Tenser Product of Representation for the Group $C_n$

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Abstract
The main objective of this paper is to compute the tenser product of representation for the group $C_n$. Also algorithms designed and implemented in the construction of the main program designated for the determination of the tenser product of representation for the group $C_n$ including a flow-diagram of the main program. Some algorithms are followed by simple examples for illustration.

Key Words: representation for the group, degree of the representation, character of representation, tenser product.

Introduction
The group of invertible $n \times n$ matrices over a field $F$ denoted by $GL(n,F)$. The matrix representation of a group $G$ is a homomorphism $T:G \rightarrow GL(n,F)$, the degree of this matrix is the degree of that representation [1], the trace for this matrix representation is the character of this representation, [2].

In this paper we consider the group $C_n = \langle x : x^n = 1 \rangle$. In section one the definition of tenser product introduced and apply that for representation of this groups by example, the main proposition introduce for the tenser product which we needed it in section two which include the algorithms designed and implemented in the construction of the main program designated for the determination of the tenser product of representation for the group $C_n$.

§.1 Preliminaries
In this section, we recall definition proposition and remark which we needed in the next section.

Definition 1.1: [3]
Let $A \in M_n(\mathbb{C})$, $B \in M_m(\mathbb{C})$, we defined a matrix $A \otimes B \in M_{nm}(\mathbb{C})$, put

$$A \otimes B = \begin{bmatrix}
\alpha_{11}B & \alpha_{12}B & \ldots & \alpha_{1n}B \\
\alpha_{21}B & \alpha_{22}B & \ldots & \alpha_{2n}B \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n1}B & \alpha_{n2}B & \ldots & \alpha_{nm}B
\end{bmatrix}_{nm \times nm},
A = \begin{bmatrix}
\alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\
\alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n1} & \alpha_{n2} & \ldots & \alpha_{nm}
\end{bmatrix}_{n \times n},
B = \begin{bmatrix}
\beta_{11} & \beta_{12} & \ldots & \beta_{1m} \\
\beta_{21} & \beta_{22} & \ldots & \beta_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{m1} & \beta_{m2} & \ldots & \beta_{mm}
\end{bmatrix}_{m \times m}$$

Thus

$$A \otimes B = \begin{bmatrix}
\delta_{11} & \delta_{12} & \ldots & \delta_{1k} \\
\delta_{21} & \delta_{22} & \ldots & \delta_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{k1} & \delta_{k2} & \ldots & \delta_{kk}
\end{bmatrix}_{nm \times nm}$$

Where $\delta_{ii} = \begin{bmatrix}
\alpha_{i1}\beta_{11} & \alpha_{i1}\beta_{12} & \ldots & \alpha_{i1}\beta_{1m} \\
\alpha_{i1}\beta_{21} & \alpha_{i1}\beta_{22} & \ldots & \alpha_{i1}\beta_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{i1}\beta_{m1} & \alpha_{i1}\beta_{m2} & \ldots & \alpha_{i1}\beta_{mm}
\end{bmatrix}_{m \times m}, \ldots, \delta_{kk} = \begin{bmatrix}
\alpha_{kn}\beta_{11} & \alpha_{kn}\beta_{12} & \ldots & \alpha_{kn}\beta_{1m} \\
\alpha_{kn}\beta_{21} & \alpha_{kn}\beta_{22} & \ldots & \alpha_{kn}\beta_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{kn}\beta_{m1} & \alpha_{kn}\beta_{m2} & \ldots & \alpha_{kn}\beta_{mm}
\end{bmatrix}_{m \times m}$
\[ \delta_{kk} = \begin{bmatrix} \alpha_{m1}\beta_{11} & \alpha_{m1}\beta_{12} & \cdots & \alpha_{m1}\beta_{1m} \\ \alpha_{m2}\beta_{21} & \alpha_{m2}\beta_{22} & \cdots & \alpha_{m2}\beta_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{mn}\beta_{m1} & \alpha_{mn}\beta_{m2} & \cdots & \alpha_{mn}\beta_{nm} \end{bmatrix}_{mn} \quad \text{and } k = nm. \]

Example 1.2:
\[ A = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}_{2 \times 2}, \quad B = \begin{bmatrix} 1 & -2 & -1 \\ 3 & 1 & 2 \\ 6 & 4 & 5 \end{bmatrix}_{3 \times 3} \]
\[ A \otimes B = \begin{bmatrix} 1 & -2 & -1 & 6 & 3 \\ 3 & 1 & 2 & -9 & -3 & -6 \\ 6 & 4 & 5 & -18 & -12 & -15 \\ 2 & -4 & -2 & 0 & 0 & 0 \\ 6 & 2 & 4 & 0 & 0 & 0 \\ 12 & 8 & 10 & 0 & 0 & 0 \end{bmatrix} \]

Proposition 1.3: [4]
Let \( A, A', B, B' \in M_n(K) \), then
1. \( (A + A') \otimes B = (A \otimes B) + (A' \otimes B) \)
2. \( (A \otimes B) (A' \otimes B') = AA' \otimes BB' \)

Remark 1.4:
Let \( S \) and \( T \) be two representations of degree \( n \) and \( m \) of the group \( C_n \), for each \( x \in C_n \) define \( U(x) = S(x) \otimes T(x) \). Then \( U \) is representation of degree \( nm \), we write \( U = S \otimes T \).

Now, let \( \chi_S, \chi_T \) be two character of \( S \) and \( T \) respectively then \( \chi_U = \chi_S \chi_T \).

§.2 The Algorithms
This section contains a collection of the computer ready Fortran algorithms for many standard methods of number theory installed in our main program.

Algorithm (1): The Number of Degree of Representation for the Group \( C_n \)
Input: \( n \) (the degree of the group \( C_n \))
Step 1: To evaluate \( m \) where \( T:C_n \longrightarrow M(K) \),
\[ M_m(K) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}_{mn} \]
Step 2: Do \( I = 1 \) to \( m \)
Do \( J = 1 \) to \( m \)
Print \( IA(I,J) \)
End J-loop
End I-loop
Output: The number of degree of representation for groups \( C_n \) is \( m \).
Example 2.1:

The representation $T: C_4 \rightarrow M_3(\mathbb{R})$, the degree of this representation for the group $C_4$ is 3.

$C_4 = \langle x : x^4 = 1 \rangle = \{1, x, x^2, x^3\}$

$T(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $T(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $T(x^2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $T(x^3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

Algorithm (2): The Tensor Product of Two Representations for the Group $C_n$

Input: $n$ (the degree of the group $C_n$)

Step 1: Do $C$ is the matrix of dimension $mn \times mn$

$C(0,0) = 0$

Do $I = 1$ to $n$

Do $J = 1$ to $n$

$T(x) = A(I,J)$

End $J$-

loop

End $I$-

loop

Step 2: Do $I = 1$ to $m$

Do $J = 1$ to $m$

Set $T(x) = B(I,J)$

End $J$-

loop

End $I$-

loop

Step 3: call algorithm 1

Step 4: To evaluate $C$ where $C(I,J) = A(I,J) \cdot B$

Step 5: Set $C(1,1) = A(1,1) \cdot B$

$C(1,2) = A(1,2) \cdot B$

$\vdots$

$C(1,n) = A(1,n) \cdot B$

Step 6: Set $C = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix}$

where $B = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1mn} \\ C_{21} & C_{22} & \cdots & C_{2mn} \\ \vdots & \vdots & \ddots & \vdots \\ C_{nm1} & C_{nm2} & \cdots & C_{nmm} \end{bmatrix}_{mn \times mn}$

Output: The tensor product of two representations of $C_n$ is $C(mn,mn)$

Example 2.2:

The representation $T: C_3 \rightarrow M_3(\mathbb{R})$, the degree of this representation for the group $C_3$ is 3.

$C_3 = \langle x : x^3 = 1 \rangle = \{1, x, x^2 \}$

$T(1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$, $T(x) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{3 \times 3}$, $T(x^2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}_{3 \times 3}$
$\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$

Algorithm (3): The Tensor Product of Three Representations for the Group $C_n$

Input: $n$ (the degree of the group $C_n$)

Step 1: Call algorithm 2

Step 2: Do $I = 1$ to $k$
  Do $J = 1$ to $k$
  $D(I,J)$
  End J-loop
End I-loop
Step 3: To evaluate R where \( R(I,J) = C(I,J) \ast D \)

Step 4: Set
\[
\begin{align*}
R(1,1) &= C(1,1) \ast D \\
R(1,2) &= C(1,2) \ast D \\
&
\end{align*}
\]

where \( D = \begin{bmatrix} D_{11} & D_{12} & \ldots & D_{1k} \\
D_{21} & D_{22} & \ldots & D_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
D_{k1} & D_{k2} & \ldots & D_{kk} \end{bmatrix} \)

Step 5: Set
\[
R = \begin{bmatrix} R_{11} & R_{12} & \ldots & R_{1s} \\
R_{21} & R_{22} & \ldots & R_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
R_{s1} & R_{s2} & \ldots & R_{ss} \end{bmatrix}
\]

where \( s = nm \times k \)

Step 6: Do I = 1 to s
Do J = 1 to s
Print R(I,J)
End J-loop
End I-loop

Output: The tensor product of three representations of \( C_4 \) is \( R(s,s) \)

**Example 2.3:**

The representation \( T : C_4 \longrightarrow M_2(\mathbb{R}) \), the degree of this representation for the group \( C_4 \) is 2.

\( C_4 = \langle x : x^4 = 1 \rangle = \{1, x, x^2, x^3\} \)

\[
T(1) = T(x^2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T(x) = T(x^3) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

Now present some tensor product for these representations of the group \( C_4 \)

\[
T(1) \otimes T(x) \otimes T(1) = \begin{bmatrix} 00 : 10 & 00 : 00 \\
00 : 01 & 00 : 00 \\
\ldots \ldots & \ldots \ldots \\
10 : 00 & 00 : 00 \\
01 : 00 & 00 : 00 \end{bmatrix}, \quad T(x) \otimes T(x^2) \otimes T(1) = \begin{bmatrix} 00 : 00 & 10 : 00 \\
00 : 00 & 01 : 00 \\
\ldots \ldots & \ldots \ldots \\
10 : 00 & 00 : 00 \\
01 : 00 & 00 : 00 \end{bmatrix}
\]
Algorithm (4): The Character of Representations for the Group C_n

Input: n (the degree of the group C_n)

Step 1: \( \gamma(0) = 0 \)

Step 2: Do I = 1 to m

\[ \gamma_i \]

End I-loop

Step 3: Do J = 1 to n

\[ \gamma_j \]

End J-loop

Step 4: Do I = 1 to m

Do J = 1 to n

\[ \gamma_{ij} = \gamma_i \ast \gamma_j \]

End J-loop

End I-loop

Print \( \gamma_i \)

Step 5: Set \( \chi_k = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_n \end{bmatrix}, s = \frac{nm}{2} \)
Step 6: Call algorithm 3
Step 7: Call algorithm 4
Output: The character of representation for \( C_n \) is \( \chi(k), k = 1 \) to s.

**Example 2.4:**

Consider the character table of \( C_3 \), where \( \omega = e^{\frac{2\pi}{3}} \)

<table>
<thead>
<tr>
<th>Class</th>
<th>1</th>
<th>x</th>
<th>( x^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_2 )</td>
<td>1</td>
<td>( \omega )</td>
<td>( \omega^2 )</td>
</tr>
<tr>
<td>( \chi_3 )</td>
<td>1</td>
<td>( \omega^2 )</td>
<td>( \omega )</td>
</tr>
</tbody>
</table>

- In 1
  - \( \chi_1 \otimes \chi_2 = (1)(1) = 1 \), \( \chi_1 \otimes \chi_3 = (1)(1) = 1 \), \( \chi_2 \otimes \chi_3 = (1)(1) = 1 \)
- In x
  - \( \chi_1 \otimes \chi_2 = (1)(\omega) = \omega \), \( \chi_1 \otimes \chi_3 = (1)(\omega^2) = \omega^2 \), \( \chi_2 \otimes \chi_3 = (\omega)(\omega^2) = 1 \)
- In \( x^2 \)
  - \( \chi_1 \otimes \chi_2 = (1)(\omega^2) = \omega^2 \), \( \chi_1 \otimes \chi_3 = (1)(\omega) = \omega \), \( \chi_2 \otimes \chi_3 = (\omega^2)(\omega) = 1 \)

\[
\chi = \begin{bmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega \\
1 & 1 & 1 \\
\end{bmatrix}
\]

**The Algorithm of the Main Program:**

**The Tensor Product of Representations for Group \( C_n \)**

Input: \( n \) (the degree of the group \( C_n \))

Step 1: Call algorithm 1
Step 2: Call algorithm 2
Step 3: Call algorithm 3
Step 4: Call algorithm

Output: \((T(I), I = 1 \) to m\) To evaluate the tensor product of representation for the group \( C_n \)

End
Flow Diagram of the Main Program

Start

Input n

Open write file "tensor product of representation for the group C_n"

Call number of representation for the group C_n

Call two representations for the group C_n

Call three representations for the group C_n

Do I = 1..n

Set T: C_n \rightarrow M_m(k)
M is the degree of matrix M(I)

Set
\[
\begin{bmatrix}
\sigma_{i1} & \sigma_{i2} & \cdots & \sigma_{in} \\
\sigma_{i1} & \sigma_{i2} & \cdots & \sigma_{in} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{i1} & \sigma_{i2} & \cdots & \sigma_{in}
\end{bmatrix}
\]

1

Print M_m(k)

Call the character of representation for the group C_n

Call p where p partition

Print \tau_{p1} = \sigma_{i1}

Do I = 1..m

\tau_{pI} = \sigma_{iI}

T_1 = \sigma_{i1}B

T_2 = \sigma_{i2}B

End

References


