Visualization of Natural Convection in a Vertical Annular Cylinder with a Partially Heat Source and Varying wall Temperature

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Abstract:
In this work, we visualize the effect of varying wall temperature on the heat transfer by supplying the heat at three different positions to the vertical annular cylinder embedded with porous medium. Finite element method has been used to solve the governing equations. Influence of Aspect ratio (Ar), Radius ratio (Rr) on Nusselt number (Nu) is presented. The effect of power law exponent effect for different values of Rayleigh number is discussed. The fluid flow and heat transfer is presented in terms of streamlines and isotherms.

Keywords- Natural Convection, Porous medium, varying wall temperature, Aspect Ratio (Ar), Radius ratio (Rr) and Rayleigh number (Ra).

I. Introduction
Convective heat transfer in a porous medium has gained considerable attention of many researchers in recent years. This is justified by the fact that porous media play a vital role in many thermal engineering applications and geophysical applications such as moisture transport in thermal insulations, ceramic processing, the extraction of geothermal energy, nuclear reactor cooling system, underground nuclear waste disposal, energy recovery of petroleum resources, ground water pollution and filtration processes.

In the natural convection aspect the steady natural convection over a semi-infinite vertical wall embedded in a saturated porous medium with wall temperature varying as power function of distance from the origin was discussed by Cheng and Minkowycz [1].

Free convection flow past a vertical flat plate embedded in a saturated porous medium for the case of non-uniform surface temperature was numerically analyzed by Na and Pop [2]. Gorla and Zinalabedini [3] applied the Merk-type series to obtain the local non-similarity solution for the heat transfer from a vertical plate with non-uniform surface temperature and embedded in a saturated porous medium. Seetharamu and Dutta [4] presented free convection in a saturated porous medium adjacent to a non-isothermal vertical impermeable wall. Similarity solution for inclined for mixed boundary region flows in porous media was obtained by Cheng [5]. Both the surface temperatures and free stream velocity must vary according to the same power function of distance from the origin. Later, non-similar solution for mixed convection adjacent to inclined flat surfaces embedded in a porous medium with constant aiding external flow and uniform surface temperature was numerically done by Jang and Ni [6]. Hsieh et al. [7] numerically investigated the non-similarity solutions for mixed convection for vertical surfaces in porous media: variable surface temperature or heat flux. Mixed convection along a non-isothermal vertical plate embedded in the porous medium was studied by Mixed convection along a non-isothermal wedge in a porous medium was numerically considered by Vargas et al. [8], Kumari and Gorla [9].

Saied [10] has considered the case of sinusoidal bottom wall temperature with top cooled and adiabatic vertical walls of a porous medium enclosed in a cavity. The heated wall was assumed to have temperature varying in a sinusoidal wave around a mean temperature greater than that of the top cooled wall. The main finding of this work is the average Nusselt number is increased when the length of the heat source or the amplitude of the temperature variations increased. In another work, Saied [11] investigated similar problem of square porous cavity but boundary conditions were swapped such that the temperature differential was maintained along vertical surfaces. A sinusoidal temperature variation was applied on left vertical surface along with cooled right surface. The bottom and top surfaces are adiabatic.

The effect of non-uniform temperature distribution on an inclined three-dimensional enclosure has been
studied by Chao et al [12]. Bottom wall is maintained at a saw-toothed temperature with distribution with different amplitude and orientation, while top wall is isothermal and other faces are adiabatic. The circulation pattern did not change significantly with inclination. Cho et al [13] in other study considered half of the Bottom surface cooled, while half of the bottom surface and other vertical surfaces were adiabatic. The results of circulation found to be in good agreement. Fu et al [14] has investigated the natural convection in an enclosure, where the heated wall of the enclosure is divided into two higher and lower temperature and the temperature of the cold wall is maintained at a constant temperature. The results show that the local Nusselt number distribution varies drastically at the intersection of the higher and lower surface temperature regions, and the flow is strongly affected by the above two parameters.

Convection motion in a square cavity with linearly varying temperature imposed along the top surface has been investigated numerically by Shukla, et.al. [15]. The side and bottom walls of the rigid cavity are assumed to be insulated. For low Rayleigh number a single convective cell is formed. With the increase in Rayleigh number, flow and temperature fields became asymmetric. The temperature field is generally stratified with lower art of cavity relatively isothermal. Oosthuizen and Paul [16] considered an enclosure with sidewall partially heated and top wall cooled. In other study, Oosthuizen [17] considered an enclosure with bottom surface heated and the top surface is inclined and maintained at uniform cold temperature. The behavior of fluid flow and heat transfer is presented in terms of streamlines and isotherms.

II. Nomenclature:

- $A_r$: Aspect ratio
- $C_p$: Specific heat
- $D_p$: Particle diameter
- $g$: Gravitational acceleration
- $H$: Height of the vertical annular cylinder
- $K$: Permeability of porous media
- $L$: Length
- $P$: Pressure
- $\bar{Nu}$: Average Nusselt number
- $q_t$: Total heat flux
- $r,z$: Cylindrical co-ordinates
- $\bar{r}, \bar{z}$: Non-dimensional co-ordinates
- $r_i, r_o$: Inner and outer radius
- $Ra$: Rayleigh number
III. Mathematical Analysis:

A vertical annular cylinder of inner radius \( r_i \) and outer radius \( r_o \) is considered to investigate the heat transfer behavior. The co-ordinate system is chosen such that the r-axis points towards the width and z-axis towards the height of the cylinder respectively. Because of the annular nature, two important parameters emerge which are Aspect ratio \( A_r \) and Radius ratio \( R_r \) of the annulus. They are defined as \( A_r = \frac{H_t}{r_o-r_i} \), \( R_r = \frac{r_o-r_i}{r_i} \), where \( H_t \) is the height of the annular cylinder. The inner surface of the cylinder is said to be power law function and it varies in the vertical direction along the height of the inner wall of the vertical annular cylinder \( T_h = T_\infty + B z^\lambda \) and the outer surface is at ambient temperature \( T_\infty \). Here \( \lambda \) and B are the constants responsible for temperature variations along the length of the vertical annular cylinder. The top and bottom surfaces of the vertical annular cylinder are adiabatic. It may be noted that, due to axisymmetry only half of the annulus is sufficient for analysis purpose, since other half is mirror image of the first half. The flow inside the porous medium is assumed to obey Darcy law and there is no phase change of fluid. The properties of the fluid and porous medium are homogeneous, isotropic and constant except variation of fluid density with temperature. The fluid and porous medium are in thermal equilibrium. With these assumptions, the governing equations are given by

Continuity Equation: \[
\frac{\partial (ru)}{\partial r} + \frac{\partial (rw)}{\partial z} = 0
\] (1)

The velocity in r and z directions can be described by Darcy law as:

Velocity in horizontal direction \( u = -\frac{k}{\mu} \frac{\partial p}{\partial r} \)

Velocity in vertical direction \( v = -\frac{k}{\mu} \left( \frac{\partial p}{\partial z} + \rho g \right) \)

The permeability \( K \) of porous medium can be expressed as Bejan \[28\], \( k = \frac{\mu^2 \phi^3}{180(1-\phi)^2} \)

Momentum Equation: \[
\frac{\partial w}{\partial r} - \frac{\partial u}{\partial z} = g k \beta \frac{\partial T}{\nu \partial r}
\] (2)
Energy Equation: \[ \frac{\alpha}{u} \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} = \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right) \]

The continuity equation (1) can be satisfied by introducing the stream function \( \psi \) as:

\[ u = \frac{1}{r} \frac{\partial \psi}{\partial z} \]

\[ w = \frac{1}{r} \frac{\partial \psi}{\partial r} \]

The variation of density with respect to temperature can be described by Boussinesq approximation as: \( \rho = \rho_\alpha [1 - \beta_T (T - T_\alpha)] \)

The corresponding boundary condition, when heat is supplied at three different locations at the inner wall of the vertical annular cylinder:

At \( r = r_1 \) and \( 0 \leq z \leq \frac{H}{6}, \frac{5H}{12} \leq z \leq \frac{7H}{12}, \frac{5H}{6} \leq z \leq H, T_w = T_\alpha + B(z)^3, \psi = 0 \)

At \( r = r_0 \)

The new parameters arising due to cylindrical co-ordinates system are:

Non-dimensional Radius \[ r = \frac{r}{L} \]

Non-dimensional Height \[ z = \frac{z}{L} \]

Non-dimensional Stream function \[ \psi = \frac{\psi}{\alpha L} \]

Non-dimensional Temperature \[ \bar{T} = \left( \frac{T - T_\alpha}{T_w - T_\alpha} \right) \]

Rayleigh Number \[ \text{Ra} = \frac{g \beta_T \Delta TLK}{\nu \alpha \lambda} \]

The non-dimensional equations for the heat transfer in vertical cylinder are:

Momentum equation: \[ \frac{\partial^2 \bar{\psi}}{\partial z^2} + \bar{T} \left( \frac{1}{r} \frac{\partial \bar{\psi}}{\partial r} \right) = \text{Ra} \frac{\partial \bar{T}}{\partial r} \]

Energy Equation: \[ \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) - \frac{\partial^2 T}{\partial z^2} \right] = \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \bar{T}}{\partial r} \right) + \frac{\partial^2 \bar{T}}{\partial z^2} \right) \]

The corresponding non-dimensional boundary conditions, when heat is supplied at three different locations at the inner wall of the vertical annular cylinder

At \( r = r_1 \) and \( 0 \leq z \leq \frac{H}{6}, \frac{5H}{12} \leq z \leq \frac{7H}{12}, \frac{5H}{6} \leq z \leq H, T = T_\alpha + B(z)^3, \bar{\psi} = 0 \)

At \( r = r_0 \)

\[ \bar{T} = 0, \quad \bar{\psi} = 0 \]

IV. Method of Solution:

Equations (12) and (13) are the coupled partial differential equations to be solved in order to predict the heat transfer behavior. These equations are solved by using finite element method. Galerkin approach is used to convert the partial differential equations into a matrix form of equations. A simple 3-noded triangular element is considered. The polynomial function for \( T \) can be expressed as \( T = \alpha_1 + \alpha_2 r + \alpha_3 z \)

The variable \( T \) has the value \( T_i, T_j \) and \( T_k \) at the nodal positions \( i, j, k \) of the element. The \( r \) and \( z \) co-ordinates at these points are \( r_i, r_j, r_k \) and \( z_i, z_j, z_k \) respectively. Since \( T = N_iT_i + N_jT_j + N_kT_k \)

Where \( N_i, N_j \) & \( N_k \) are shape functions given by
\[ N_m = \frac{a_m + b_m r + c_m z}{2a}, \quad m = 1, 2, 3 \quad \text{Where } a_m, b_m, c_m \text{ are matrix coefficients.} \]

Following the same procedure the Eq(12) becomes:

\[
\frac{2\pi R}{4A} \begin{bmatrix} b_1^2 & b_1 b_2 & b_1 b_3 \\ b_1 b_2 & b_2^2 & b_2 b_3 \\ b_1 b_3 & b_2 b_3 & b_3^2 \end{bmatrix} + \begin{bmatrix} c_1^2 & c_1 c_2 & c_1 c_3 \\ c_2 c_1 & c_2^2 & c_2 c_3 \\ c_3 c_1 & c_3 c_2 & c_3^2 \end{bmatrix} \begin{bmatrix} \bar{\psi}_1 \\ \bar{\psi}_2 \\ \bar{\psi}_3 \end{bmatrix} = \frac{2\pi R^2 Ra}{6} \begin{bmatrix} \frac{b_1 T_1 + b_2 T_2 + b_3 T_3}{T_1} \\ \frac{b_1 T_1 + b_2 T_2 + b_3 T_3}{T_2} \\ \frac{b_1 T_1 + b_2 T_2 + b_3 T_3}{T_3} \end{bmatrix}
\]

The stiffness matrix of Energy Equation is:

\[
\frac{2\pi}{12A} \begin{bmatrix} c_1 \bar{\psi}_1 + c_2 \bar{\psi}_2 + c_3 \bar{\psi}_3 \\ c_1 \bar{\psi}_1 + c_2 \bar{\psi}_2 + c_3 \bar{\psi}_3 \\ c_1 \bar{\psi}_1 + c_2 \bar{\psi}_2 + c_3 \bar{\psi}_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} - \frac{2\pi}{12A} \begin{bmatrix} b_1 \bar{\psi}_1 + b_2 \bar{\psi}_2 + b_3 \bar{\psi}_3 \\ b_1 \bar{\psi}_1 + b_2 \bar{\psi}_2 + b_3 \bar{\psi}_3 \\ b_1 \bar{\psi}_1 + b_2 \bar{\psi}_2 + b_3 \bar{\psi}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \bar{T}_1 \\ \bar{T}_2 \\ \bar{T}_3 \end{bmatrix}
\]

\[
+ \frac{2\pi R}{4A} \begin{bmatrix} b_1^2 & b_1 b_2 & b_1 b_3 \\ b_1 b_2 & b_2^2 & b_2 b_3 \\ b_1 b_3 & b_2 b_3 & b_3^2 \end{bmatrix} \begin{bmatrix} \bar{T}_1 \\ \bar{T}_2 \\ \bar{T}_3 \end{bmatrix} + \begin{bmatrix} c_1^2 & c_1 c_2 & c_1 c_3 \\ c_2 c_1 & c_2^2 & c_2 c_3 \\ c_3 c_1 & c_3 c_2 & c_3^2 \end{bmatrix} \begin{bmatrix} \bar{T}_1 \\ \bar{T}_2 \\ \bar{T}_3 \end{bmatrix} = 0
\]

V. Results and Discussion:

Results are obtained in terms of Nusselt number at hot wall for various parameters such as power law exponent 'n', Aspect Ratio(A), Radius ratio(R) and Rayleigh number(Ra), when heat is supplied at three different locations of the hot wall of the vertical annular cylinder.

The average Nusselt number is given by \[ \bar{Nu} = -\frac{1}{L} \int_0^L \left( \frac{\partial T}{\partial y} \right)_{y=0} dy \]

Where \( L \) is the length of the heated wall of the vertical annular cylinder. i.e., \( L = L_1 + L_2 + L_3 \)

\[
\begin{array}{c}
\text{a}) \quad A_r = 0.5 \\
\text{b}) \quad A_r = 1 \\
\text{c}) \quad A_r = 2
\end{array}
\]

Fig:1: Streamlines(left) and Isotherms(Right) for \( \lambda = 1, R = 1, Ra = 50 \)

\[ a) \quad A_r = 0.5 \quad \text{b) } A_r = 1 \quad \text{c) } A_r = 2 \]
Fig. 2: Streamlines (left) and Isotherms (Right) for $\lambda = 1$, $A_r = 0.5$, $Ra = 100$

- a) $R_r = 1$
- b) $R_r = 5$
- c) $R_r = 10$

b)

![Streamlines and Isotherms](image)

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Fig:3: Streamlines(left) and Isotherms(Right) for \( A_r = 0.5, R_r = 1, Ra=100 \)

a) \( \lambda = 0 \) b) \( \lambda = 0.5 \) c) \( \lambda = 1 \)
Fig:4: Streamlines (left) and Isotherms(Right) for A_r =0.5, R_r =1, λ=1
a) Ra =25  b) Ra =50  c) Ra =100
Fig: 5: Nu Variations with $A_r$ at hot wall for different values of $\lambda$ at $R_r=1$, $Ra=50$

Fig: 6: Nu Variations with $R_r$ at hot wall for different values of $\lambda$ at $A_r=0.5$, $Ra=100$
Fig. 1 shows the evolution of streamlines and isothermal lines inside the porous medium for various values of Aspect Ratio $A_r$ for $\lambda = 1$, $R_r = 1$, $Ra = 50$. It is clear from the streamlines and isothermal lines that the thermal boundary layer thickness decreases as the Aspect ratio($A_r$) increases. At low Aspect ratio($A_r$) the streamlines tend to occupy the whole domain of the vertical annular cylinder as compared to the higher values of Aspect ratio($A_r$).

It is clearly seen that more convection heat transfer takes place at the upper portion of the vertical annular cylinder. The streamlines and isothermal lines shift from the left upper portion of the cold wall of vertical annular cylinder.

Fig. 2 illustrates the distribution of streamlines and isothermal lines inside the porous medium of vertical annular cylinder for various values of Radius Ratio $R_r$, corresponding to $\lambda = 1$, $A_r = 0.5$, $Ra = 100$. The streamlines and isothermal lines tend to move towards the hot wall and move away from the cold wall of the vertical annular cylinder as Radius Ratio($R_r$) increases. Which in turn, increases the thermal gradient at hot wall and decreases the same at cold wall of the vertical annular cylinder. This alludes that the heat transfer rate increases at the hot wall and decreases at cold wall with increase in Radius Ratio($R_r$). Thus the thermal boundary layer becomes thinner with increase in Radius Ratio($R_r$).

Fig. 3 depicts the evolution of streamlines and isothermal lines inside the porous medium for various values of power law exponent $\lambda$ for $A_r = 0.5$, $R_r = 1$, $Ra = 100$. The fluid gets heated up near the hot wall and moves up towards the cold wall due to high buoyancy force and then returns back to hot wall of the vertical annular cylinder. For the case of isothermal wall temperature ($\lambda = 0$), the magnitude of the streamlines is high as compared to the non-isothermal temperature ($\lambda \neq 0$). It can be seen from streamlines and isothermal lines that the fluid movement shifts towards the cold wall of the vertical annular cylinder, when the value of power law exponent $\lambda$ is varied from 0 to 1. Thus the circulation of the fluid decreases with the increase in power law exponent $\lambda$.

Fig. 4 shows the evolution of streamlines and isothermal lines inside the porous medium for various values of Rayleigh number (Ra) for $A_r = 0.5$, $R_r = 1$, $\lambda = 1$. The increases Rayleigh number(Ra) promotes the fluid movement due to higher buoyancy force, which in turn allows the convection heat transfer to be dominant. So the magnitude of the streamlines increases with the increase in Rayleigh number (Ra).

Fig. 5 illustrates the average Nusselt number $\bar{Nu}$ at hot wall with respect to Aspect ratio($A_r$) for various values of power law exponent($\lambda$), corresponding to $R_r = 1$, $Ra = 50$. For a given value of Aspect ratio($A_r$), the average Nusselt number ($\bar{Nu}$) decreases with the increase in power law exponent($\lambda$). This happens due to the reason that the heat content of the wall is more at wall temperature ($\lambda = 0$) as compared to the non-isothermal.

Fig. 7: $\bar{Nu}$ Variations with $\lambda$ at hot wall for different values of Ra at $A_r=0.5$, $R_r=1$.
temperature ($\lambda \neq 0$), which leads to increased fluid movement at the hot wall, which in turn increases the average Nusselt number ($\overline{Nu}$). Whereas, it is found that the average Nusselt number ($\overline{Nu}$) decreases by 71%, when the power law exponent ($\lambda$) increases from 0 to 1. So, there is a decrease in average Nusselt number ($\overline{Nu}$) for higher values of Aspect ratio ($A_r$).

Fig.6 visualize, the average Nusselt number ($\overline{Nu}$) at hot wall of the vertical annular cylinder, with respect to Radius Ratio ($R_s$) for various values of power law exponent ($\lambda$), corresponding to $A_r = 0.5$, $Ra = 100$. It is found that the average Nusselt number ($\overline{Nu}$) increases with increase in Radius Ratio ($R_s$). But, it is seen that the average Nusselt number ($\overline{Nu}$) decreases with increase in power law exponent ($\lambda$).

So, for a given Radius Ratio ($R_s$), the difference between the average Nusselt number ($\overline{Nu}$) at two different values of power law exponent ($\lambda$) increases with increase in power law exponent ($\lambda$).

For instance, at $R_s = 1$, the average Nusselt number ($\overline{Nu}$), decreased by 69.32%, when power law exponent ($\lambda$) increased from 0 to 1.

Fig.7 shows the variation of the average Nusselt number ($\overline{Nu}$) at hot wall of the vertical annular cylinder, with respect to power law exponent ($\lambda$), for various values of Rayleigh number (Ra) corresponding to $A_r = 0.5$, $R_s = 1$. It is found that the average Nusselt number ($\overline{Nu}$), decreases with the increase in power law exponent ($\lambda$). It can be seen that the average Nusselt number ($\overline{Nu}$) increases with increase in Rayleigh number (Ra) for small values of power law exponent ($\lambda$). As the value of power law exponent ($\lambda$) increases beyond 0.5, practically there is no effect of Rayleigh number (Ra) on the average Nusselt number ($\overline{Nu}$).

At the isothermal wall temperature ($\overline{Nu}$) = 0, the average Nusselt number ($\overline{Nu}$), decreased by 9% , when Rayleigh number (Ra) is increased from 25 to 100. The corresponding decrease in the average Nusselt number ($\overline{Nu}$) at $\lambda = 0.5$, is found to be 0.37%. This shows that the difference between the average Nusselt number ($\overline{Nu}$) at two different values of Rayleigh number (Ra) decreases for non-isothermal temperature ($\lambda \neq 0$) as compared to that of isothermal temperature ($\lambda = 0$).

VI. Conclusion:

It is found that more convection heat transfer takes place at the three positions of heated wall of the vertical annular cylinder at higher values of varying wall temperature ($\lambda'$ and Rayleigh number(Ra)). The magnitude of streamlines decreases with increase in Radius ratio ($R_s$). The effect of Rayleigh number on the average Nusselt number ($\overline{Nu}$) becomes insignificant with the increase of varying wall temperature, i.e., power law exponent '$\lambda$'.

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