_s = L_q + \rho$ by using Little's formula.

VIII. SPECIAL CASES

Case 1. No balking

In this case, we let $a_1 = a_2 = 1$ in the equations (48), (49) and L_q

$$Q = 1 - \frac{\phi E_2^*(M\lambda + \phi U^*)}{M\phi + \lambda\phi\,[M - E(I)]E_2^*} \quad (57)$$

$$\rho = \frac{\phi E_2^*(M\lambda + \phi U^*)}{M\phi + \lambda\phi\,[M - E(I)]E_2^*} \quad (58)$$

$$N'(1) = \phi E_2^*(M\lambda a_1 Q + \phi U^*) \quad (59)$$

$$D'(1) = M\phi - \lambda a_2 E(I) - \phi\lambda a_1 E(I)E_1^* \quad (60)$$

$$N''(1) = \phi(M\lambda a_1 Q + \phi U^*)\begin{bmatrix}\lambda a_1 E(I)E_3^* \\ -2ME_1^*\end{bmatrix} + \phi E_2^*\left[2\lambda a_1 E(I)E_1^*\begin{pmatrix}M\lambda a_1 Q \\ +\phi U^*\end{pmatrix} + U^{**}\right] \quad (61)$$

$$D''(1) = \begin{bmatrix}M(M-1)\phi - \lambda a_2 E\big(I(I-1)\big) \\ -2M\lambda a_2 E(I)\end{bmatrix} - \phi\begin{bmatrix}\lambda^2 a_1^2 \big(E(I)\big)^2 E_3^* \\ +\lambda a_1 E\big(I(I-1)\big)E_1^*\end{bmatrix} \quad (62)$$

$$\frac{d}{dz}C(z)|_{z=1} = E_1^*(M\lambda a_1 Q + \phi U^*) \quad (63)$$

Then, if we substitute the values $N'(1)$, $D'(1)$, $N''(1)$, $D''(1)$, $\frac{d}{dz}C(z)|_{z=1}$ from equations (59) to (63) into equation (51), we obtain L_q in the closed form.

Case 2. Single arrival, only essential service (one by one general service), no balking and no vacation

In this case, we let $E(I) = 1; E\big(I(I-1)\big) = 0;$ $\theta = 0\,; M = 1;\ a_1 = a_2 = 1$ and $\frac{1}{\phi} \longrightarrow 0$ in the equations (48), (49) and L_q

$$Q = 1 - \lambda E(S_1) \quad (64)$$

$$\rho = \lambda E(S_1) \quad (65)$$

$$L_q = \frac{\lambda^2 E(S_1^2)}{2\{1 - \lambda E(S_1)\}} \quad (66)$$

Note that (64) and (66) are known results from [3]. Again (66) is the well-known Pollaczek-Khinchine formula.

IX. CONCLUSION

In this paper the single server queueing system with bulk arrival and bulk service with second optional service, balking and compulsory server vacation is analyzed. The transient solution, steady state results and the various performance measures of the system are discussed.

REFERENCES

[1]. M. L. Chaudhry and J. G. C. Templeton, *A first course in bulk queues* (John Wiley and sons, New York, 1983).

[2]. R. B. Cooper, *Introduction to queueing theory* (2nd Edition, Elsevier, North Holland, New York, 1981).

[3]. J. Medhi, *Stochastic Processes* (2nd edition, New Age International (P) Limited Publishers, India, 1994).

[4]. B. T. Doshi, Queueing systems with vacations – a survey, *Queueing Systems, 1,* 1986, 29-66.

[5]. S. H. Chang and T. Takine, Factorization and Stochastic decomposition properties in bulk queues with generalized vacation, *Queueing Systems, 50,* 2005, 165-183.

[6]. K. C. Madan, An M/G/1 Queueing system with compulsory server vacations, *Trabajos De InvestigacionOperativa, 7,* 1992, 105-115.

[7]. V. Thangaraj and S. Vanitha, M/G/1 queue with two stage heterogeneous service compulsory server vacation and random breakdowns, *Int. J. Contemp. Math. Sciences, 5,* 2010, 307-322.

[8]. K. Punniyamoorthy and S. Uma, Single server bulk queue with second optional service and compulsory vacation, *International Journal of Applied Engineering Research, 11 (1),* 2016, 379-383.

[9]. J. Medhi, A single server poisson input queue with a second optional channel, *Queueing Systems, 42,* 2002, 239-242.

[10]. MonitaBaruah, Kailash C. Madan and TillalEldabi, A batch arrival queue with second optional service and reneging during vacation periods, *Revista Investigation operacional, 34,* 2013, 244-258.

[11]. J. R. Artalejo and M.J. Herrero, On the single server retrial queue with balking, *INFOR, 38(1),* 2000, 33-50.

[12]. R. O. Al-Seedy, A. A. EI-Sherbiny, S. A.-Shehawy and S. I. Ammar, Transient solution of the M/M/C queue with balking and reneging, *Computers and Mathematics with Applications,57,* 2009, 1280-1285.

[13]. X. Wu, P. Brill, M. Hlynka and J. Wang, AnM/G/1 retrial queue with balking and retrials during service, *International Journal of Operations Research,1,* 2005, 30-51.

[14]. R. Kalyanaraman and S. PazhanibalaMurugan, A single server queue with additional optional service batches and server vacation, *Applied*

Mathematical Sciences, 2(56), 2008, 2765-2776.

[15]. Kailash C. Madan, On a $M^{[X]}/(G_1, G_2)/1$ queue with third-stage optional service and deterministic server vacations, *J. Math. Comput. Sci.,5(2),* 2015,195-206.

[16]. J. Wang, An$M/G/1$ queue with second optional service and server breakdowns, *Computers and Mathematical Applications*, *47 (10)*, 2004, 1713-1723.

[17]. G. Choudhury, Some aspects of an$M/G/1$ queueing system with optional second service, *TOP*, *11 (1)*, 141-150.

[18]. M. Jain, G. C. Sharma and R. Sharma, Unreliable server$M/G/1$ queue with multi-optional vacation, *International Journal of Mathematical Operation Research*, *5(2),* 2013*,* 145-169.